Vertex deletion and edge deletion problems for homomorphisms of signed graphs

Dimitri Lajou
LaBRI, université de Bordeaux

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HOSIGRA 2021

Overview

Overview

Signed graphs

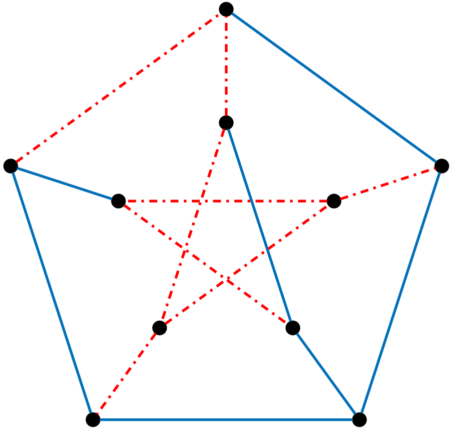


Figure: A signed graph having the Petersen graph as underlying graph.

Balance

Definition (Balance)

Let (G, σ) be a (simple) signed graph and W be a walk $s_0, \ldots, s_n$ in G. The *sign* of W is

$$\sigma(W) = \sigma(s_0 s_1)\sigma(s_1 s_2) \ldots \sigma(s_i s_{i+1}) \ldots \sigma(s_{n-1} s_n)$$

We say that W is a *balanced walk* if $\sigma(W) = +$ and an *unbalanced walk* otherwise.

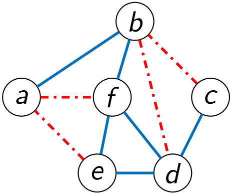
Definition (Balance signed graph)

A signed graph (G, σ) is *balanced* if and only if every closed walk of (G, σ) is balanced. It is *unbalanced* otherwise.
A signed graph (G, σ) is *antibalanced* if and only if $(G, -\sigma)$ is balanced.

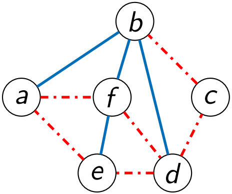
Switching

Definition (Switching)

Let (G, σ) be a signed graph and v be a vertex of G.
To switch v is to create the signed graph (G, σ') where $\sigma'(e) = -\sigma(e)$ when e is incident to v and $\sigma'(e) = \sigma(e)$ otherwise.



(a) A signed graph (G, σ).



(b) The signed graph (G, σ') obtained from (G, σ) by switching d.

Homomorphisms

Definition (Homomorphisms of undirected graphs)

A *homomorphism* from a graph G to a graph H is a function $\varphi : V(G) \to V(H)$ such that for every edge $uv \in E(G)$, $\varphi(u)\varphi(v) \in E(H)$.
When there is a homomorphism from G to H, we note $G \to H$.

Definition (Sign-preserving homomorphisms)

A *sign-preserving homomorphism* from a signed graph (G, σ) to a signed graph (H, π) is a homomorphism φ from G to H such that for every edge $uv \in E(G)$, $\pi(\varphi(u)\varphi(v)) = \sigma(uv)$.
When there is a sign-preserving homomorphism from (G, σ) to (H, π), we note $(G, \sigma) \longrightarrow^{p}_{s} (H, \pi)$.

Homomorphisms

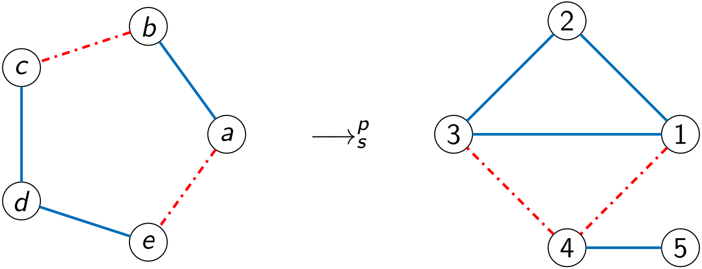


Figure: A sign-preserving homomorphism φ such that $\varphi(a) = 1$, $\varphi(b) = 3$, $\varphi(c) = \varphi(e) = 4$ and $\varphi(d) = 5$.

Homomorphisms

Definition (Homomorphisms of signed graphs)

A *homomorphism* from a signed graph (G, σ) to a signed graph (H, π) is a homomorphism φ from G to H which maps balanced (resp. unbalanced) closed walks of (G, σ) to balanced (resp. unbalanced) closed walks of (H, π).
Equivalently, there is a homomorphism from (G, σ) to (H, π) if and only if there exists (G, σ') obtained by switching some vertices of (G, σ), such that $(G, \sigma') \longrightarrow_s^p (H, \pi)$.
When there is a homomorphism from (G, σ) to (H, π), we note $(G, \sigma) \longrightarrow_s (H, \pi)$.

Homomorphisms

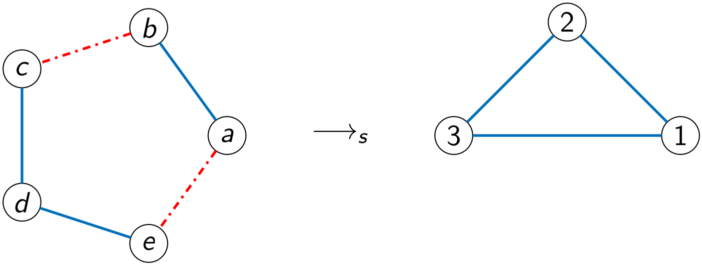


Figure: A homomorphism φ such that $\varphi(a) = 1$, $\varphi(b) = \varphi(d) = 3$ and $\varphi(c) = \varphi(e) = 2$.

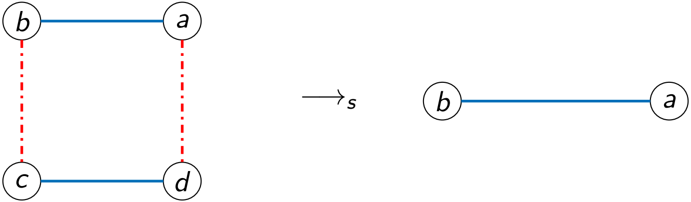
Cores

Definition (s-cores)

The *s-core* of a signed graph (H, π) is the smallest subgraph (C, π_C) of (H, π) for which $(H, \pi) \longrightarrow_s (C, \pi_C)$. An *s-core* is a signed graph (H, π) which is its own s-core.

Observation

If (C, π_C) *is the s-core of* (H, π)*, then:* $(G, \sigma) \longrightarrow_s (H, \pi)$ *if and only if* $(G, \sigma) \longrightarrow_s (C, \pi_C)$.



Overview

HOSIGRA 2019

HOSIGRA 2019: joint work with F. Foucaud, H. Hocquard, V. Mitsou and T. Pierron.

Let (H, π) be a signed graph.

> (H, π)-Coloring
> **Input:** A signed graph (G, σ).
> **Question:** Does $(G, \sigma) \longrightarrow_s^p (H, \pi)$?

Can we "modify" (G, σ) such that the instance becomes a positive instance?

HOSIGRA 2019

VERTEX DELETION (H, π)-COLORING
Input: A signed graph (G, σ), an integer k.
Parameter: k.
Question: Is there a set S of at most k vertices of (G, σ) such that $(G, \sigma) - S \longrightarrow_s^p (H, \pi)$?

EDGE DELETION (H, π)-COLORING
Input: A signed graph (G, σ), an integer k.
Parameter: k.
Question: Is there a set S of at most k edges of (G, σ) such that $(G, \sigma) - S \longrightarrow_s^p (H, \pi)$?

HOSIGRA 2019

Limited Switchings (H,π)-Coloring
Input: A signed graph (G,σ), an integer k.
Parameter: k.
Question: Is there a set S of k vertices of G such that the signed graph (G,σ') obtained from (G,σ) by switching every vertex of S satisfies $(G,\sigma') \longrightarrow_s^p (H,\pi)$?

Something is missing

What about modifications problems for **homomorphisms of signed graphs**?

> SIGNED-(H, π)-COLORING
> **Input:** A signed graph (G, σ).
> **Question:** Does $(G, \sigma) \longrightarrow_s (H, \pi)$?

Overview

Our problems

Let (H, π) be a signed graph.

Vertex Deletion Signed-(H, π)-Coloring
Input: A signed graph (G, σ), an integer k.
Parameter: k.
Question: Is there a set S of at most k vertices of (G, σ) such that $(G, \sigma) - S \longrightarrow_s (H, \pi)$?

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Questions: P/NP-complete dichotomy? FPT algorithms?

Some already known complexities

Theorem (Brewster, Foucaud, Hell, Naserasr, Siggers)

Let (H, π) be a signed graph. The problem Signed-(H, π)-Coloring *is in* P *if the s-core of (H, π) has at most two edges, and is* NP*-complete otherwise.*

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Theorem (Brewster, Foucaud, Hell, Naserasr, Siggers)

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This implies that if (H, π) is an s-core with at least three edges then $\textsc{Vertex Deletion Signed-}(H,\pi)\textsc{-Coloring}$ and $\textsc{Edge Deletion Signed-}(H,\pi)\textsc{-Coloring}$ are NP-complete even for $k = 0$.

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The only open questions are for the following s-cores: H^1_- (), H^1_{rb} (), H^1_b (), H^1_r (), $H^{2b}_{-,-}$ (), $H^{2rb}_{-,-}$ () and $H^{2-}_{r,b}$ ().

Undirected graphs cases

For H^1_- ($\bullet$), H^1_{rb} () and $H^{2rb}_{-,-}$ (), the sign of the edges do not matter. Hence:

- Vertex Deletion Signed-H^1_--Coloring is Vertex Cover,
- Edge Deletion Signed-H^1_--Coloring boils down to counting the number of edges,
- Vertex Deletion Signed-H^1_{rb}-Coloring and Edge Deletion Signed-H^1_{rb}-Coloring are trivial (always answer yes),
- Vertex Deletion Signed-$H^{2rb}_{-,-}$-Coloring is Odd Cycle Transversal,
- Edge Deletion Signed-$H^{2rb}_{-,-}$-Coloring is Edge Bipartization.

Undirected graphs cases

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The only open questions are for the following s-cores: H^1_b (), H^1_r (), $H^{2b}_{-,-}$ () and $H^{2-}_{r,b}$ ().

Cases which reduces to sign-preserving homomrphisms

Brewster, Foucaud, Hell, Naserasr:

$(G,\sigma) \longrightarrow_s$  if and only if $(G,\sigma) \longrightarrow_s^p$  .

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The remaining cases are NP-complete

Theorem

The problem VERTEX DELETION SIGNED-(H,π)-COLORING *is* NP*-complete when* (H,π) *is one of* $H^{2b}_{-,-}$ () *or* $H^{2-}_{r,b}$ ().

Theorem

The problem EDGE DELETION SIGNED-(H,π)-COLORING *is* NP*-complete when* (H,π) *is one of* $H^{2b}_{-,-}$ () *or* $H^{2-}_{r,b}$ ()

Overview

Main result

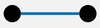
Theorem

The problem VERTEX/EDGE DELETION SIGNED-(H,π)-COLORING *is* FPT *when* (H,π) *is one of* $H^{2b}_{-,-}$ ( ) *and* $H^{2-}_{r,b}$ ( ).

Vertex Deletion Signed-$H^{2b}_{-,-}$-Coloring ()

The problem can be reformulated as follows:

Vertex Deletion Signed-$H^{2b}_{-,-}$-Coloring
Input: A signed graph (G, σ) and an integer k.
Parameter: k
Question: Is there a set S of at most k vertices of G such that $(G, \sigma) - S$ is a **bipartite balanced** signed graph?

VERTEX DELETION SIGNED-$H^{2b}_{-,-}$-COLORING ()

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We need to solve two problems at the same time.
Adapt the FPT algorithm for:

> ODD CYCLE TRANSVERSAL
> **Input:** A graph G and an integer k.
> **Parameter:** k
> **Question:** Is there a set S of at most k vertices of G such that $G - S$ is a **bipartite** graph?

Vertex Deletion Signed-$H^{2-}_{r,b}$-Coloring

Step 1: Apply iterative compression.

VERTEX DELETION SIGNED-$H^{2-}_{r,b}$-COLORING ()

Step 1: Apply iterative compression.

Step 2: Find some B_+ and B_- such that $(G, \sigma) - (B_+ \cup B_-) \longrightarrow_s$ .

Vertex Deletion Signed-$H^{2-}_{r,b}$-Coloring ( )

Step 1: Apply iterative compression.

Step 2: Find some B_+ and B_- such that $(G, \sigma) - (B_+ \cup B_-) \longrightarrow_s$.

Goal: Find a solution of size k among $V(G) - (B_+ \cup B_-)$ such that B_+ maps to and B_- maps to .

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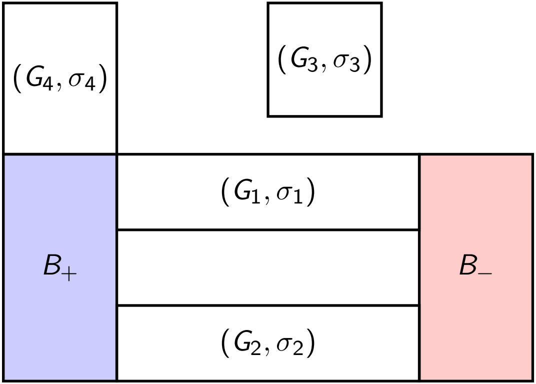

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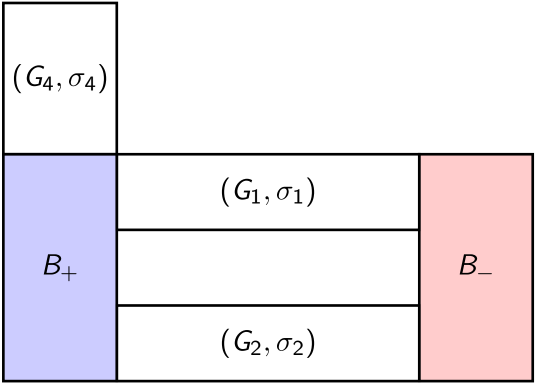

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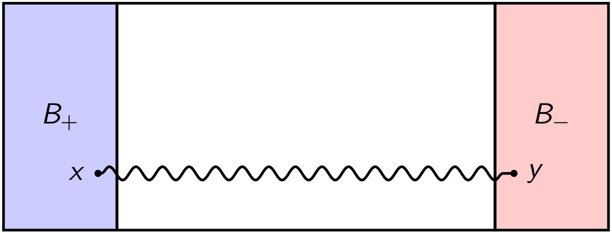
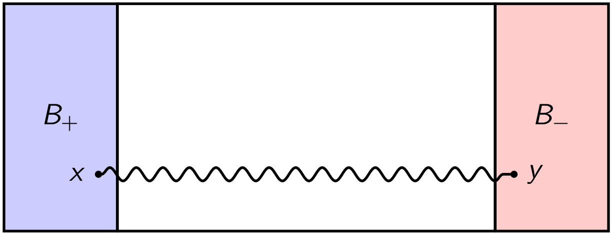


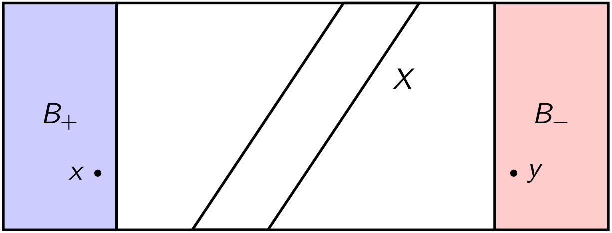
Vertex Deletion Signed-$H_{r,b}^{2-}$-Coloring



Observation

A solution S is a (B_+, B_-)-separator.

Vertex Deletion Signed-$H_{r,b}^{2-}$-Coloring



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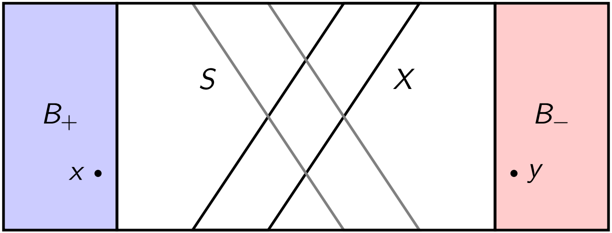

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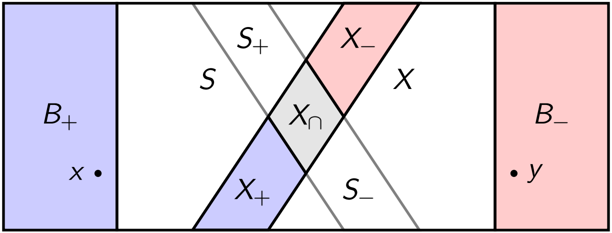

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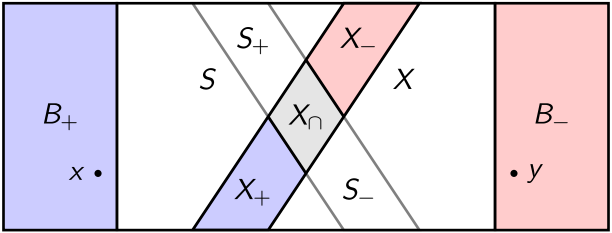
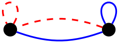
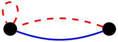
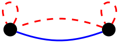


Observation

A solution S is a (B_+, B_-)-separator.

We separate our problem into two sub-problems: the left part and the right part.
Problem: how to find a good (B_+, B_-)-separator X?

Thanks for your attention

(H, π)	Vertex Deletion Signed-(H, π)-Coloring	Edge Deletion Signed-(H, π)-Coloring
	P	P
	NP-hard but FPT	NP-hard but FPT
	NP-hard but FPT	P
	NP-hard but FPT	NP-hard but FPT
	NP-hard but FPT	NP-hard but FPT
	NP-hard even for $k = 0$	NP-hard even for $k = 0$
	NP-hard but FPT	NP-hard but FPT
	NP-hard even for $k = 0$	NP-hard even for $k = 0$
	NP-hard even for $k = 0$	NP-hard even for $k = 0$
	NP-hard even for $k = 0$	NP-hard even for $k = 0$