Switchable 2-Colouring is Polynomial

Arnott Kidner
Joint work with Rick Brewster, Gary MacGillivray

University of Victoria

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Homomorphism

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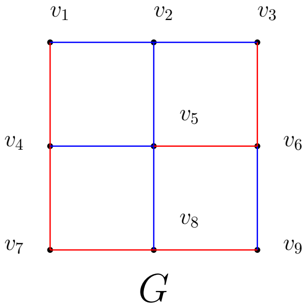

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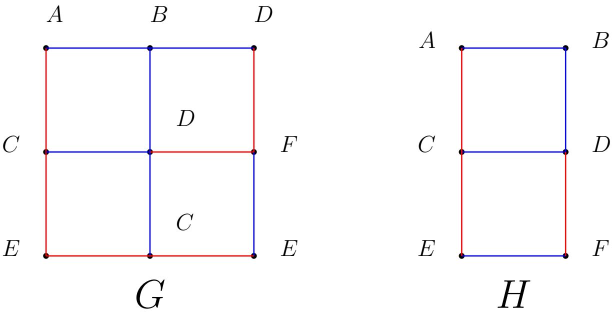

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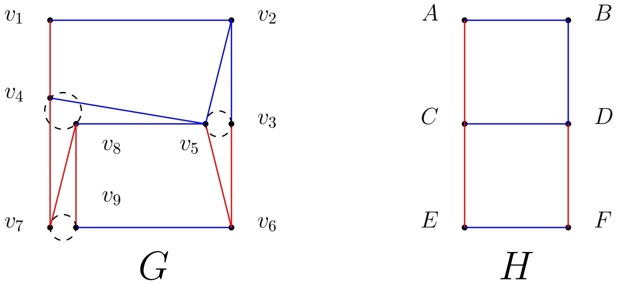

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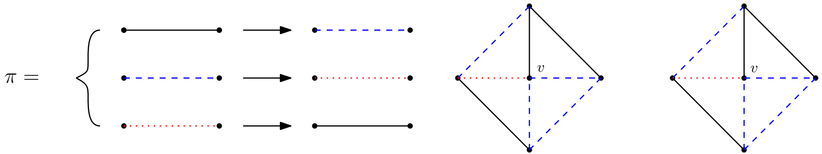

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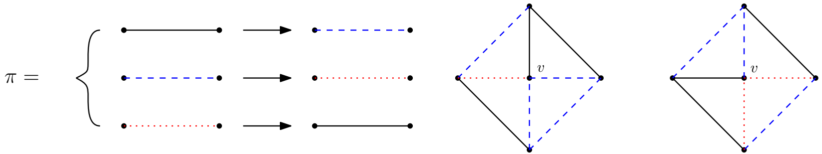

Switch Equivalence

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Two graphs m-edge coloured graphs G and H are switch equivalent with respect to a group Γ if there exists a sequence of switches that can be applied to vertices of G, after which the resulting graph is isomorphic to H.

It is important to note that the order of switches matters. This follows as Γ is not necessarily Abelian.

Switch Equivalence Example

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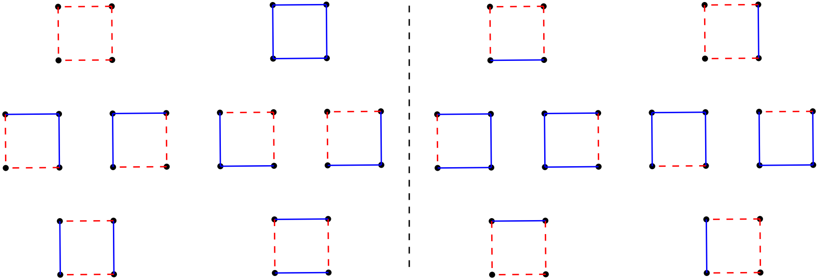

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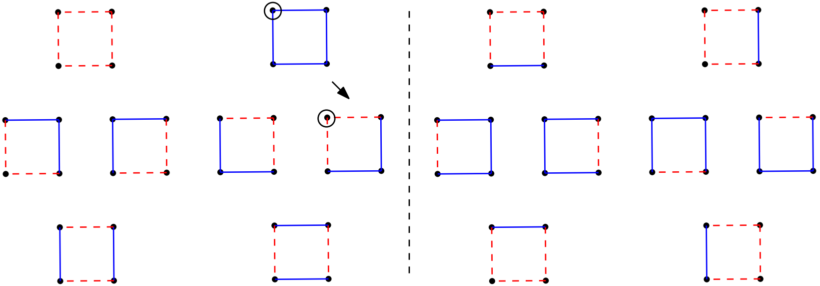

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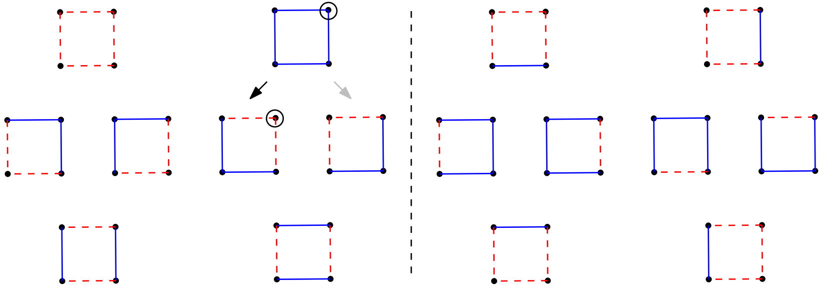

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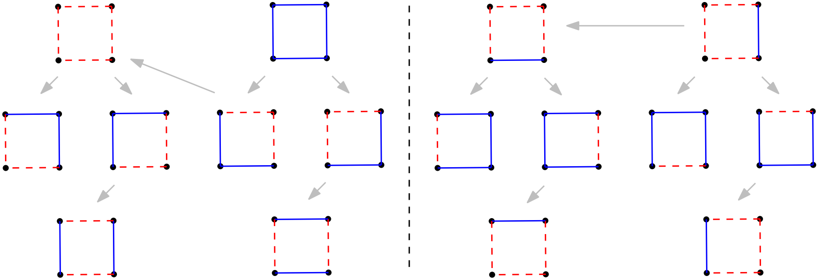

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A m-edge coloured graph G is switchably homomorphic to a m-edge coloured graph H with respect to a group Γ if there exists sequence of switches at vertices of G such that the resulting graph has a homomorphism to H. This is denoted as $G \to_\Gamma H$.

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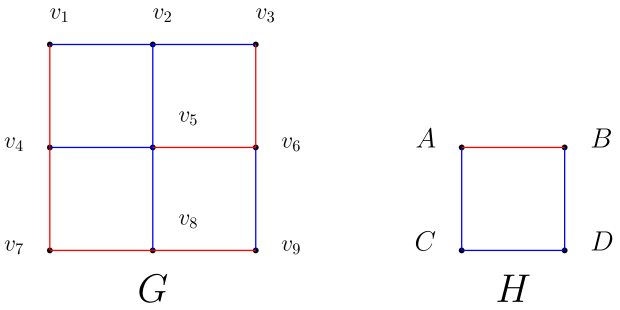

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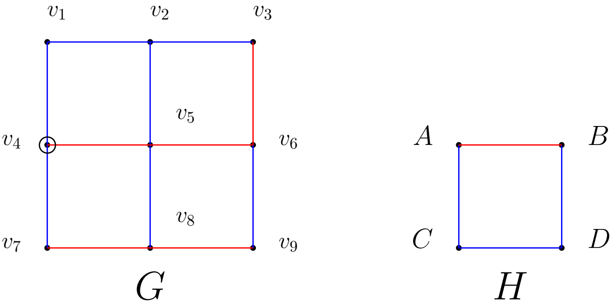

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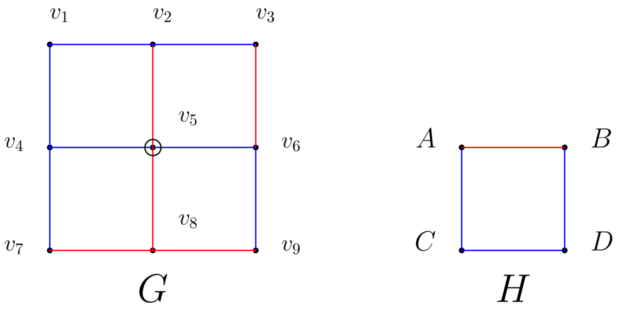

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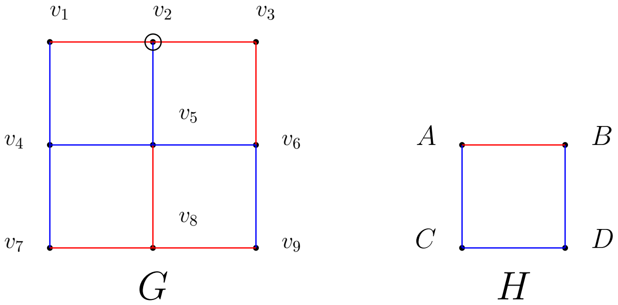

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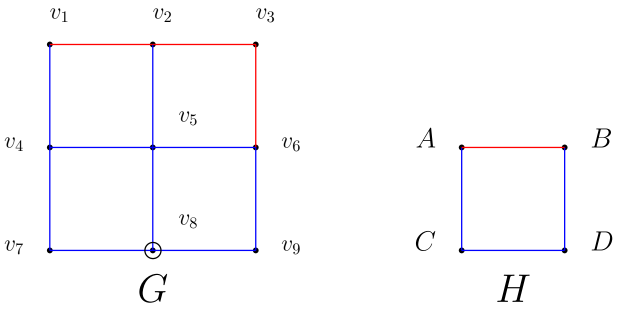

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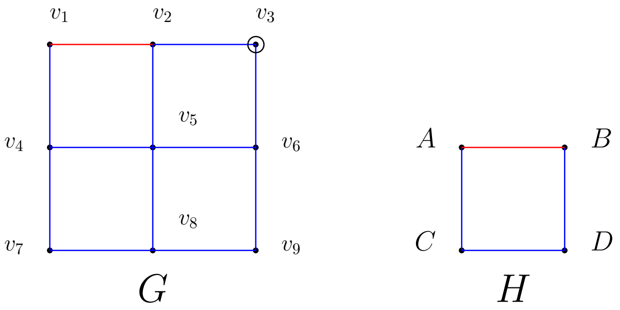

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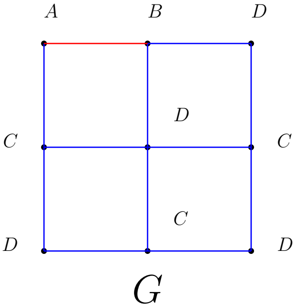

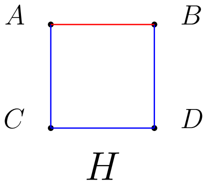

Switchable Homomorphism DP

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Switchable Homomorphism Decision Problem ($\Gamma -$ HOM(H))
Input: A m-edge coloured graph G.
Question: Does G admit a switchable homomorphism to H with respect to Γ?

We note that a 2-colouring of an m-edge coloured graph is a homomorphism to a monochromatic K_2.

Statement of Main Result

Theorem

Let H be a monochromatic K_2. Then for any finite group Γ, $\Gamma - HOM(H)$ is in P.

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-Γ can be assumed to be transitive.

Cycle Result

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Theorem

Let G be a m-edge coloured graph, $\mathcal{C}(G)$ be the set of all cycles of G, and H be a monochromatic K_2. Then $G \to_\Gamma H$ if and only if for each $C \in \mathcal{C}(G)$, $C \to_\Gamma H$.

Agreeance Class Definition

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Let G be a m-edge coloured graph, Γ be a group, and ϕ and ϕ' be edge colours of G. We define the relation $\sim_{2k}$ on the edge colours of G as $\phi \sim_{2k} \phi'$ if and only if when C_{2k} has $2k - 1$ edges of colour ϕ and 1 edge of colour ϕ', it can be switched to be monochromatic of colour ϕ. This is an equivalence relation.

We denote the equivalence class with respect to $\sim_{2k}$ by $[\phi]_{2k}$. And for an element $\phi' \in [\phi]_{2k}$ we say ϕ' agrees with ϕ or that ϕ' belongs to the agreeance class of ϕ.

Agreeance Class Statement

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Theorem

For a group Γ *and an edge colour* ϕ*, the agreeance class of* ϕ *is independent of cycle length. That is,* $[\phi]_4 = [\phi]_{2k}$ *for all* $k \in \{2, 3, \ldots\}$.

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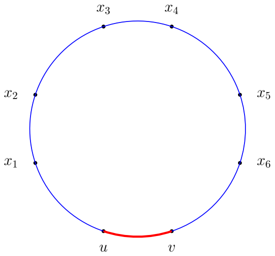

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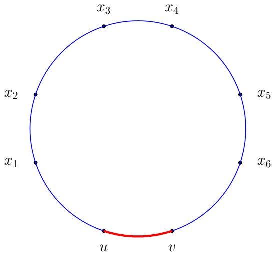

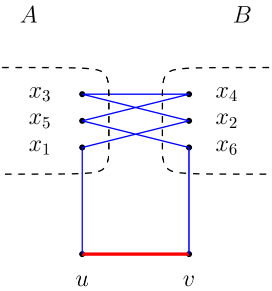

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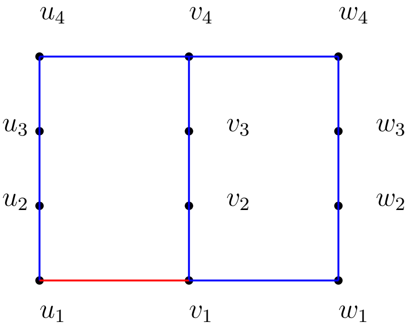

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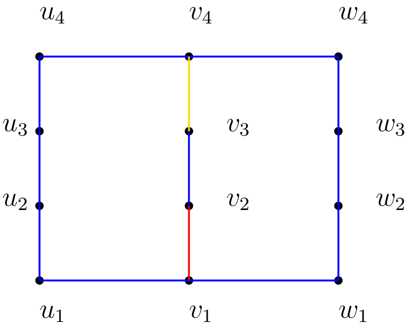

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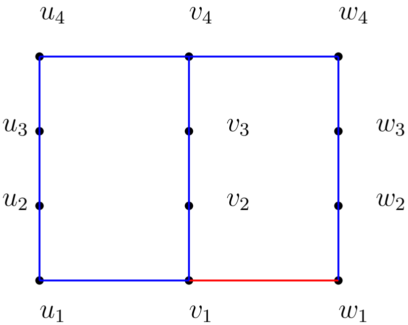

Cotree Result

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Theorem

Let G be a m-edge coloured graph, T be a spanning tree of G, and H be a monochromatic K_2. Then $G \to_\Gamma H$ if and only if after G is switched such that T is monochromatic of colour ϕ, each cotree edge agrees with ϕ.

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The agreeance classes depend only on Γ and can be found in advance.

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What about (m, n)-mixed graphs?

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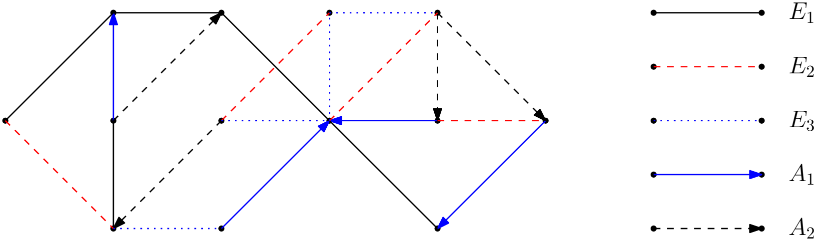

Figure: A $(3, 2)$-mixed graph

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Thank You!