Separating two signatures in signed planar graphs

Weiqiang Yu

(A joint work with Reza Naserasr)

Université de Paris, IRIF

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Definitions

- A *signed graph* (G, σ) is a graph together with an assignment σ of signs to the edges called signature.
- A *switching* at a vertex v is to reserve the sign of each edge incident to v.
- Two signatures σ_1 and σ_2 on G are *equivalent* if one can be obtained from the other by a sequence of switchings.

Definition

The *packing number* of a signed graph (G, σ), denoted $\rho(G, \sigma)$, is defined to be the maximum number of signatures $\sigma_1, \sigma_2, \cdots, \sigma_l$ such that each σ_i is switching equivalent to σ and the sets of negative edges are pairwise disjoint.

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Packing number

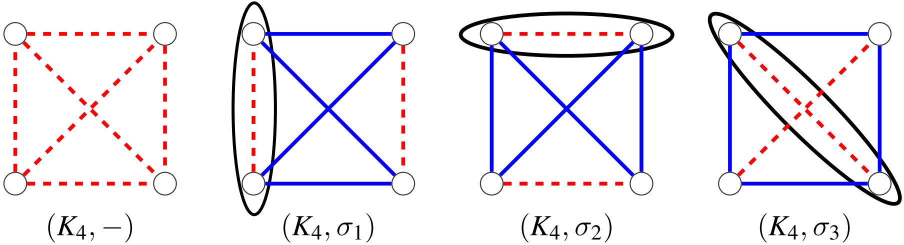

Figure: A 3-packing of $(K_4, -)$

Packing number

Theorem

Given a non negative integer k, for a signed graph (G, σ), we have $\rho(G, \sigma) \geq k + 1$ if and only if $(G, \sigma) \to \mathcal{SPC}_k^o$.

Figure: $\mathcal{SPC}_d^o$ for $d \in \{0, 1, 2, 3\}$

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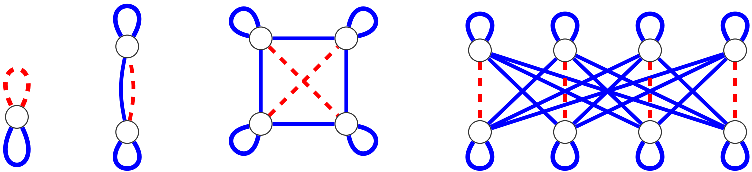

Figure: $\mathcal{SPC}_d^o$ for $d \in \{0, 1, 2, 3\}$

Motivation

- In packing number, we consider one signature σ and its equivalent signatures. But how about k signatures $\sigma_1, \sigma_2, \cdots, \sigma_k$ (not necessarily switching equivalent) and ask whether there exist signatures $\sigma'_1, \sigma'_2, \cdots, \sigma'_k$, where σ'_i is a switching of σ_i, such that the sets of negative edges $E^-_{\sigma'_i}$ are pairwise disjoint.
- In particular, if we choose these k signatures to be switching equivalent to σ, then separating k signatures implies $\rho(G, \sigma) \geq k$.

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Motivation

- It is known that there exists signed planar graph whose packing number is 1, which follows from the results of Kardoš and Narboni. Thus for a general planar graph separating two signatures is not always possible even if $\sigma_1 = \sigma_2$.
- We want to give some sufficient conditions for a planar graph to have two disjoint signatures.

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For planar graph G without 4-cycle and any two signatures σ and π, there are switchings σ' and π' of σ and π, respectively, such that $E^-_{\sigma'} \cap E^-_{\pi'} = \varnothing$.

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Sketch of the proof

- Let G be a smallest counterexample with minimum number of edges.
- For any subgraph of G and two signatures σ and π, we could separate them.
- Find some properties of graph G.
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Notation

- For two signatures σ and π on G, and for an edge $uv \in E(G)$, let $s_{\sigma\pi}(uv) = \{\sigma(uv)\pi(uv)\} \subseteq \{+,-\} \times \{+,-\}$.
- For a vertex u define $S_{\sigma\pi}(u) = \{s_{\sigma\pi}(e) | e \in E_u\}$, where E_u is the set of edges incident to u.
- Let $S^* = \{++,+-,-+\}$. We say a vertex v is *saturated* by σ and π if $S^* \subseteq S_{\sigma\pi}(v)$.
- If all the vertices of a path(resp. face) in G are of degree 4, then we call it a *light path(resp. face)* in G.

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Properties of graph G

- G is 2-connected.
- Let $uv \in E(G)$ and $G' = G - uv$. For any two signatures σ and π on G, we have switchings σ_{uv} and π_{uv} of σ and π on G', such that $E^-_{\sigma_{uv}} \cap E^-_{\pi_{uv}} = \varnothing$. Then both u and v are saturated with respect to σ_{uv} and π_{uv} on G'.
  - The minimum degree of G is at least 4. So for any 3-degenerated graph, we could separate two signatures.
  - Let P be a light path of G, $e \in P$ and $G' = P - e$. we have switchings σ_e and π_e of σ and π on G', such that $E^-_{\sigma_e} \cap E^-_{\pi_e} = \varnothing$. Then by switching the vertices on P we can find switchings $\sigma_{e'}$ and $\pi_{e'}$ of σ and π, such that $E^-_{\sigma_{e'}} \cap E^-_{\pi_{e'}} = e'$, where $e' \in P$.

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Discharging Part

- We define a weight function:

$$\forall v \in V(G), \omega(v) = d(v) - 4;$$
$$\forall f \in F(G), \omega(f) = d(f) - 4.$$

By Euler's formula and Handshake lemma, we derive that

$$\sum_{x \in V(G) \cup F(G)} \omega(x) = -8.$$

- After applying discharging rules, we obtain that:

$\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$.

And the contradiction follows:

$$-8 = \sum_{x \in V(G) \cup F(G)} \omega(x) = \sum_{x \in V(G) \cup F(G)} \omega^*(x) \geq 0.$$

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