Discriminating Codes in Geometric Setups

Sanjana Dey

With Florent Foucaud, Subhas C Nandy and Arunabha Sen

May 6, 2021

Problem Description

Minimum Discriminating Code (MIN-DISC-CODE)

Input: A bipartite graph $G = (U \cup V, E)$

Output: A minimum-size subset $U^* \subseteq U$ such that

- $U^* \cap N(v) \neq \emptyset$ for all $v \in V$, and
- $U^* \cap N(v) \neq U^* \cap N(v')$ for every pair $v, v' \in V$, $v \neq v'$.

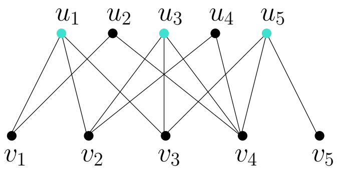

Figure: MIN-DISC-CODE

Problem Description

Geometric Minimum Discriminating Code (G-Min-Disc-Code)

Input: Points P and objects S

Output: To choose a subset $S^* \subseteq S$ of minimum cardinality such that the subsets $S_i^* \subseteq S^*$ covering p_i, $i = 1, 2, \ldots n$ satisfy

- $S_i^* \neq \emptyset$ for all i, and
- $S_i^* \neq S_j^*$ for all $i \neq j$.

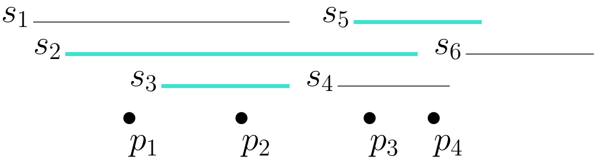

Figure: G-Min-Disc-Code

Variations

Discrete-G-Min-Disc-Code

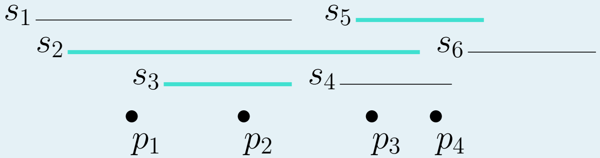

Figure: Discrete-G-Min-Disc-Code

Variations

Continuous-G-Min-Disc-Code

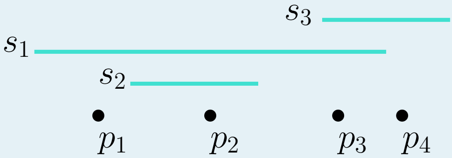

Figure: Continuous-G-Min-Disc-Code

Motivation

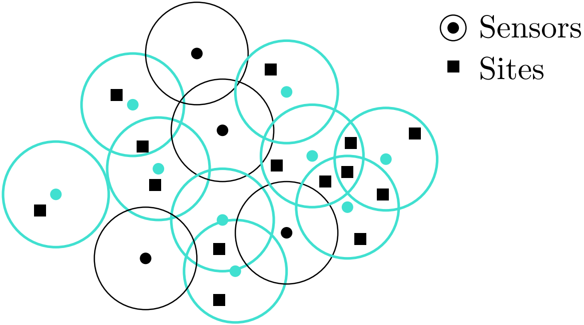

Figure: Network sensing for difficult terrains

Our Contribution

Object Type	Continuous-G-Min-Disc-Code		Discrete-G-Min-Disc-Code	
	Hardness	Algorithm	Hardness	Algorithm
Intervals	-	Polynomial	**NP-hard**	**2-factor**
Unit intervals	Open	**PTAS**	Open	**PTAS**
Axis parallel unit squares	**NP-hard**	**$(4+\epsilon)$-factor**	**NP-hard**	**$(4+\epsilon)$-factor**

Table: Our contribution

Reduction

NP-hardness reduction from the 3-SAT-$2l$ problem.

3-SAT-$2l$

Input: A collection of m clauses $C = \{c_1, c_2, \ldots, c_m\}$ where each clause contains at most three literals, over a set of n Boolean variables $X = \{x_1, x_2, \ldots, x_n\}$, and each literal appears at most twice.

Output: A truth assignment of X such that each clause is satisfied.

Gadgets

Covering Gadget

A *covering gadget* Π consists of three intervals I, J, K and four points p_1, p_2, p_3 and p_4 satisfying $p_1 \in I$, $p_2 \in I \cap J$, $p_3 \in I \cap J \cap K$ and $p_4 \in J \cap K$.

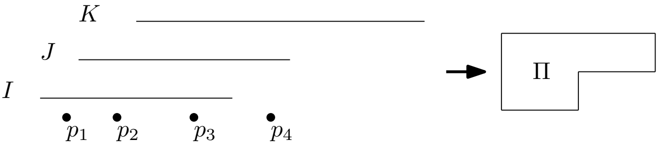

Figure: A covering gadget Π, and its schematic representation.

Gadgets

Clause Gadget

Let c_i be a clause of C. The *clause gadget* for c_i, denoted $G_c(c_i)$, is defined by a covering gadget $\Pi(c_i)$ along with two points p_{c_i}, p'_{c_i} placed in $K \setminus \{I \cup J\}$.

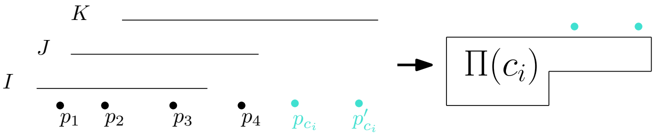

Figure: A clause gadget $G_c(c_i)$, and its schematic representation.

Gadgets

Variable Gadget

Let x_j be a variable of X. The *variable gadget* for x_j, denoted $G_v(x_j)$, is defined by a covering gadget $\Pi(x_j)$, and five points $p^1_{x_j}, \dots, p^5_{x_j}$ placed consecutively in $K \setminus \{I \cup J\}$. It also contains six intervals $I^0_{x_j}$, $I^1_{x_j}$, $I^2_{x_j}$, $I^0_{\bar{x}_j}$, $I^1_{\bar{x}_j}$, $I^2_{\bar{x}_j}$. The right end points will depend on the formula.

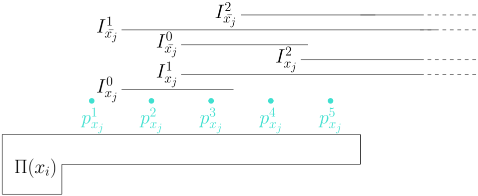

Figure: A variable gadget $G_v(x_j)$.

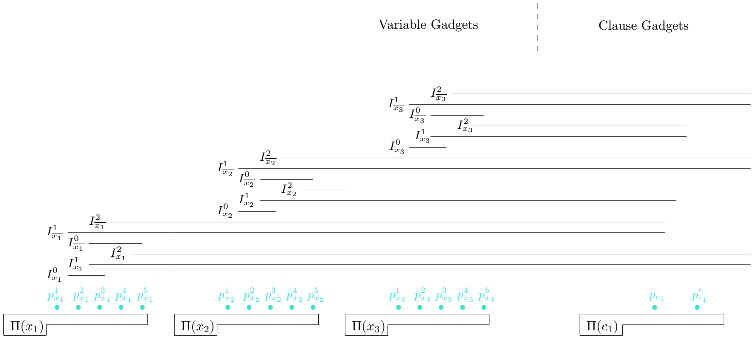

Figure: Clause $(\bar{x_1} \vee x_2 \vee x_3)$.

Satisfiability

The 3-SAT-$2l$ instance is satisfiable if and only if the minimum discriminating code is of size $6n + 3m$.

Theorem

Discrete-G-Min-Disc-Code *with intervals is NP-complete.*

2-factor approximation algorithm comes from edge-cover.

Edge-Cover

Input: An undirected graph $G = (V, E)$.
Output: A minimum cardinality subset $E' \subseteq E$ such that every vertex is incident to at least one edge of E'.

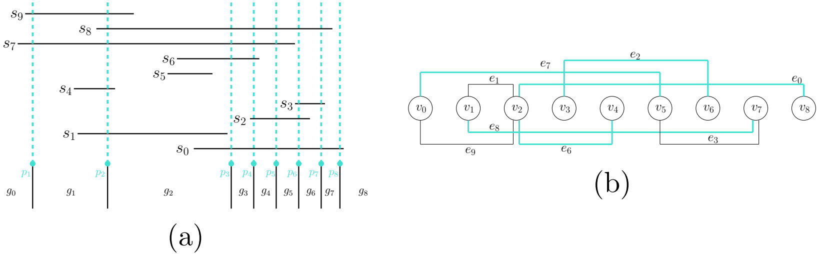

Figure: (a) An input instance, (b) corresponding graph $G = (V, E)$ with Minimum Edge Cover highlighted.

Final step

Minimum Discriminating Code = Edge Cover + Additional Intervals

|Additional Intervals| $\leq$ |Edge Cover|

Theorem

The proposed algorithm produces a 2-factor approximation for DISCRETE-G-MIN-DISC-CODE *in 1D, and runs in time* $O(\min(n^2, m\sqrt{n}))$.

Observation (Unit Intervals)

Discriminating pairs of *consecutive* points in P is equivalant to discriminating P.

Lemma

The shortest weight of an s-t path in the multipartite graph H (created while processing blocks and free regions) is a lower bound on the size of the optimum discriminating code for (P, S).

Lemma

$|SOL| \leq (1+\epsilon)OPT$.

Theorem

DISCRETE-G-MIN-DISC-CODE *in 1D for unit interval objects has a PTAS with time complexity* $O(2^{O(1/\epsilon^2)}n^2)$.

Hardness

A reduction from the P_3-Partition-Grid problem.

P_3-Partition-Grid

Input: A grid graph G.
Output:A partition of the vertices of G into disjoint P_3-paths, where a P_3-path is a path with three vertices.

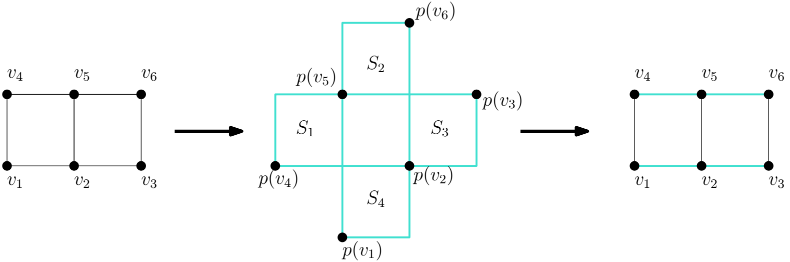

Figure: Reduction from P_3-PARTITION-GRID

Lemma

A P_3-partition for $G=(V,E)$ exists if and only if there exists a set of $\frac{2|V|}{3}$ axis-parallel unit squares discriminating P_G.

Theorem

G-MIN-DISC-CODE *for axis-parallel unit squares is NP-complete.*

Algorithm

Segment-Stabbing

Input: A set L of segments in 2D.
Output: A minimum-size set S of axis-parallel unit squares in 2D such that each segment is intersected exactly once by some square of S.

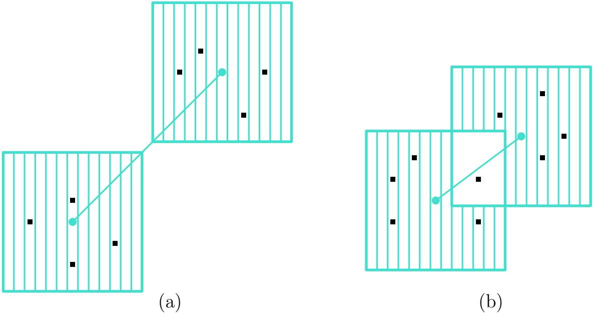

Figure: Object for segment $\ell = [a, b]$, where (a) $\lambda(\ell) \geq 1$ and (b) $\lambda(\ell) < 1$

Block representation

Segment Stabbing	=	Seg-Hit	$\times$	L-Hit	$\times$	U-Hit (Continuous)
		2-factor		2-factor		$(1+\epsilon)$-factor

Theorem

CONTINUOUS-G-MIN-DISC-CODE *for axis-parallel unit squares in 2D has a polynomial-time* $(4+\epsilon)$*-factor approximation algorithm, for every fixed* $\epsilon > 0$.

Theorem

Discrete-G-Min-Disc-Code *for axis-parallel unit squares in 2D has a polynomial-time* $(4+\epsilon)$*-factor approximation algorithm, for every fixed* $\epsilon > 0$..

Open Problem

Question

Is G-Min-Disc-Code (both discrete and continuous) in P for unit intervals?

Thank You!