On \((m,n)\)-absolute clique number of planar graphs

Susobhan Bandopadhyay
Joint work with Sandip Das
Sagnik Sen & Soumen Nandi

NISER, Bhubaneswar

04/05/2021
This work
This work

Based on the paper:
This work

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Colored mixed graphs
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- Type1 ARC
Colored mixed graphs

- Type 1 ARC
- Type 2 ARC
Colored mixed graphs

- Type 1 ARC
- Type 2 ARC
- Type 1 EDGE
Colored mixed graphs

- Type1 ARC
- Type2 ARC
- Type1 EDGE
- Type2 EDGE

2 types of arcs & 2 types of edges...
Colored mixed graphs

- Type1 ARC
- Type2 ARC
- Type1 EDGE
- Type2 EDGE

(2,2)-colored mixed graph
Colored mixed graphs

- Type 1 ARC
- Type 2 ARC
- Type 1 EDGE
- Type 2 EDGE

(2,2)-mixed graph
Colored mixed graphs

(m,n)-mixed graph
Colored mixed graphs

(m,n)-mixed graph [generalization by Nešetřil and Raspaud, J. Combin. Theory Ser. B(2000)]

- m types of arcs
- n types of edges
Colored mixed graphs

- Type 1 ARC
- Type 2 ARC
- Type 1 EDGE
- Type 2 EDGE

homomorphism?
Colored mixed graphs

- Type1 ARC
- Type2 ARC
- Type1 EDGE
- Type2 EDGE
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**Homomorphism:** A vertex mapping $f$ such that an edge/arc $uv$ implies that $f(u)f(v)$ is an edge/arc of the same type.
Colored mixed graphs

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chromatic number $(\chi_{(m,n)})$: minimum $|H|$ such that $G \rightarrow H$. 
Colored mixed graphs

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\*vertex images=colors

chromatic number (\( \chi_{(m,n)} \)): minimum \(|H|\) such that \( G \rightarrow H \).
Special 2-path uvw
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- $uv, vw$ different types of edges
- $vu, vw$ different types of arcs
- $uv, wv$ different types of arcs
- In $uv, vw$ exactly one is an edge
Special 2-path $uvw$

- $uv$, $vw$ different types of edges
- $uv$, $vw$ are arcs (maybe same type)
Special 2-path $uvw$

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- $uv$, $wv$ are different types of arcs

\[ \begin{tikzpicture}[>=stealth, shorten >=1pt]
  \node (a) at (0,0) [circle,draw] {};
  \node (b) at (1,0) [circle,draw] {};
  \node (c) at (2,0) [circle,draw] {};
  \draw [->,dotted] (a) to (b);
  \draw [->] (b) to (c);
\end{tikzpicture} \quad \begin{tikzpicture}[>=stealth, shorten >=1pt]
  \node (a) at (0,0) [circle,draw] {};
  \node (b) at (1,0) [circle,draw] {};
  \node (c) at (2,0) [circle,draw] {};
  \draw [->] (a) to (b);
  \draw [->] (b) to (c);
  \node (d) at (1.5,0) [circle,draw] {};
  \draw [->,red] (d) to (b);
\end{tikzpicture} \]
Special 2-path $uvw$

- $uv$, $vw$ different types of edges
- $uv$, $vw$ are arcs (maybe same type)
- $uv$, $wv$ are different types of arcs
- $vu$, $vw$ are different types of arcs

![Diagram showing the special 2-path $uvw$ with different types of edges and arcs.]
Special 2-path $uvw$

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- $uv$, $vw$ are arcs (maybe same type)
- $uv$, $wv$ are different types of arcs
- $vu$, $vw$ are different types of arcs
- in $uv$, $vw$ exactly one is an edge

\[\text{Diagram:} \quad \begin{array}{c}
\begin{array}{c}
\circ \rightarrow \circ \circ \rightarrow \circ \\
\circ \circ \circ \rightarrow \circ \circ \circ \rightarrow \circ
\end{array}
\end{array} \]
Absolute \((m,n)\)-clique
Absolute \((m,n)\)-clique

\((m,n)\)-mixed graph whose non-adjacent vertices are connected by a special 2-path
Absolute $(m,n)$-clique

$(m,n)$-mixed graph whose non-adjacent vertices are connected by a special 2-path

$$
\omega_{a(m,n)}(F) = \max_{C \in F} \{ \#\text{vertices in } C: C \text{ is } (m,n)\text{-clique} \}$$
Known related results
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Theorem (Bensmail, Duffy and Sen, Graphs Combin., 2017): For the family $O$ of outerplanar graphs
\[ \omega_{a(m,n)}(O) = 3(2m+n)+1, \text{ for all } (m,n)\neq(0,1). \]

Theorem (Bensmail, Duffy and Sen 2017): For the family $P$ of planar graphs
\[ 3(2m+n) \leq \omega_{a(m,n)}(P) \leq 9(2m+n)^2 + 2(2m+n) + 2, \text{ for all } (m,n)\neq(0,1). \]

Conjecture (Bensmail, Duffy and Sen 2017): For the family $P$ of planar graphs
\[ \omega_{a(m,n)}(P) = 3(2m+n)^2 + (2m+n) + 1, \text{ for all } (m,n)\neq(0,1). \]
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$$3(2m+n)^2+(2m+n)+1 \leq \omega_{a(m,n)}(P) \leq 9(2m+n)^2+2(2m+n)+2, \text{ for all } (m,n)\neq(0,1).$$
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Our result (Bandopadhyay, Das, Naandi and Sen)

Here we settle the conjecture.
Our result (Bandopadhyay, Das, Nandi and Sen)

Theorem: For the family $P$ of planar graphs

$$\omega_{a(m,n)}(P) = 3(2m+n)^2+(2m+n)+1,$$

for all $(m,n)\neq(0,1)$. 

Theorem: For the family $P_{tf}$ of triangle-free planar graphs

$$\omega_{a(m,n)}(P_{tf}) = (2m+n)^2+2,$$

for all $(m,n)\neq(0,1)$. 

Theorem: For the family $SP$ of series-parallel graphs

$$\omega_{a(m,n)}(SP) = 3(2m+n)+1,$$

for all $(m,n)\neq(0,1)$. 

time for proof...
Theorem: For the family $P$ of planar graphs $\omega_{a(m,n)}(P) = 3(2m+n)^2+(2m+n)+1$, for all $(m,n)\neq(0,1)$.

Proof (sketch)
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Proof (sketch)

- We will use the notion of Dominating Set (D) and Domination Number in this proof.
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**Proof (sketch)**

- We will use the notion of Dominating Set ($D$) and Domination Number in this proof.

- Goddard and Henning (*J. Graph. Th.*, 2002) showed that any planar graph with at least 10 vertices and diameter 2 has domination number at most 2.
Theorem: For the family $P$ of planar graphs
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- Goddard and Henning (J. Graph. Th., 2002) showed that any planar graph with at least 10 vertices and diameter 2 has domination number at most 2.

- We have to proof the theorem for domination number 1 and 2.
Theorem: For the family $P$ of planar graphs $\omega_{a(m,n)}(P) = 3(2m+n)^2 + (2m+n) + 1$, for all $(m,n) \neq (0,1)$.

Proof (sketch)

- We adapt the technique of Nandy, Sen and Sopena (J. Graph Th., 2016) for our proof.
Theorem: For the family $P$ of planar graphs 
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- We adapt the technique of Nandy, Sen and Sopena (J. Graph Th., 2016) for our proof.

**Lemma:** For triangulated planar $(m, n)$-absolute clique $G$ with domination number 1 
\[ \omega_{a(m,n)}(G) \leq 3(2m+n)^2+(2m+n)+1 \], for all $(m,n) \neq (0,1)$. 

3p+1
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Proof (sketch)

For triangulated planar $(m, n)$–absolute clique $G$ with domination number 2,
Theorem: For the family $P$ of planar graphs $\omega_{a(m,n)}(P) = 3(2m+n)^2 + (2m+n) + 1$, for all $(m,n)\neq(0,1)$.

Proof (sketch)

For triangulated planar $(m, n)$-absolute clique $G$ with domination number 2,

Lemma: $2 \leq |C| \leq 3(2m+n)^2$, where $C$ is the common neighbor of $x$, $y$ and $D=\{x, y\}$. 
Theorem: For the family $P$ of planar graphs $\omega_{a(m,n)}(P) = 3(2m+n)^2 + (2m+n) + 1$, for all $(m,n) \neq (0,1)$. 

Proof (sketch)

For triangulated planar $(m, n)$-absolute clique $G$ with domination number $2$,

Lemma: $2 \leq |C| \leq 3(2m+n)^2$, where $C$ is the common neighbor of $x, y$ and $D = \{x, y\}$.

Lemma: If $3 \leq |C| \leq 3(2m+n)^2$ then,

$$\omega_{a(m,n)}(G) \leq 3(2m+n)^2 + (2m+n) + 1,$$

for all $(m,n) \neq (0,1)$. 
Theorem: For the family $P$ of planar graphs
\[ \omega_{a(m,n)}(P) = 3(2m+n)^2 + (2m+n) + 1, \text{ for all } (m,n) \neq (0,1). \]

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Lemma: If $3 \leq |C| \leq 3(2m+n)^2$ then,
\[ \omega_{a(m,n)}(G) \leq 3(2m+n)^2 + (2m+n) + 1, \]
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Lemma: If $|C| = 2$ then,
\[ \omega_{a(m,n)}(G) \leq 3(2m+n)^2 + (2m+n) + 1, \]
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Theorem: For the family $P$ of planar graphs 
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Lemma: For triangulated planar $(m, n)$-absolute clique $G$ with domination number 2 
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Proof (sketch)

Lemma: For triangulated planar $(m, n)$-absolute clique $G$ with domination number 1
\[ \omega_{a(m,n)}(G) \leq 3(2m+n)^2 + (2m+n) + 1, \text{ for all } (m,n) \neq (0,1). \]
Theorem: For the family $P$ of planar graphs
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Proof (sketch)

Lemma: For triangulated planar $(m, n)$-absolute clique $G$ with domination number 1
\[ \omega_{a(m,n)}(G) \leq 3(2m+n)^2 + (2m+n) + 1 \],
for all $(m,n) \neq (0,1)$.

Lemma: For triangulated planar $(m, n)$-absolute clique $G$ with domination number 2
\[ \omega_{a(m,n)}(G) \leq 3(2m+n)^2 + (2m+n) + 1 \],
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Theorem: For the family $P$ of planar graphs
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Theorem (Bensmail, Duffy and Sen, Graphs Combin., 2017):
For the family $P$ of planar graphs
\[ 3(2m+n)^2+(2m+n)+1 \leq \omega_{a(m,n)}(P), \text{ for all } (m,n) \neq (0,1). \]
Thank You