Complexity for signed graph homomorphisms

Selected topics

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Classic homomorphism complexity

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Theorem (Grohe, 2007)

If $FPT \neq W[1]$ and let $\mathcal{C}$ be a recursively enumerable class of graphs. Then, HOMOMORPHISM for input graphs in $\mathcal{C}$ is polynomial if and only if all cores of graphs in $\mathcal{C}$ have bounded tree-width.

(Explicit lower bounds under ETH by Marx, 2010)

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H-Colouring is polynomial if the core of H has at most one edge.
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List H-Colouring

H-List-Colouring

INSTANCE: A graph G and a list function $L : V(G) \rightarrow 2^{V(H)}$.

QUESTION: is it true that $G \xrightarrow{L} H$? ($\forall x \in V(G)$, the image of x is in $L(x)$)

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Theorem (Feder, Hell, Huang, 2003)

Let H be a graph. If H is a bi-arc graph, H-List-Colouring is polynomial. Otherwise, NP-complete.

List H-Colouring, continued

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Other nice studied cases:

- Connected lists Feder-Hell, 1998 (Polynomial for reflexive chordal graphs)
- Minimum Cost Homomorphisms Gutin-Hell-Rafiey-Yeo, 2008 (Polynomial for reflexive proper interval graphs and irreflexive proper interval bigraphs)
- Digraphs Hell-Rafiey, 2011 (Polynomial for Digraph Asteroidal Triple free digraphs)
- Minimum Cost Homomorphisms for digraphs Hell-Rafiey, 2012 (Polynomial for digraphs with a MIN-MAX ordering)

Instance restrictions - planar graphs

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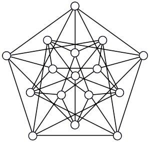

Theorem (Naserasr, 2007)

Let C_{16} be the Clebsch graph. C_{16}-Colouring is poly-time when the input graph is planar.

Girth dichotomy and hypothetical NP-completeness

Conjecture (Jaeger, 1988 - Stockmeyer, Zhang, 2000)

If G is a planar graph with (odd-)girth at least $4k+1$, then $G \to C_{2k+1}$.

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Theorem (Esperet, Montassier, Ochem, Pinlou, 2013)

Let k, g be positive integers. Either all planar graphs of (odd-)girth g are C_{2k+1}-colourable, or C_{2k+1}-COLOURING is NP-complete, even when the input graph is planar.

Instance restrictions - bounded max. degree

Theorem (Holyer, 1981)

K_3-COLOURING is NP-complete, even when the input graph has max. degree 4.

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Whenever H is loop-free and non-bipartite, there exists an integer $b(H)$ such that H-Colouring is NP-complete, even when the input graph has max. degree $b(H)$. Moreover, $b(H) \leq (2\Delta(H)+1)2^{E(\overline{H})}$.

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Theorem (Emden-Weinert, Hougardy, Kreuter, 1998 + Molloy, Reed, 2001)

We have $b(K_k) = O(k + \sqrt{k})$ and for infinitely many values of k, this is tight.

Complexity dichotomy for CSPs

Conjecture (Feder-Vardi, 1998 - Dichotomy Conjecture)

For any **relational structure** S, S-Colouring (i.e. S-CSP) is either NP-complete or polynomial-time solvable.

(**Equivalently:** dichotomy for digraphs, 2-edge-coloured graphs, and for MMSNP)
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Theorem (Bulatov, 2017 + Zhuk, 2017)

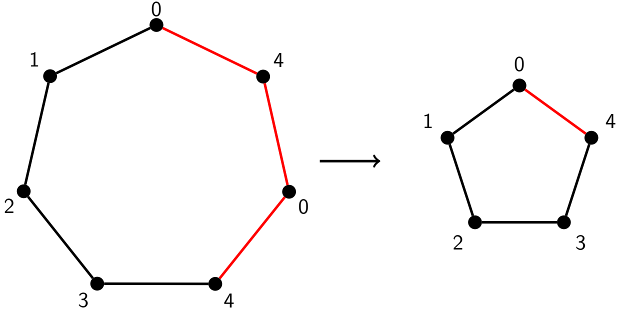
The Algebraic Dichotomy Conjecture is true.

2-edge-coloured graph homomorphisms

Studied by Alon & Marshall, Brewster, Nešetřil & Raspaud...

Definition - Edge-coloured homomorphism of (G, Σ) to (H, Π)

Homomorphism $f : G \to H$ that preserves edge-colors (no re-signing!). We write $(G, \Sigma) \xrightarrow{ec} (H, \Pi)$.



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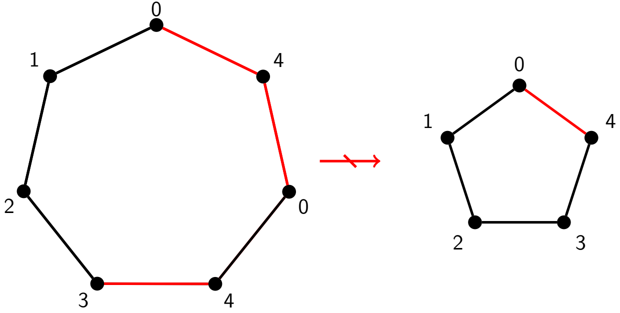

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Concept developed by Naserasr, Rollova, Sopena in 2012.

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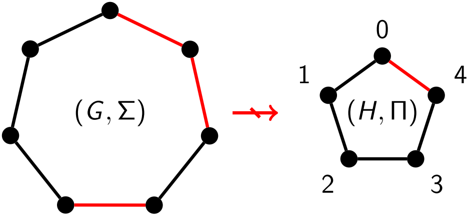

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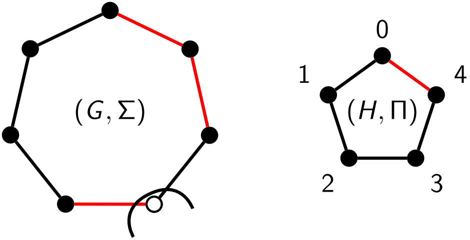

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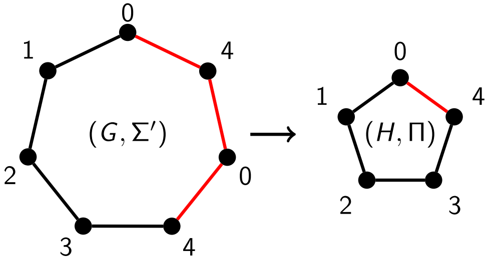

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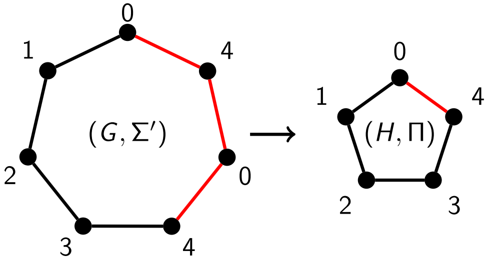

Note: we can assume target signature is fixed $\to$ no re-signing at target.

Dichotomy for signed graphs

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UC_{2k}: unbalanced $2k$-cycle

Theorem (F., Naserasr, 2014)

UC_{2k}-Colouring is NP-complete for every $k \geq 2$
(even for inputs of max. degree 6)

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Theorem (Brewster, F., Hell, Naserasr, 2016 + Brewster, Siggers, 2018+)

If the core of (H,Π) has at most 2 edges, (H,Π)-COLOURING is polynomial. Otherwise, NP-complete.

Poly-time cases

(H, Π)-Colouring

INSTANCE: A signed graph (G, Σ).
QUESTION: does $(G, \Sigma) \xrightarrow{s} (H, \Pi)$?

Proposition

(H, Π)-Colouring is poly-time if:

(a) H is bipartite and $\Pi \equiv \emptyset \equiv E(H)$ (i.e. (H, Π) retracts to an edge);

(b) (H, Π) retracts to a single vertex with loop(s);

(c) H is bipartite and (H, Π) contains (retracts to) a negative digon.

A tool to capture them all

Construction of targets that are invariant under re-signing

(Zaslavsky, Brewster and Graves, Klostermeyer and MacGillivray ...)

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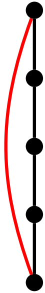

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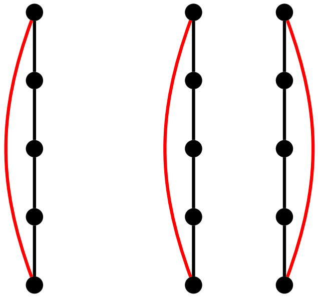

$P(H, \Pi)$

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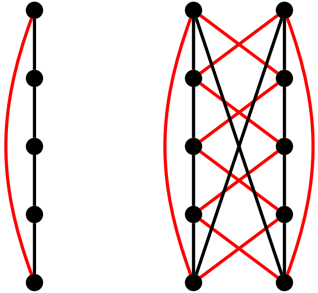

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$\rightarrow$ For every $\Pi' \equiv \Pi$, $P(H, \Pi)$ contains (H, Π') as a subgraph.

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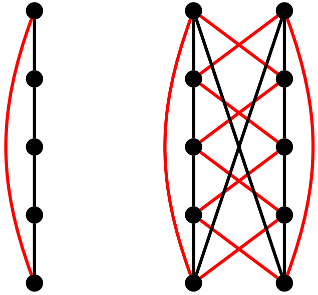

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Theorem

$$(G, \Sigma) \xrightarrow{s} (H, \Pi) \text{ if and only if } (G, \Sigma) \xrightarrow{ec} P(H, \Pi).$$

Hard cases: bipartite graphs

Theorem (Brewster, F., Hell, Naserasr)

If (H, Π) is digon-free and has an unbalanced even cycle, then (H, Π)-Colouring is NP-complete. True even with the presence of (single) loops.

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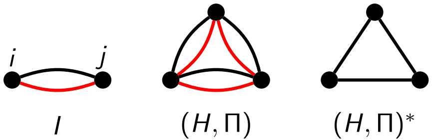
Indicator-construction: classic tool for reductions (see e.g. Hell-Nešetřil).

- Indicator: subgraph I with two distinguished vertices i, j.
- H^*: graph on $V(H)$ with an edge uv iff $f : I \to H$ with $f(i) = u$, $f(j) = v$.
- ${}^*G$: replace each edge of G by a copy of I.



Proposition

$$G \to H^* \iff {}^*G \to H$$

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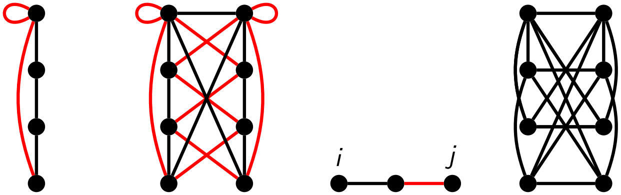
Proof: Use indicator-gadget on $P(H, \Pi)$:



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Hard cases: odd cycles

Theorem (Brewster, F., Hell, Naserasr)

(H,Π)-Colouring is NP-complete if (H,Π) is non-bipartite and:
(a) no loops, no digon;
(b) (single) loops but no digon;
(c) a digon and only one kind of (single) loops.

Proof: Either

- $P(H,\Pi)$ has a $+$ odd cycle and no $+$ loop (or vice-versa), or
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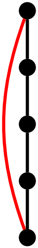

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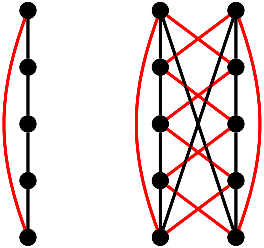

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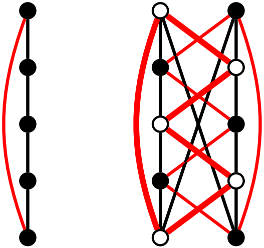

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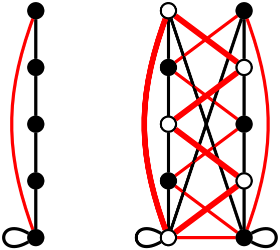

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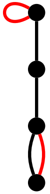

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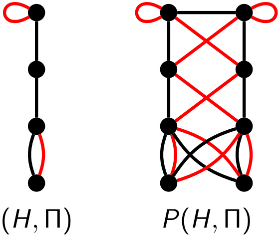

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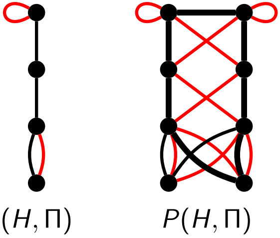
(b) (single) loops but no digon;
(c) a digon and only one kind of (single) loops.

Proof: Either

- $P(H, \Pi)$ has a $+$ odd cycle and no $+$ loop (or vice-versa), or
- $P(H, \Pi)^*$ has an odd cycle and no loop



Hard cases: odd cycles

Theorem (Brewster, F., Hell, Naserasr)

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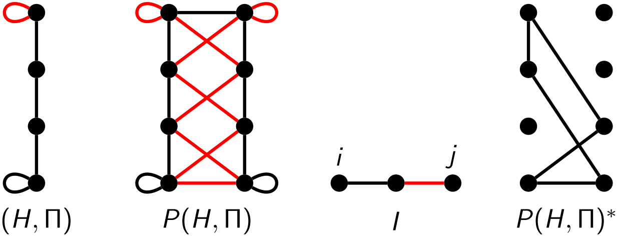

Open problems

- complexity of UC_{2k}-Colouring for planar instances
- hypothetical complexity for planar graphs with given girth
- complexity of (H, Π)-Colouring for bounded max. degree instances
- study complexity of (H, Π)-List-Colouring
- ...