Switchable 2-Colouring is Polynomial

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A homomorphism from a \( m \)-edge coloured graph \( G \) to a \( m \)-edge coloured graph \( H \) is a function \( h: V(G) \rightarrow V(H) \) such that the image of an edge of colour \( \phi \) in \( G \) is an edge of colour \( \phi \) of \( H \).
A **homomorphism** from a $m$-edge coloured graph $G$ to a $m$-edge coloured graph $H$ is a function $h : V(G) \rightarrow V(H)$ such that the image of an edge of colour $\phi$ in $G$ is an edge of colour $\phi$ of $H$. 
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Homomorphism

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![Diagram](image-url)
Vertex Switch

Let $G$ be a $m$-edge coloured graph, $\Gamma$ be a group acting on the edge colours, and $\pi \in \Gamma$ be a permutation. We define switching at a vertex $v$ with respect to $\pi$ as follows. Replace each edge $vw$ of colour $\phi$ by an edge $vw$ of colour $\pi(\phi)$.
Let $G$ be a $m$-edge coloured graph, $\Gamma$ be a group acting on the edge colours, and $\pi \in \Gamma$ be a permutation.
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[Diagram of the switching process]
Switch Equivalence

Two graphs $m$-edge coloured graphs $G$ and $H$ are switch equivalent with respect to a group $\Gamma$ if there exists a sequence of switches that can be applied to vertices of $G$, after which the resulting graph is isomorphic to $H$.

It is important to note that the order of switches matters. This follows as $\Gamma$ is not necessarily Abelian.
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Switch Equivalence Example
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Switchable Homomorphism

A $m$-edge coloured graph $G$ is switchably homomorphic to a $m$-edge coloured graph $H$ with respect to a group $\Gamma$ if there exists a sequence of switches at vertices of $G$ such that the resulting graph has a homomorphism to $H$. This is denoted as $G \rightarrow \Gamma H$. 
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![Diagram of graphs G and H with vertices v1 to v9 and edges colored in blue and red, showing a switchable homomorphism between the two graphs.]
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\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{A} \\
\text{B}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{C} \\
\text{D}
\end{array}
\end{array}
\end{array}

G \quad H
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{v}_1 \\
\text{v}_2
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{v}_3
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{v}_4 \\
\text{v}_5
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{v}_6
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{v}_7 \\
\text{v}_8
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{v}_9
\end{array}
\end{array}
\end{array}
\end{array}

\]
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Switchable Homomorphism

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\begin{center}
\begin{tikzpicture}
    \node[fill] (v1) at (0,0) {}; \node (v1) at (0,0) {$v_1$};
    \node[fill] (v2) at (1,0) {}; \node (v2) at (1,0) {$v_2$};
    \node[fill] (v3) at (2,0) {}; \node (v3) at (2,0) {$v_3$};
    \node[fill] (v4) at (0,1) {}; \node (v4) at (0,1) {$v_4$};
    \node[fill] (v5) at (1,1) {}; \node (v5) at (1,1) {$v_5$};
    \node[fill] (v6) at (2,1) {}; \node (v6) at (2,1) {$v_6$};
    \node[fill] (v7) at (0,2) {}; \node (v7) at (0,2) {$v_7$};
    \node[fill] (v8) at (1,2) {}; \node (v8) at (1,2) {$v_8$};
    \node[fill] (v9) at (2,2) {}; \node (v9) at (2,2) {$v_9$};
    \draw[blue, thick] (v1) -- (v2) -- (v3);
    \draw[red, thick] (v2) -- (v5) -- (v3);
    \draw[blue, thick] (v4) -- (v5) -- (v6);
    \draw[red, thick] (v5) -- (v8) -- (v6);
    \draw[blue, thick] (v7) -- (v8) -- (v9);
    \draw[red, thick] (v8) -- (v1) -- (v9);
    \node at (2.5,0) {$G$};
    \node at (2.5,2) {$H$};
    \node[fill] (A) at (4,0) {}; \node (A) at (4,0) {$A$};
    \node[fill] (B) at (5,0) {}; \node (B) at (5,0) {$B$};
    \node[fill] (C) at (4,1) {}; \node (C) at (4,1) {$C$};
    \node[fill] (D) at (5,1) {}; \node (D) at (5,1) {$D$};
    \draw[blue, thick] (A) -- (B) -- (D);
    \draw[red, thick] (A) -- (C) -- (B);
\end{tikzpicture}
\end{center}
Switchable Homomorphism

A $m$-edge coloured graph $G$ is switchably homomorphic to a $m$-edge coloured graph $H$ with respect to a group $\Gamma$ if there exists a sequence of switches at vertices of $G$ such that the resulting graph has a homomorphism to $H$. This is denoted as $G \rightarrow_{\Gamma} H$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_diagram.png}
\caption{Example of switchable homomorphism.}
\end{figure}
Switchable Homomorphism DP

Input: A $m$-edge coloured graph $G$.

Question: Does $G$ admit a switchable homomorphism to $H$ with respect to $\Gamma$?

We note that a 2-colouring of an $m$-edge coloured graph is a homomorphism to a monochromatic $K_2$. 

Switchable Homomorphism Decision Problem (Γ − HOM(H))

Input: A $m$-edge coloured graph $G$.

Question: Does $G$ admit a switchable homomorphism to $H$ with respect to $\Gamma$?

We note that a 2-colouring of an $m$-edge coloured graph is a homomorphism to a monochromatic $K_2$. 
Statement of Main Result

Theorem

Let $H$ be a monochromatic $K_2$. Then for any finite group $\Gamma$, $\Gamma - \text{HOM}(H)$ is in $P$. 
Some Observations

What can we say about $\Gamma - \text{HOM}(H)$ when $H$ is a monochromatic $K_2$?

- $G$ can be assumed to be bipartite.

- $\Gamma$ can be assumed to be transitive.
Some Observations

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Theorem

Let $G$ be a $m$-edge coloured graph, $C(G)$ be the set of all cycles of $G$, and $H$ be a monochromatic $K_2$. Then $G \rightarrow \Gamma H$ if and only if for each $C \in C(G)$, $C \rightarrow \Gamma H$. 
Cycle Result

**Theorem**

Let $G$ be a $m$-edge coloured graph, $\mathcal{C}(G)$ be the set of all cycles of $G$, and $H$ be a monochromatic $K_2$. Then $G \rightarrow_{\Gamma} H$ if and only if for each $C \in \mathcal{C}(G)$, $C \rightarrow_{\Gamma} H$. 
Agreeance Class Definition

Let $G$ be a $m$-edge coloured graph, $\Gamma$ be a group, and $\phi$ and $\phi'$ be edge colours of $G$. We define the relation $\sim_{2^k}$ on the edge colours of $G$ as $\phi \sim_{2^k} \phi'$ if and only if when $C_{2^k}$ has $2^k - 1$ edges of colour $\phi$ and 1 edge of colour $\phi'$, it can be switched to be monochromatic of colour $\phi$. This is an equivalence relation.

We denote the equivalence class with respect to $\sim_{2^k}$ by $[\phi]_{2^k}$. And for an element $\phi' \in [\phi]_{2^k}$ we say $\phi'$ agrees with $\phi$ or that $\phi'$ belongs to the agreeance class of $\phi$. 

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We denote the equivalence class with respect to $\sim_{2k}$ by $[\phi]_{2k}$. And for an element $\phi' \in [\phi]_{2k}$ we say $\phi'$ agrees with $\phi$ or that $\phi'$ belongs to the agreeance class of $\phi$. 
Theorem

For a group $\Gamma$ and an edge colour $\phi$, the agreeance class of $\phi$ is independent of cycle length. That is, $[\phi]^4 = [\phi]^2$ for all $k \in \{2, 3, ...\}$. 
Agreement Class Statement

Theorem
For a group $\Gamma$ and an edge colour $\phi$, the agreement class of $\phi$ is independent of cycle length. That is, $[\phi]_4 = [\phi]_{2k}$ for all $k \in \{2, 3, \ldots \}$. 
Suppose $\phi' \in [\phi]_4$. Our goal is to show $\phi' \in [\phi]_{2^k}$.
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\[
\begin{array}{c}
\text{u}_4 & \text{v}_4 & \text{w}_4 \\
\text{v}_3 & \text{v}_3 & \text{w}_3 \\
\text{u}_2 & \text{v}_2 & \text{w}_2 \\
\text{u}_1 & \text{v}_1 & \text{w}_1 \\
\end{array}
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Suppose $\phi' \in [\phi]_{2k+2}$. Our goal is to show $\phi' \in [\phi]_{2k}$. 
Cotree Result

Theorem

Let $G$ be a $m$-edge coloured graph, $T$ be a spanning tree of $G$, and $H$ be a monochromatic $K_2$. Then $G \rightarrow \Gamma H$ if and only if after $G$ is switched such that $T$ is monochromatic of colour $\phi$, each cotree edge agrees with $\phi$. 
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Main result

Theorem

Let \( H \) be a monochromatic \( K_2 \). Then for any group \( \Gamma \), \( \Gamma - \text{HOM}(H) \) is in \( P \).

Determining if \( G \) is bipartite is \( O(|V(G)| + |E(G)|) \).

Building a monochromatic tree in \( G \) is \( O(|V(G)| + |E(G)|) \).

There are at most \( O(|E(G)|) \) cotree edges.

The agreeance classes depend only on \( \Gamma \) and can be found in advance.
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Let $H$ be a monochromatic $K_2$. Then for any group $\Gamma$, $\Gamma - HOM(H)$ is in $P$. 

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There are at most $O(|E(G)|)$ cotree edges.

The agreeance classes depend only on $\Gamma$ and can be found in advance.
Or is it?
Or is it?

What about \((m, n)\)-mixed graphs?
Mixed Graph Definition

Let $m$ and $n$ be non-negative integers. A $(m,n)$-mixed graph is a mixed graph whose edge set is partitioned into $m$ colour classes and whose arc set is partitioned into $n$ colour classes.
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Let \( m \) and \( n \) be non-negative integers. A \((m,n)\)-mixed graph is a mixed graph whose edge set is partitioned into \( m \) colour classes and whose arc set is partitioned into \( n \) colour classes.

Figure: A \((3, 2)\)-mixed graph
Some More Observations

What can we say about $\Gamma - H_{\text{HOM}}(H)$ when $H$ is a monochromatic $K_2$?

- $G$ can be assumed to be bipartite.
- $\Gamma$ can be assumed to be transitive.
- $G$ can be assumed to have only edges
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- What can we say about $\Gamma - \text{HOM}(H)$ when $H$ is a monochromatic $T_2$?
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- $\Gamma$ can be assumed to be transitive.
- $G$ has only arcs.
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- $G$ has only arcs.
Putting it together

If $\Gamma$ is a group acting on the $n$-colours and arc directions of a $n$-arc coloured oriented graph $G$, then we can model $G$ as a $2n$-edge coloured graph.

Theorem

The problem of deciding whether a given $(m,n)$-mixed graph is switchable 2-colourable with respect to a finite group $\Gamma$ is in $P$.

What about the agreeance classes?
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