On Deeply Critical Oriented Cliques

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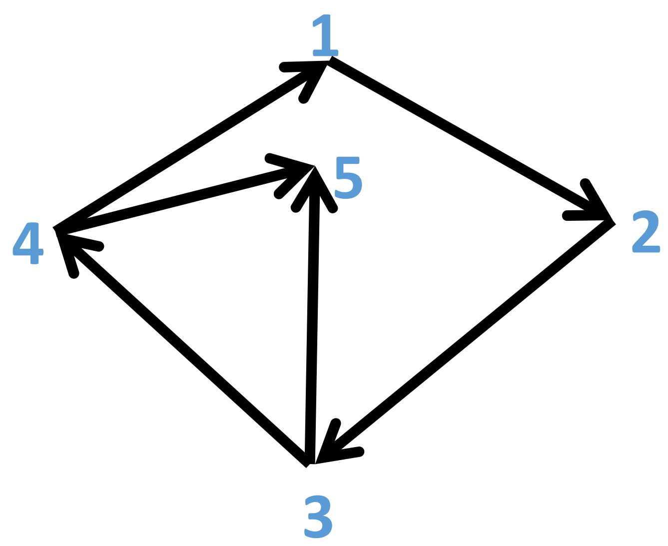
Oriented graphs

An *oriented graph* is a digraph with no loops and no opposite arcs. An oriented graph is thus an *orientation* of a simple undirected graph U, obtained from U by giving to each edge of U one of its two possible orientations.
We denote by $V(G)$ and $A(G)$ the set of vertices and the set of arcs, respectively, of an oriented graph G.

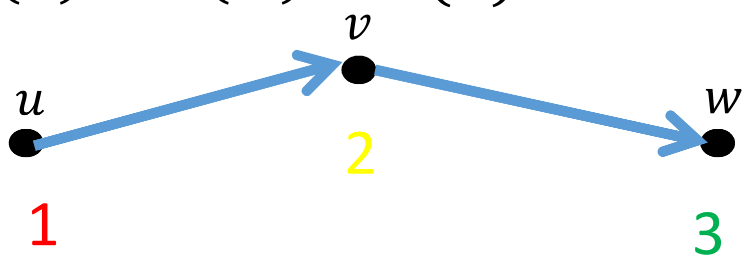


Oriented colouring

An *oriented k-colouring* of an oriented graph G is a mapping $c: V(G) \to C$, where C is a set of k colors (usually $C = \{1, \dots, k\}$), such that:

- $c(u) \neq c(v)$ for every arc uv in $A(G)$,
- $c(u) \neq c(y)$ for every two arcs uv and xy with $c(v) = c(x)$.

Note in particular that if uvw is a directed path in G (that is $uv, vw \in A(G)$), then $c(u) \neq c(v) \neq c(w) \neq c(u)$.



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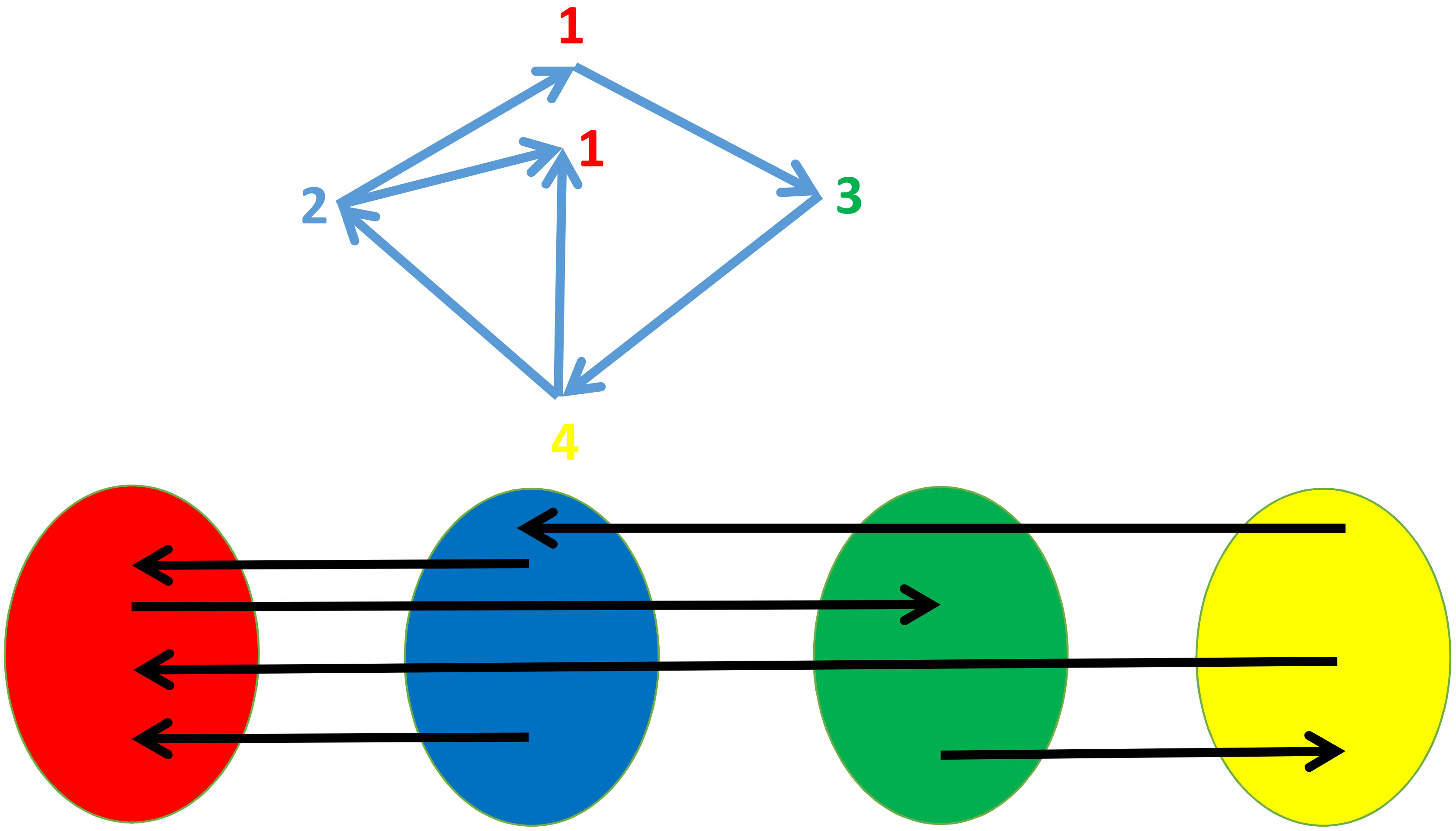
An oriented k -coloring of an oriented graph G thus corresponds to a partition of the vertex set $\mathrm{V}(G)$ of G into k independent subsets such that all the arcs linking any two of these subsets have the same direction.

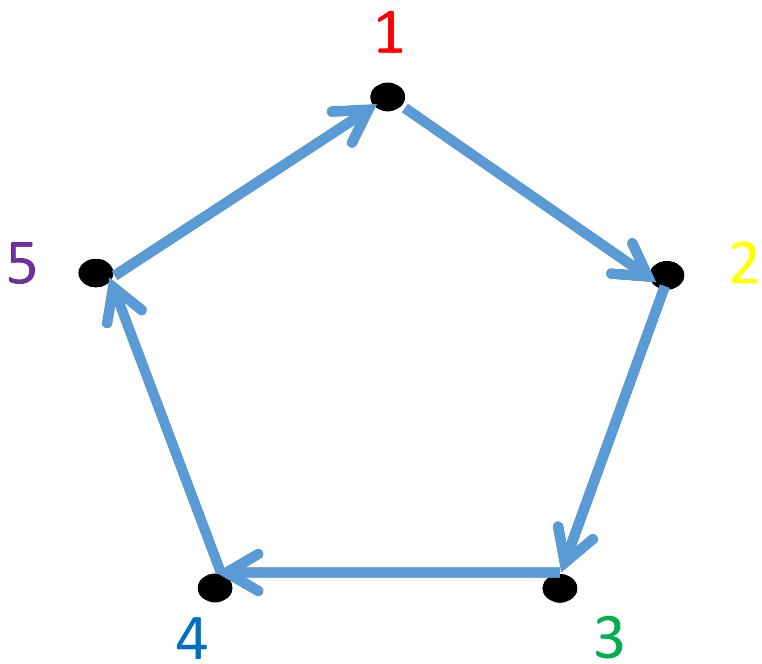
Examples:

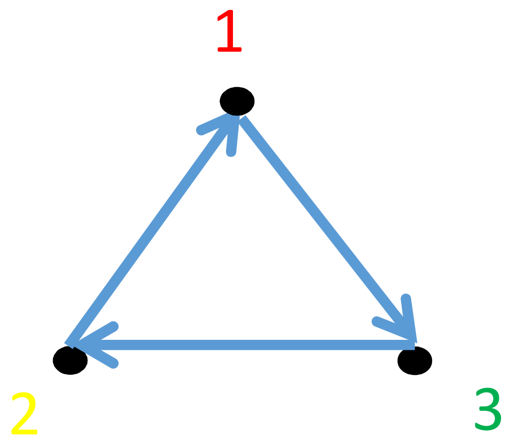


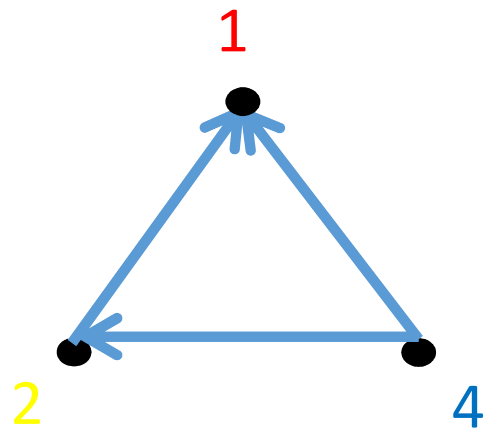
Oriented chromatic number

The *oriented chromatic number* $\chi_0(G)$ of an oriented graph *G* is defined as the smallest k for which G admits an oriented k-colouring.

Examples



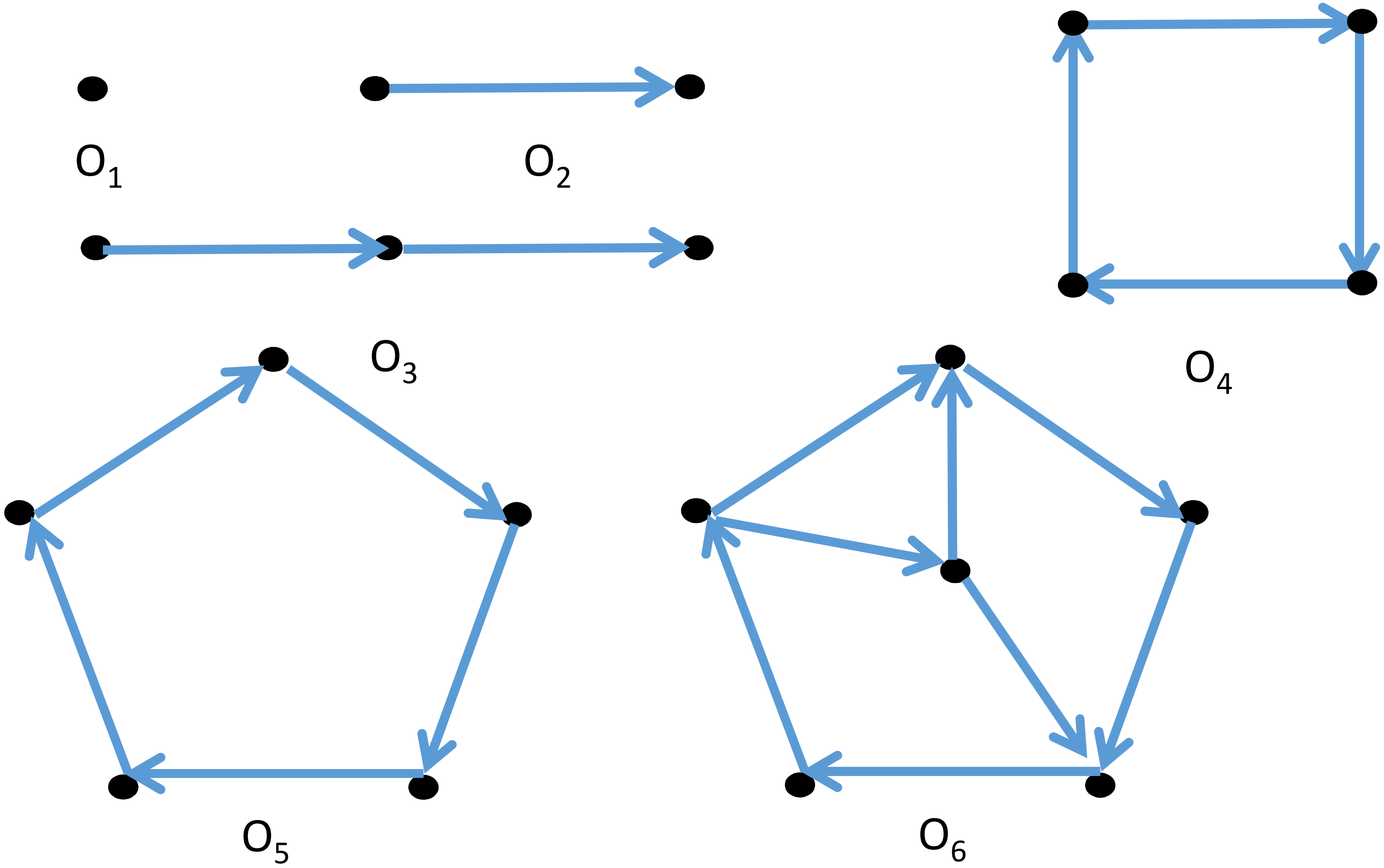




Oriented cliques

An oriented graph G is an *oriented clique* (or, simply, an *o-clique*) if $\chi_0(G) = |V(G)|$. Therefore, an oriented graph is an o-clique if and only if every two of its vertices are linked (in either direction) by a directed path of length 1 or 2.

Examples of o-cliques



Observation

(i) If for some oriented graph H and some $v \in V(H)$, $o(H - v) \leq k$, then $o(H) \leq 2k + 1$. On the other hand, for every positive integer k, there exists an oriented graph H_k and a vertex $v \in V(H_k)$ such that $o(H_k) = 2k + 1$ and $o(H_k - v) = k$.

(ii) For every oriented graph H and $(v, u) \in E(H)$,

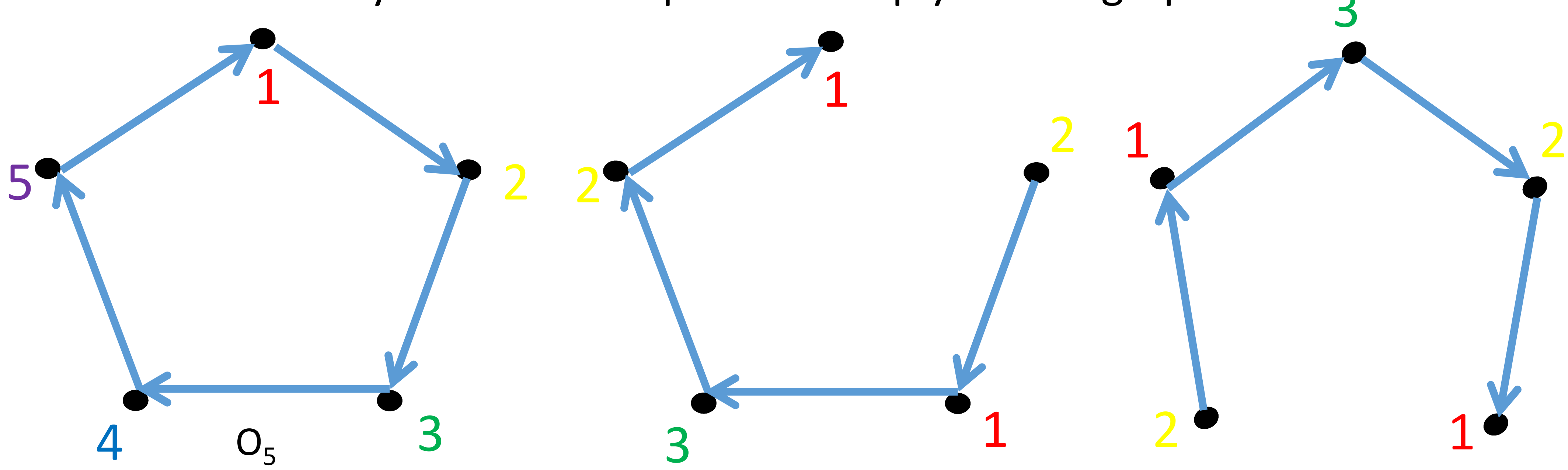
$$\mathrm{o}(H - (v, u)) \geq o(H) - 2.$$

O.V. Borodin, D. Fon-der-Flaass, A.V. Kostochka, A. Raspaud and E. Sopena. On deeply critical oriented graphs. *J. Combin. Theory, Ser. B.* 81:150-155 (2001), doi:10.1006/jctb.2000.1984

Deeply critical graphs

We call an oriented graph H deeply critical if, for every $(v, u) \in E(H)$, $o(H) - o\big(H - (v, u)\big) = 2$.

The oriented 5-cycle is an example of a deeply critical graph:

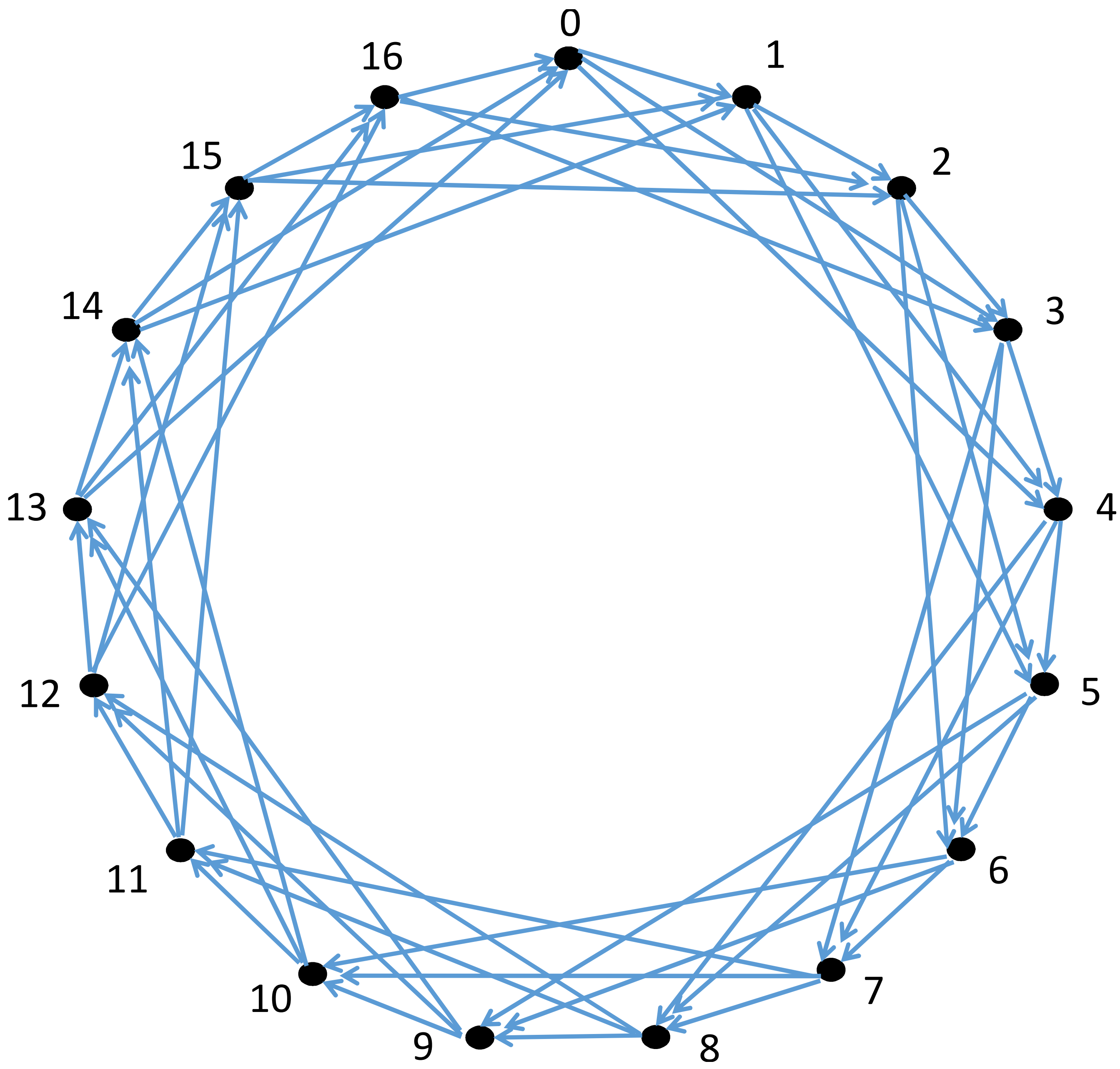


Theorem [Borodin, Fon-der-Flaass, Kostochka, Raspaud, Sopena,]:

For every positive integer k, there exists a deeply critical graph G_k such that $o(G_k) - o(G_k - v) \geq k$ for every $v \in V(G_k)$.

They give the construction for deeply critical graphs with number of vertices $q = 2.3^m - 1$.

O.V. Borodin, D. Fon-der-Flaass, A.V. Kostochka, A. Raspaud and E. Sopena. On deeply critical oriented graphs. *J. Combin. Theory, Ser. B.* 81:150-155 (2001), doi:10.1006/jctb.2000.1984



$q = 17$

Comments in the paper

The paper presents deeply critical graphs only with a very specific number of vertices (namely, with the number of the form $2.3^m - 1$). The authors stated that experimenting on a computer indicated that deeply critical graphs on q vertices with oriented chromatic number q might exist for all odd $q > 31$.

Work progress

Question: For what number of vertices q, does there exist deeply critical graphs with q vertices?

$q = 1$, $q = 2$ and $q = 3$ $\longrightarrow$ Trivial

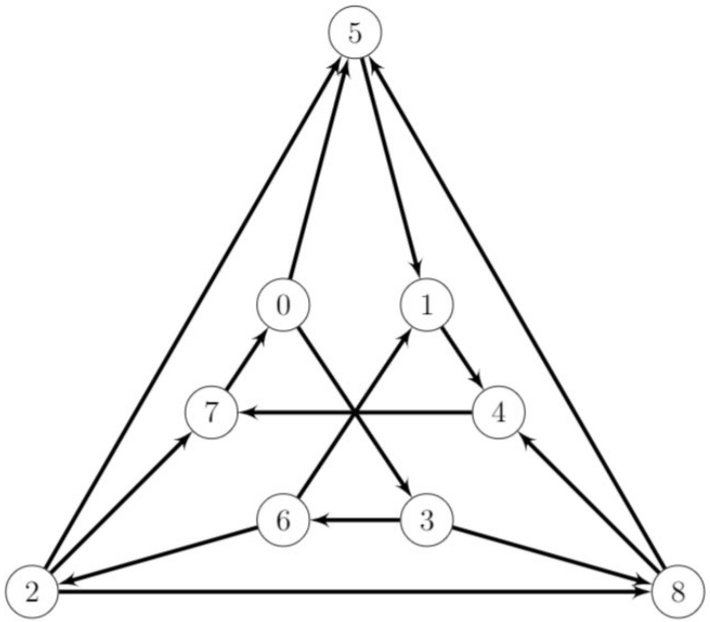
$q = 4$?

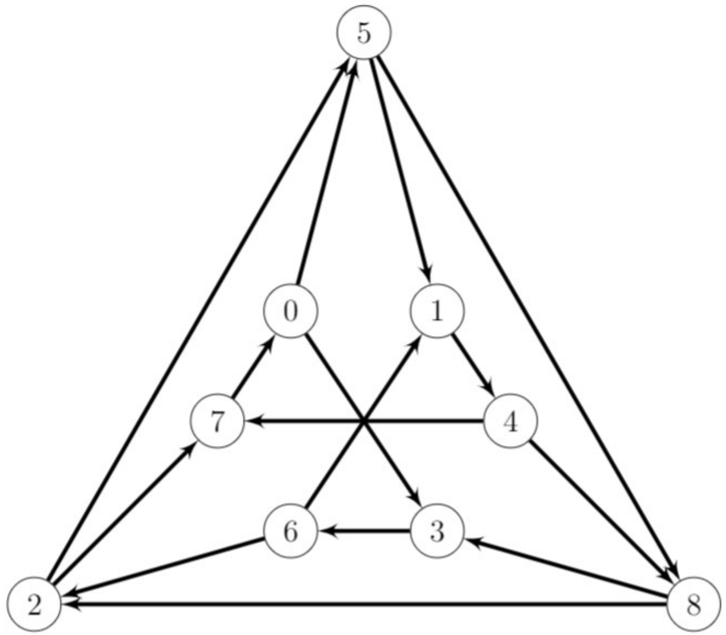
$q = 5$? $-\!\!\rightarrow$ Oriented 5-cycle

$q = 6$? $-\!\!\rightarrow$ No, verified using a computer program.

Further using the program, the following results were found:

- No deeply critical graphs on 7 or 8 vertices.
- There exists deeply critical graphs on 9 vertices, two of which are known but with same underlying graph.



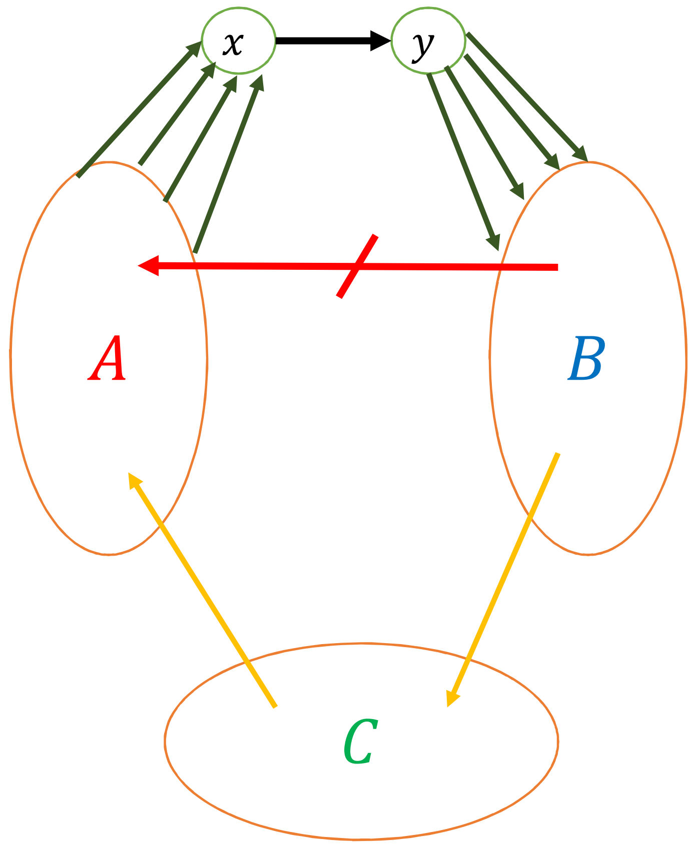


ABC - Partitioning

Let G be a deeply critical oriented clique. Partition the vertices into three nonempty sets A, B and C such that the following conditions are satisfied:

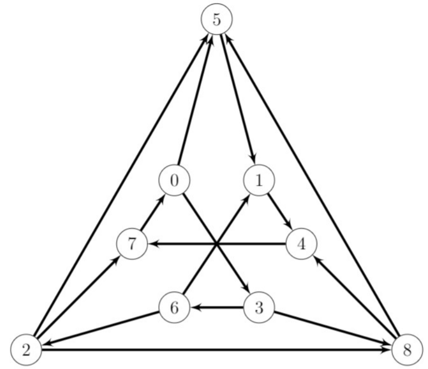
- for every vertex $c \in C$ there is at least one arc to some vertex $a \in A$.
- for every vertex $c \in C$ there is at least one arc from some vertex $b \in B$.
- there are no arcs from any vertex in B to any vertex in A.
- for every vertex $a \in A$ there exists at least one vertex $c \in C$ such that there is only one arc from c to A which is ca.
- for every vertex $b \in B$, there exists at least one vertex $c \in C$ such that there is only one arc from B to c which is bc.

Then, the new graph G' obtained by introducing two new vertices x and y and the arcs xy, arcs of the form $ax\ \forall a \in A$, and arcs of the form $yb\ \forall b \in B$ is a deeply critical oriented clique.



Examples

Starting with 9 vertex deeply critical oriented clique: (in the order C, A, B)



1. set([0, 1, 2, 8]) set([4, 5]) set([3, 6, 7])
2. set([0, 2, 3, 4]) set([8, 9, 5]) set([1, 10, 6, 7])
3. set([0, 1, 10, 12, 8]) set([3, 4, 6]) set([9, 2, 11, 5, 7])
4. set([0, 1, 2, 3, 11, 14]) set([8, 12, 4, 5]) set([9, 10, 13, 6, 7]
5. set([0, 2, 4, 10, 12, 15]) set([8, 16, 3, 13, 6]) set([1, 5, 7, 9, 11, 14])
6. set([0, 1, 2, 10, 12, 16, 17]) set([8, 18, 3, 4, 6]) set([5, 7, 9, 11, 13, 14, 15])
7. set([0, 2, 5, 10, 12, 16, 17, 19]) set([1, 3, 6, 8, 18, 20]) set([4, 7, 9, 11, 13, 14, 15])

Extendable graphs

Let G be an oriented graph. An *extending partition* of G is a partition of its set of vertices $V(G) = X_1 \sqcup X_2 \sqcup X_3$ so that

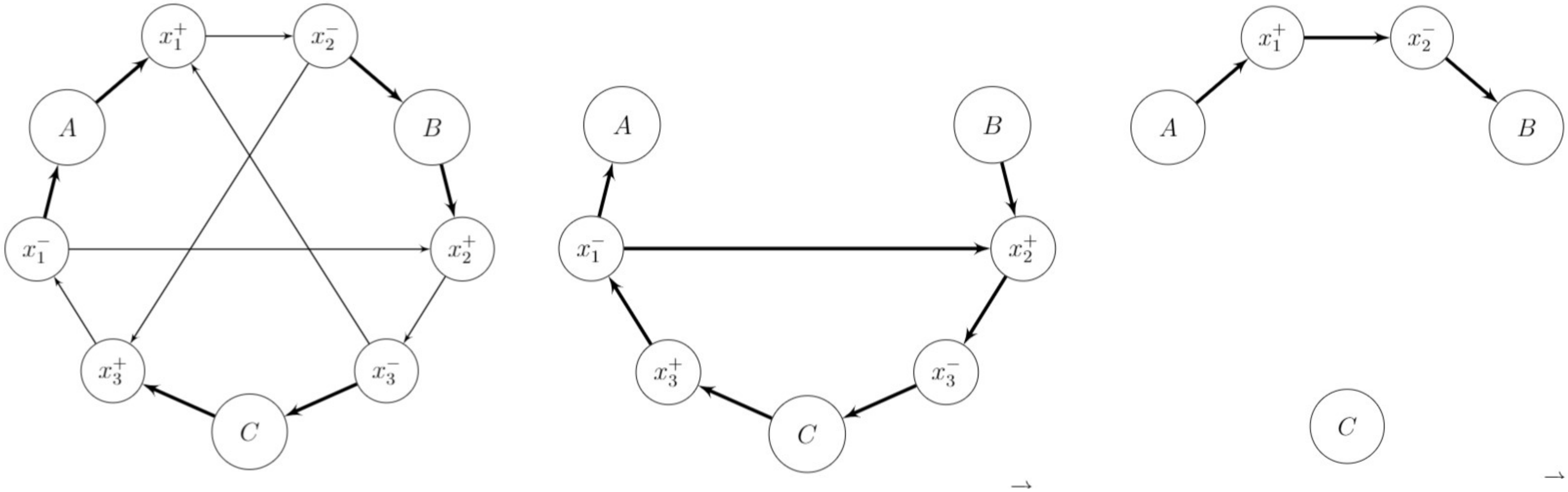
i) there is no arc from a vertex of X_{i+1} to a vertex of X_i, $\forall i \in \{1,2,3\}$;

ii) for each $u \in X_i$, there exists a vertex $v \in X_{i+1}$ such that $N^-(v) \cap X_i = \{u\}$, $\forall i \in \{1,2,3\}$; and

iii) for each $v \in X_{i+1}$, there exists a vertex $u \in X_i$ such that $N^+(u) \cap X_{i+1} = \{v\}$, $\forall i \in \{1,2,3\}$

where addition in indices is taken modulo 3. We say an oriented graph is *extendable* when it admits an extending partition.

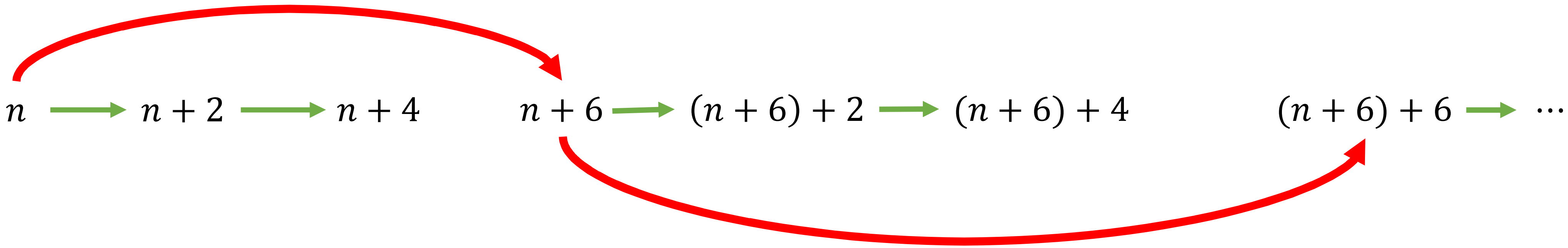
6-extension, 4-extension and 2-extension



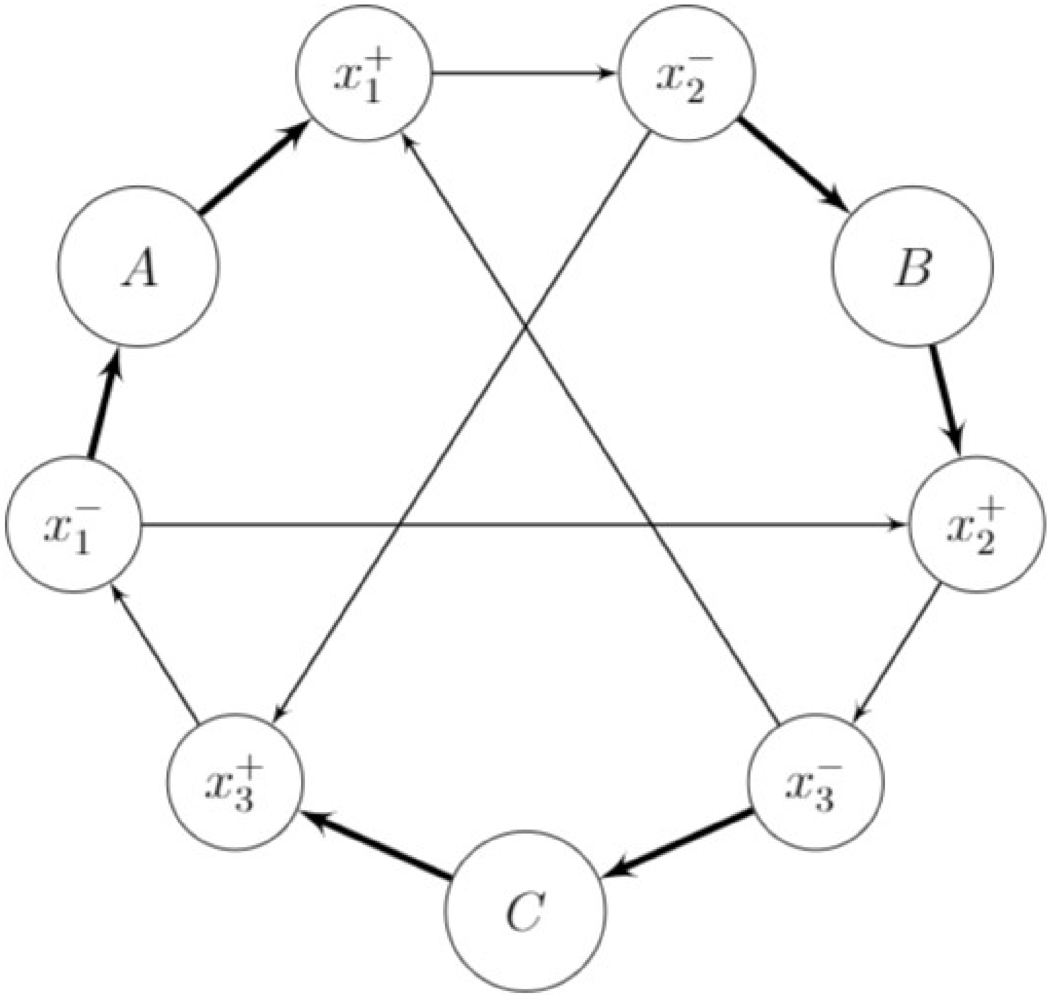
G_6 G_4 G_2

$Lemma$: Let G be an extendable deeply critical oriented clique. The 2-extension, 4-extension, and the 6-extension of G are deeply critical oriented cliques.

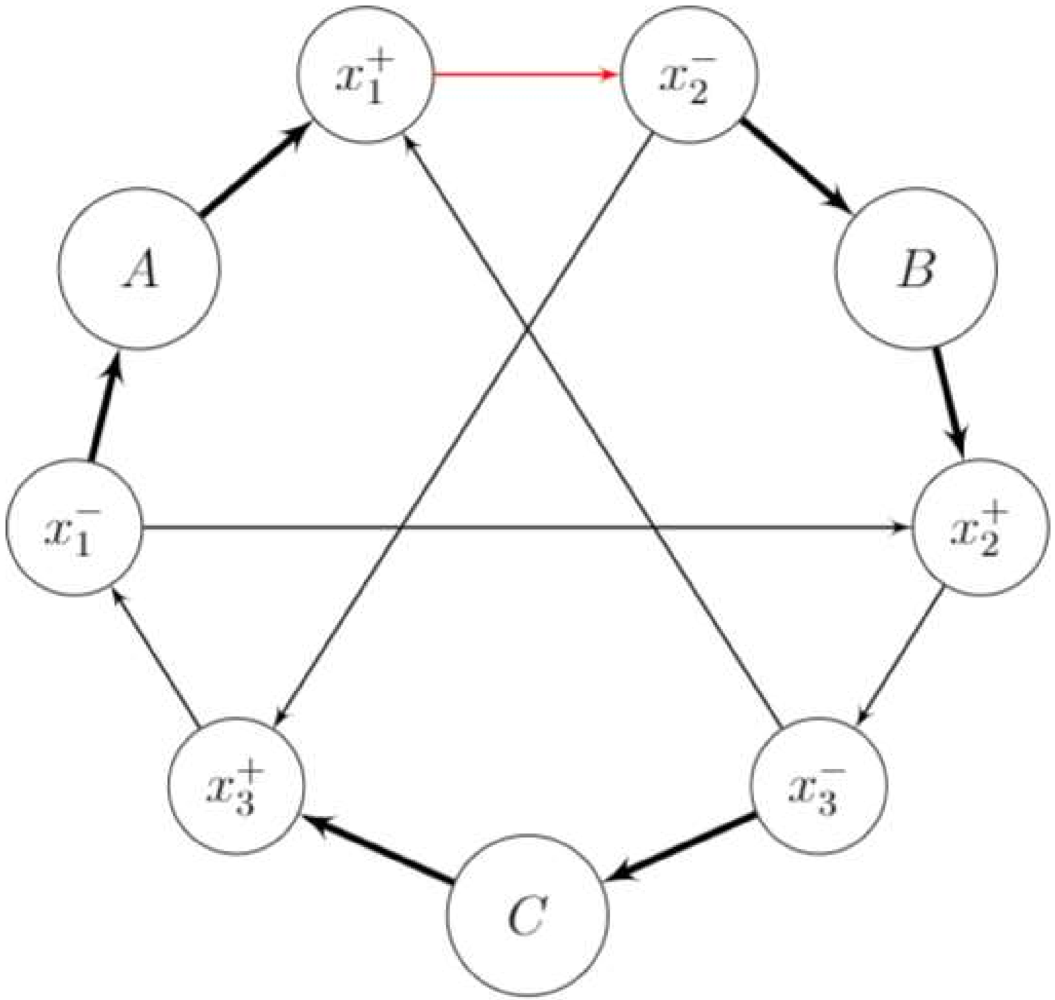
$Lemma$: The 6-extension of an extendable deeply critical oriented clique is extendable.



Proof



Proof



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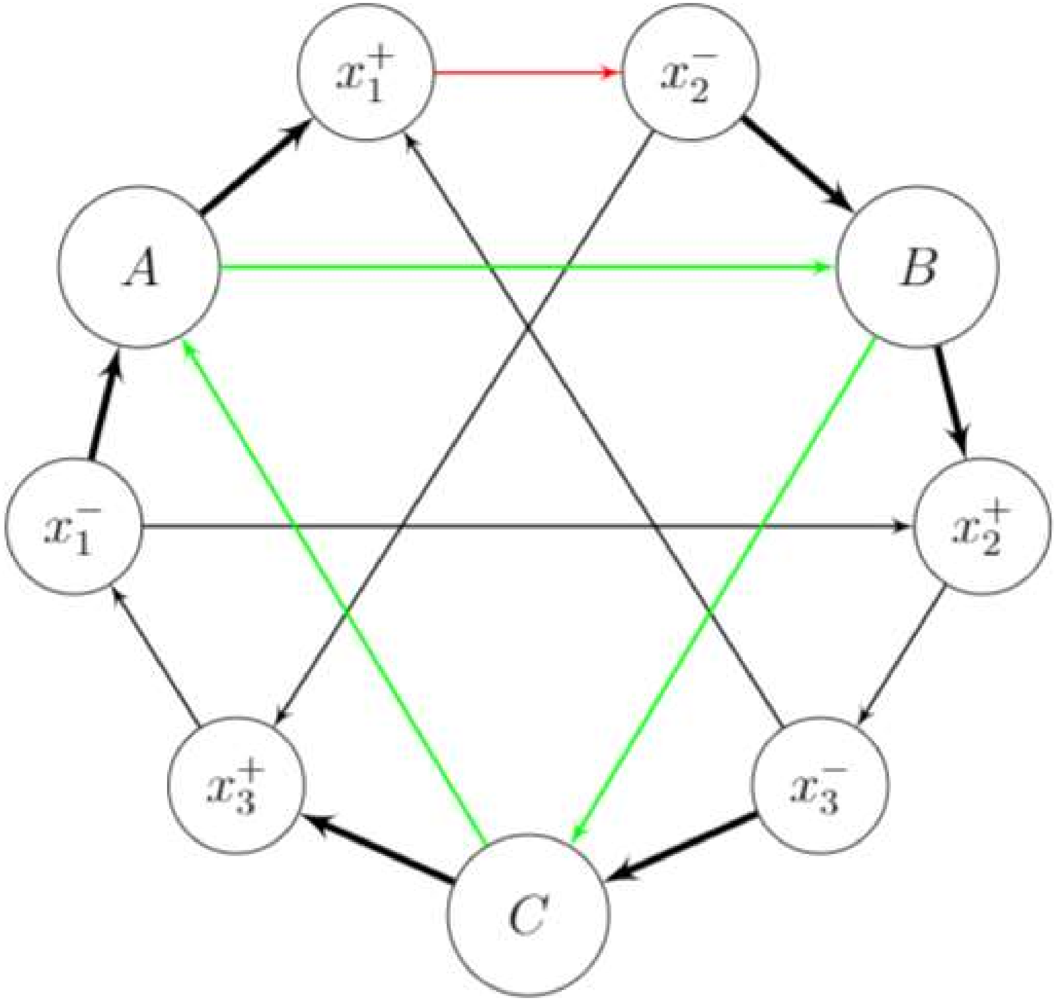

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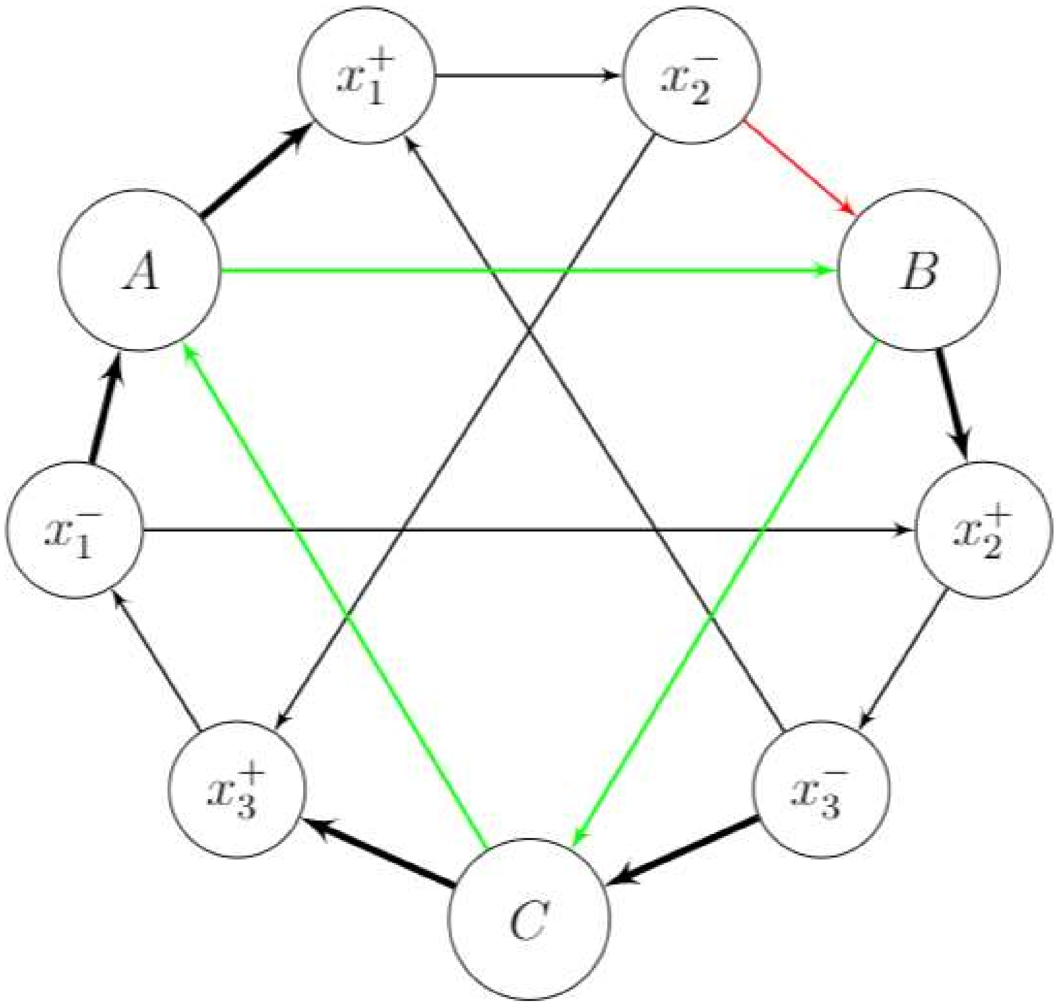

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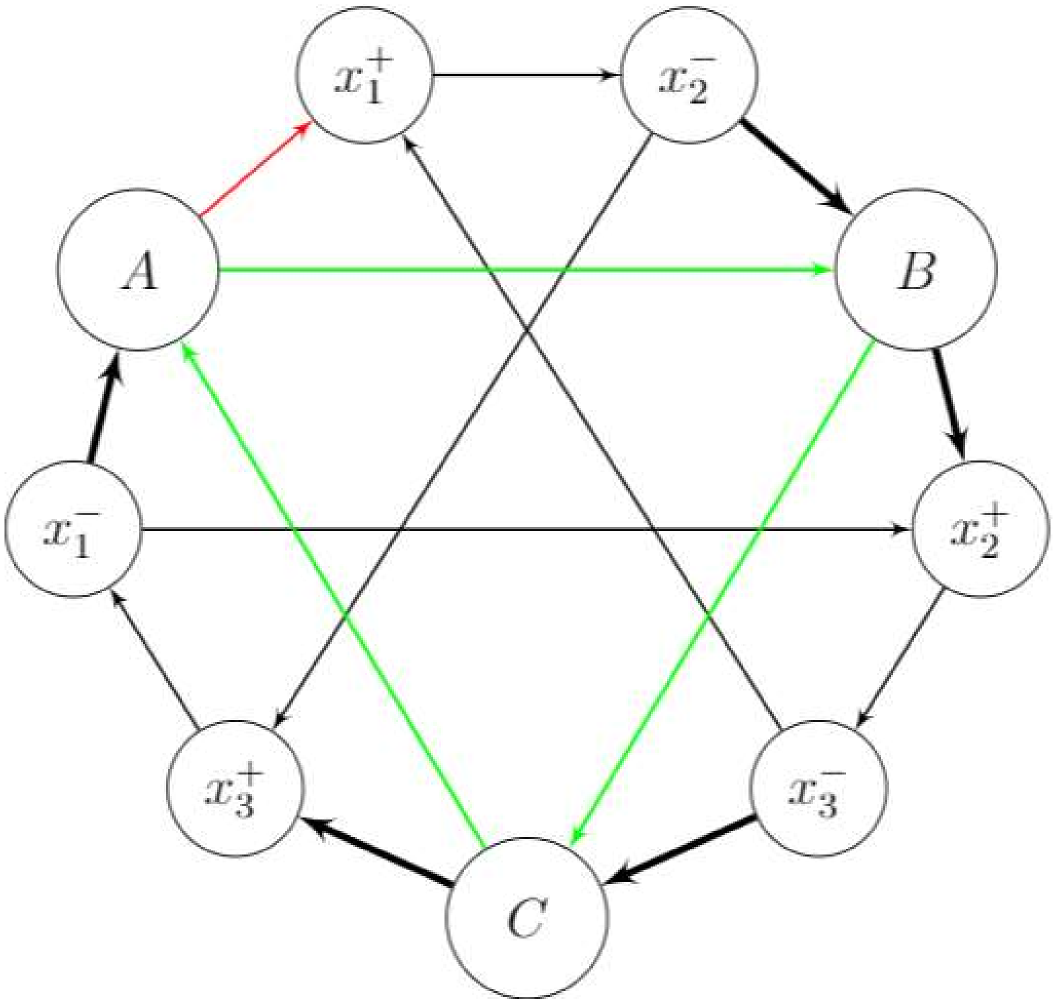

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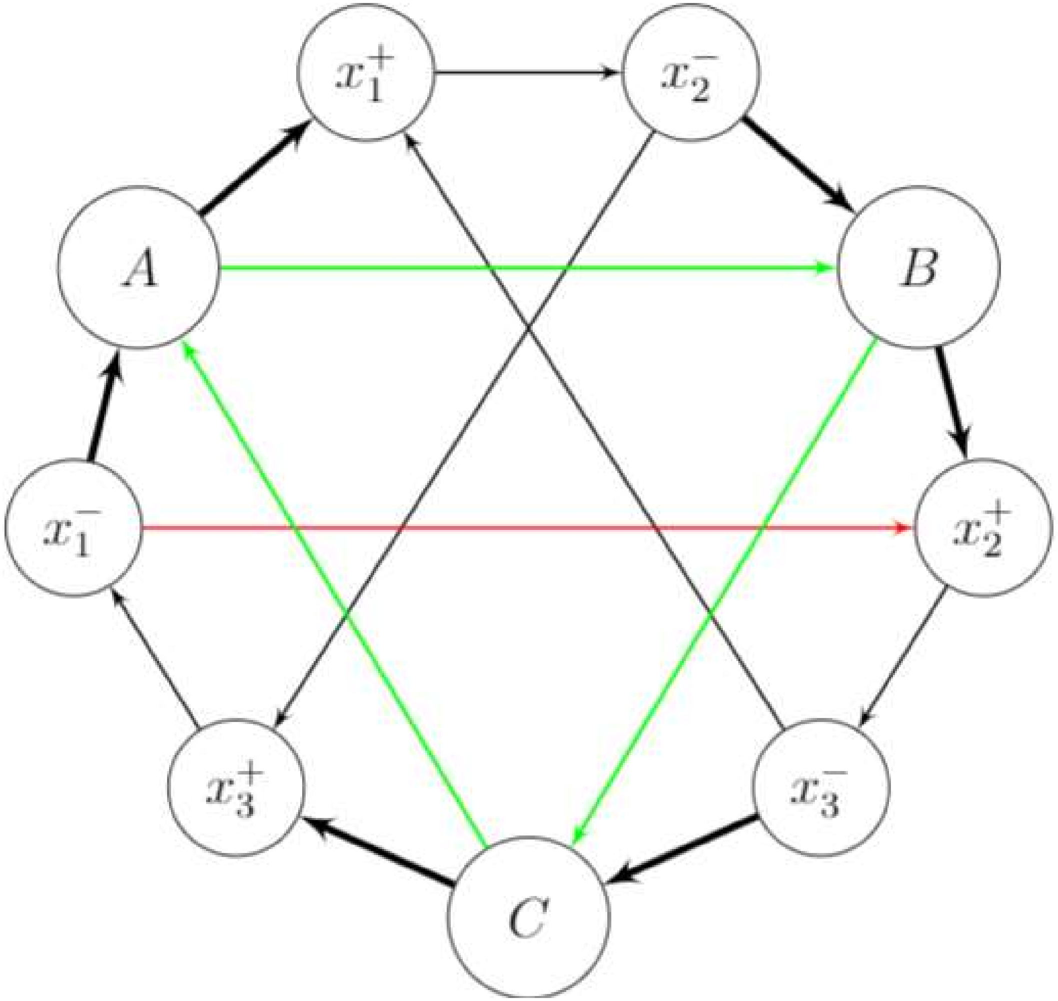
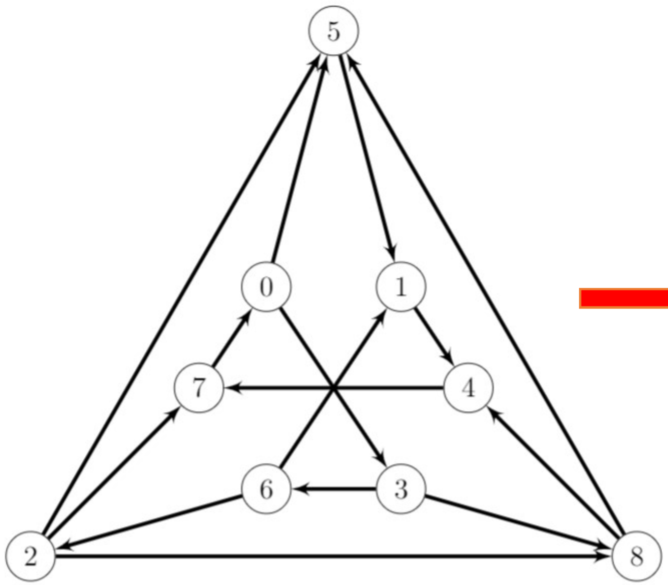
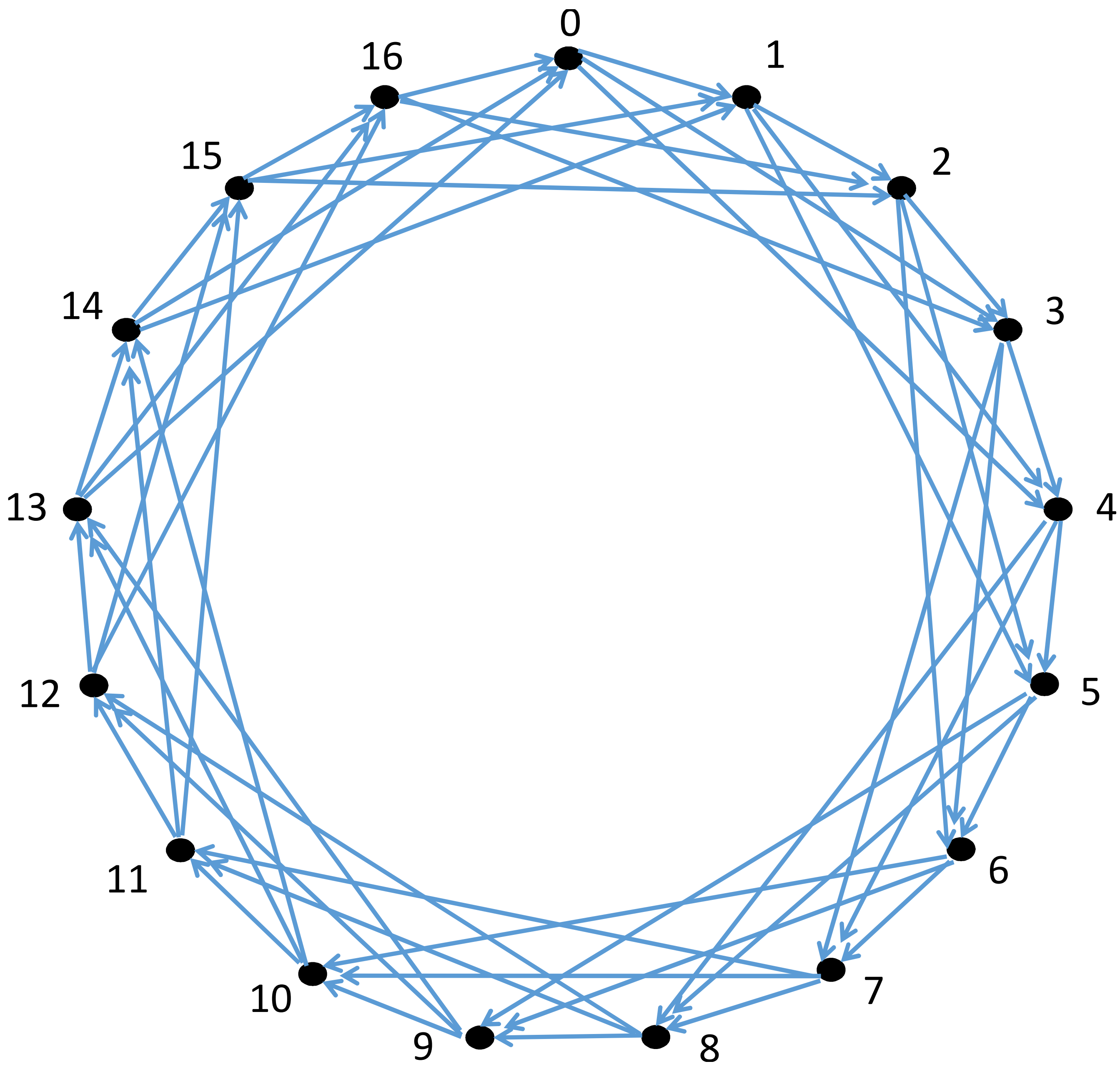




This deeply critical oriented clique is extendable.

There exist deeply critical oriented cliques for all odd $q \geq 9$ number of vertices!!!

q even?



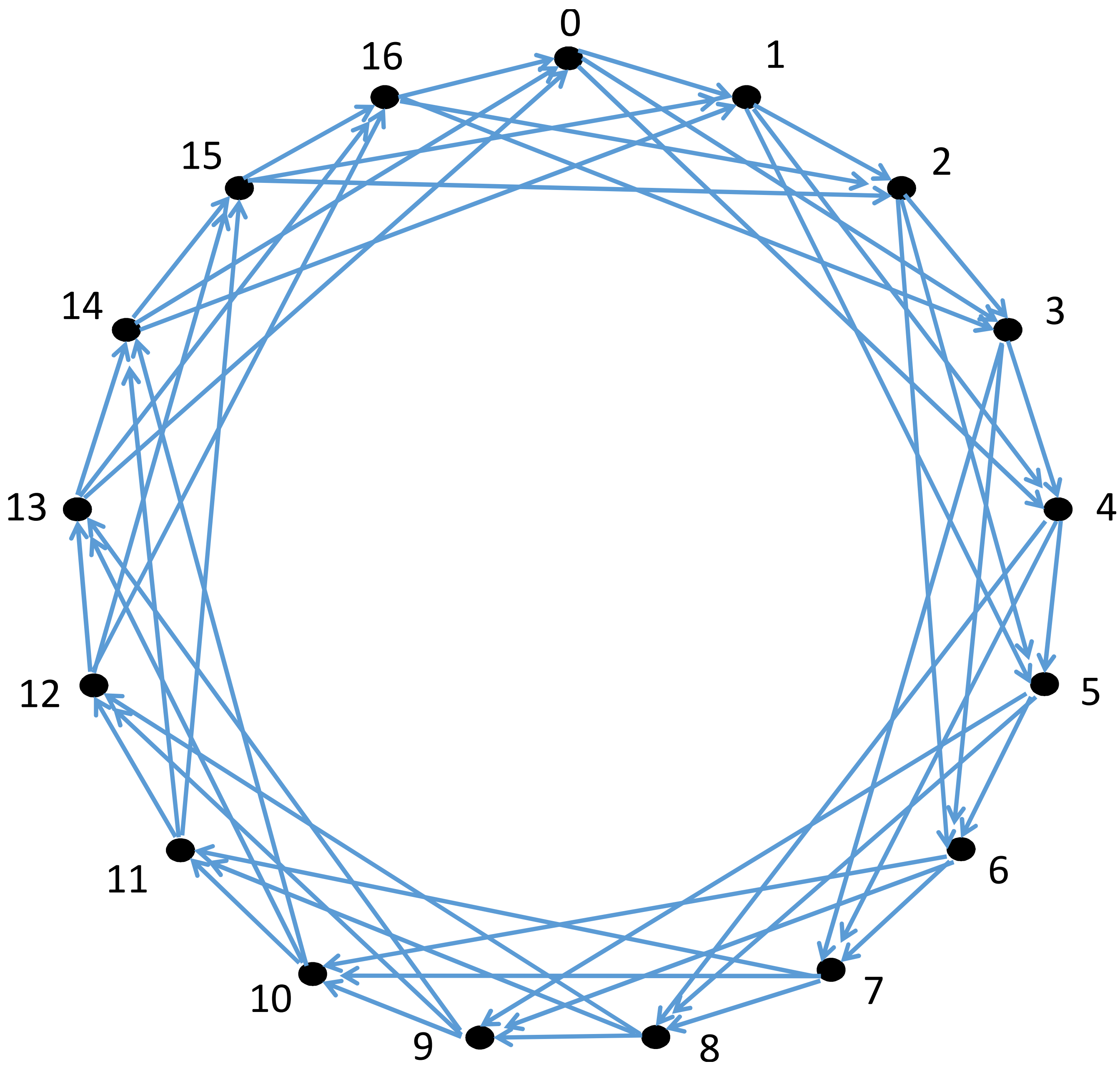
$q = 17$

Circulant graphs

In graph theory, a circulant graph is an undirected graph acted on by a cyclic group of symmetries which takes any vertex to any other vertex.

The circulant graph $C_n^{\{s_1,\ldots,s_k\}}$ with jumps $s_1, \ldots, s_k$ is defined as the graph with n nodes labelled $0, 1, \ldots, n - 1$ where each node i is adjacent to $2k$ nodes $i \pm s_1, \ldots, i \pm s_k \bmod n$

The circulant graphs here are represented as $G = \{n, (s_1, s_2, \ldots, s_k)\}$, where n is the number of vertices in G.



$G = \{17, (1, 3, 4)\}$

Known deeply critical circulant graphs

- $G = \{5, (1)\}$
- $G = \{11, (1, 3)\}$
- $G = \{13, (1, 3, 9)\}$
- $G = \{17, (1, 3, 4)\}$
- $G = \{19, (1, 3, 7)\}$
- $G = \{23, (1, 4, 6, 13)\}$
- $G = \{27, (1, 6, 8, 22)\}$
- $G = \{27, (1, 8, 20, 22)\}$
- $G = \{29, (1, 3, 8, 11, 17)\}$
- $G = \{33, (1, 3, 9, 13, 29)\}$
- $G = \{33, (1, 4, 6, 10, 18)\}$
- $G = \{35, (1, 3, 4, 9, 10)\}$
- $G = \{35, (1, 3, 4, 9, 12)\}$
- $G = \{35, (1, 3, 4, 10, 13)\}$
- $G = \{41, (1, 3, 7, 8, 10, 23, 30)\}$

There are no circulant deeply critical graphs on:

- 15 vertices
- 21 vertices
- 25 vertices
- 31 vertices

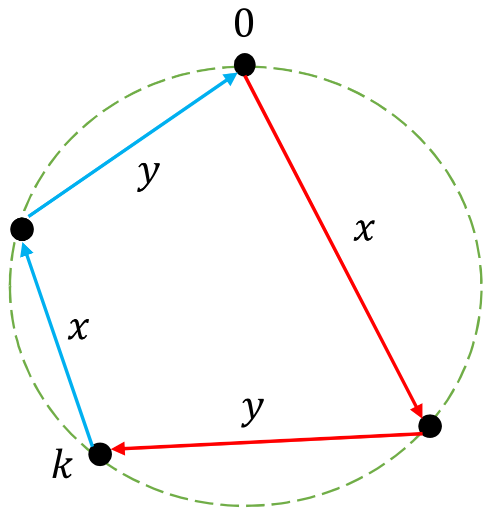
Lemma: The circulant graph $G = \{n, (s_1, s_2, \dots, s_r)\}$ is deeply critical if:

a) any integer in $S = \{1, 2, \dots, n - 1\}$ can be expressed as $(x + y)$ or $-(x + y)$ where $x \in (s_1, s_2, \dots, s_r)$ and $y \in (s_1, s_2, \dots, s_r) \cup \{0\}$.

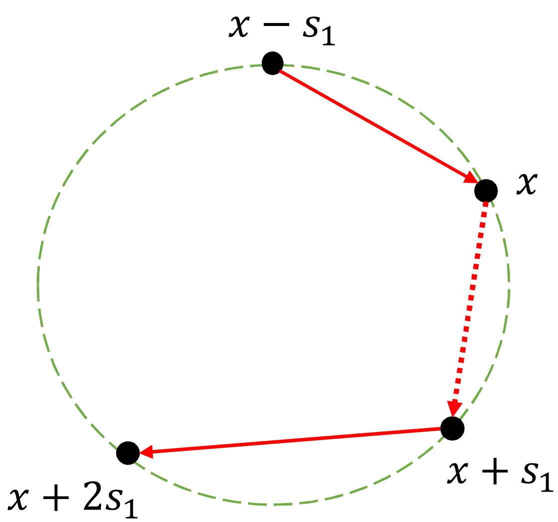
b) also, the only way to express the integer $2i$ as above, where $i \in (s_1, s_2, \dots, s_r)$, is to have $x = y = i$.

Proof: Let $G = \{n, (s_1, s_2, \dots, s_r)\}$ satisfy the given conditions.

The first condition, that is any integer k in $S = \{1, 2, \dots, n-1\}$ can be expressed as $(x + y)$ or $-(x + y)$ where $x \in (s_1, s_2, \dots, s_r)$ and $y \in (s_1, s_2, \dots, s_r) \cup \{0\}$ implies that the graph is a clique.



To show that the graph is deeply critical, consider the graph $G - (x, x + s_1)$.



We observe that x and $x + 2s_1$ are neither adjacent nor are connected by a 2-dipath. Similarly, $x - s_1$ and $x + s_1$ are neither adjacent nor are connected by a 2-dipath. Hence, giving the same colour for both vertices in each pair reduces the number of colours used by 2. Hence, the graph is deeply critical.

Example

We check for the graph $G = \{19, (1, 3, 7)\}$.

The numbers of the form $(x + y)$ or $-(x + y)$ where $x \in (s_1, s_2, \dots, s_k)$ and $y \in (s_1, s_2, \dots, s_k) \cup \{0\}$ are:

$1 = 1 + 0.$
$2 = 1 + 1.$
$3 = 3 + 0.$
$4 = 1 + 3.$
$5 = -(7 + 7).$
$6 = 3 + 3.$ etc

$$2 = 1 + 1$$
$$6 = 3 + 3$$
$$14 = 7 + 7$$

Hence, G is deeply critical.

Lemma: There exist no circulant deeply critical oriented cliques on even number of vertices.

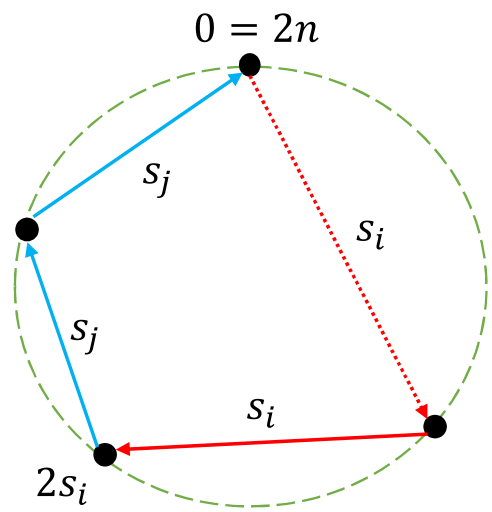
Proof: Let G be a circulant oriented clique with $2n$ vertices. Clearly $n \notin \{s_1, s_2, \cdots, s_r\}$.

Let $n = s_i + s_j$. Consider $G \setminus \{0\ s_i\}$.

0 and $2s_i$ must not be connected by a 2-dipath so that they can be colored with the same colour.

But $0 = 2n = (2s_i + s_j) + s_j$.

Hence, it is not deeply critical.



Final Comments and Future direction

No deeply critical clique on even number of vertices has been found so far. We conjecture that none exist, but the reason is not yet clear.

No circulant deeply critical graphs on even number of vertices. It is yet to be checked if there are other families of deeply critical graphs which prove to be non existent on even number of vertices.

The relation between number of vertices and size of the deeply critical oriented clique can also be studied.

We have only studied the existence of deeply critical graphs. Properties of deeply critical graphs can be compared with other graph classes.

References

- https://www.labri.fr/perso/sopena/pmwiki/index.php?n=TheOrientedColoringPage.TheOrientedColoringPage

- O.V. Borodin, D. Fon-der-Flaass, A.V. Kostochka, A. Raspaud and E. Sopena. On deeply critical oriented graphs. *J. Combin. Theory, Ser. B.* 81:150-155 (2001), doi:10.1006/jctb.2000.1984

- https://en.wikipedia.org/wiki/Circulant_graph#:~:text=In%20graph%20theory%2C%20a%20circulant,this%20term%20has%20other%20meanings.

- 'On Deeply Critical Oriented Cliques'; https://arxiv.org/abs/2103.17140