Twin-width and generalized coloring numbers

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The *twin-width* of $G$ is the minimum integer $d$ such that $G$ admits a $d$-contraction sequence $G_n, \ldots, G_1$, which we dually see as a $d$-construction sequence:

- $G_i$ with nodes $V_1, \ldots, V_i$, red and black edges;
- $G_i \to G_{i+1}$ by splitting some $V_a$ into $V_a$ and $V_{i+1}$;
- black edge $V_aX \to$ black edges $V_aX$ and $V_{i+1}X$;
- red edge $V_aX \to$ cannot give both $V_aX$ and $V_{i+1}X$ black or $V_aX$ and $V_{i+1}X$ non-edges;
- between $V_a$ and $V_{i+1}$: anything;
- maximum degree in red is $\leq d$;
- $G_n = G$ and has no red edge.
The twin-width of $G$ (tww($G$)): minimum $d$.

\[ V_1 = V(G) \]

\[ G_1 \rightarrow G_2 \rightarrow \ldots \rightarrow G_i \rightarrow G_{i+1} \rightarrow \ldots \rightarrow G_n = G \]

$d$-construction sequence of $G$
**Twin-width**

The twin-width of $G$: minimum $d$.

$d$-construction sequence of $G$

$G_1 \rightarrow G_2 \rightarrow \ldots \rightarrow G_i \rightarrow G_{i+1} \rightarrow \ldots \rightarrow G_n = G$

$V_1 = V(G)$

X $V_a$ $V_{i+1}$

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The **twin-width** of $G$ (tww($G$)): minimum $d$. 

\[ V_1 = V(G) \]

\[ G_1 \rightarrow G_2 \rightarrow \ldots \rightarrow G_i \rightarrow G_{i+1} \rightarrow \ldots \rightarrow G_n = G \]

\[ X \rightarrow V_a \rightarrow X \rightarrow X \rightarrow X \rightarrow X \]

\[ V_a \rightarrow V_i+1 \rightarrow V_{i+1} \rightarrow V_{i+2} \rightarrow V_{i+3} \rightarrow V_{i+4} \]
Generalized coloring number

\[
\text{adm}_r \leq \text{scol}_r \leq wcol_r
\]
Generalized coloring number

\[ \text{adm}_r(G) \leq \text{scol}_r(G) \leq \text{wcol}_r(G) \leq r \]
Generalized coloring number

\[ \text{adm}_r(G) \leq \text{scol}_r(G) \leq \text{wcol}_r(G) \]
Excluding a biclique

Biclique free classes with bounded twin-width have bounded expansion (Bonnet, Geniet, Kim, Thomassé, Watrigant). These include

- proper minor closed classes of graphs
- some classes of expanders
- but not the class of cubic graphs because every class with bounded twin-width is small.
Excluding a biclique

Biclique free classes with bounded twin-width have bounded expansion (Bonnet, Geniet, Kim, Thomassé, Watrigant). These include

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\( \mathcal{C} \) has bounded expansion.

\[ \Leftrightarrow \sup \{\text{wcol}_r(G) : G \in \mathcal{C}\} < \infty \text{ for every integer } r. \quad (\text{Zhu}) \]

\[ \Leftrightarrow \sup \{\text{scol}_r(G) : G \in \mathcal{C}\} < \infty \text{ for every integer } r. \quad (\text{Zhu}) \]

\[ \Leftrightarrow \sup \{\text{adm}_r(G) : G \in \mathcal{C}\} < \infty \text{ for every integer } r. \quad (\text{Dvořák}) \]
Main Problem

\( b\omega(G) \) (biclique number): the maximum integer \( s \) such that \( K_{s,s} \subseteq G \).

Problem

Bound generalized coloring numbers of graphs with biclique number \( s \) and twin-width at most \( d \).
Upper bounds:

- exponential upper bound for $\text{scol}_r \lesssim d^r s$ (and $\text{adm}_r$);
- deduce exponential upper bound for $\text{wcol}_r$ by

\[
\text{wcol}_r(G) \leq 2^{r-1} \max_{1 \leq k \leq r} \text{scol}_k(G)^{r/k}
\]

(but now $\text{wcol}_r \lesssim (ds)^r$)
Results

Upper bounds:

- exponential upper bound for \( \text{scol}_r \lesssim d^r s \) (and \( \text{adm}_r \));
- deduce exponential upper bound for \( \text{wcol}_r \) by
  \[
  \text{wcol}_r(G) \leq 2^{r-1} \max_{1 \leq k \leq r} \text{scol}_k(G)^{r/k}
  \]
  (but now \( \text{wcol}_r \lesssim (ds)^r \))

Lower bounds:

- with high-girth \( d/2 \)-regular graphs, \( d \geq 14 \), and \( r = 2^k \) we get \( \text{scol}_r \gtrsim (d/8)^r s \).
- with \( K_n^{(r-1)} \) plus blowing we get \( \text{adm}_r \gtrsim (\log \log \log d)^{2r} s \);
Upper bound of $scol_r$.

**Theorem - [Dreier, Gajarsky, J, Ossona de Mendez, and Raymond]**

For every graph $G$ and every positive integer $r$ we have

$$scol_r(G) \leq \left( 3 + d \sum_{i=0}^{r-1} (d - 1)^i \right) s \leq (d^r + 3)s$$

where $d = \text{tww}(G)$ and $s = \text{b}\omega(G)$. 
Bounding for \( \text{scol}_r \) - Computing the ordering

**d-construction sequence of** \( G \)

\[ G_1, G_2, \ldots, G_i, \ldots, G_n \]
Bounding for $scol_r$ - Computing the ordering

At step $i$

resolved

$\leq s$ vertices

$\leq d$

$\leq s$
Bounding for $\text{scol}_r$ - Computing the ordering

resolved

split

$\leq d$
Bounding for $\text{scol}_r$

At step $i$

$\leq s$ vertices

split

$\leq d$

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Bounding for $\text{scol}_r$

At step $i$

$\leq s$ vertices

$\leq d$

$\nu$
Case 1
Bounding for $\text{scol}_r$

Case 1

At step $i$

Red BFS-tree

$\leq s$ vertices

$\leq d$

$\leq d(d-1)$

$\leq d(d-1)^{r-2}$

$\leq d(d-1)^{r-1}$

$\leq d$
Bounding for $\text{scol}_r$

Case 1

At step $i$

$\leq s$ vertices

$S$

$\nu$

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$\leq d(d-1)^{r-2}$

Red BFS-tree

Leaves are

$\mathcal{P}_{\leq r}$

$\mathcal{U}$
Bounding for $scol_r$

Case 1

At step $i$

$\leq s$ vertices

$\leq d$

$\leq d(d - 1)$

$\leq d(d - 1)^{r-2}$

$\leq d(d - 1)^{r-1}$

Red BFS-tree

Leaves are

$X_u$
Case 1

At step $i$

- $S$
- $\leq s$ vertices
- $\leq d$
- $\leq d(d-1)$
- $\leq d(d-1)^{r-2}$
- $\leq d(d-1)^{r-1}$

Red BFS-tree

Leaves are $X_u$

Bounding for $scol_r$
Bounding for $scol_r$

Case 1

At step $i$

$u \in S$
$X_u$ (black edge)
$X_u$ is a leaf.

Red BFS-tree

$\leq d$

$\leq d(d - 1)$

$\leq d(d - 1)^{r-2}$

Leaves are

$\leq d(d - 1)^{r-1}$
Bounding for $\text{scol}_r$

Case 1

$|\text{Sreach}[G, L, v]| \leq 2s + (1 + d + \cdots + d(d - 1)^{r-2})s + (d(d - 1)^{r-1})s$

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Case 2

Bounding for $\text{scol}_r$

At step $i$

$\leq s$ vertices

Red BFS-tree

$\leq d$

$\leq d(d-1)$

$\leq d(d-1)^{r-2}$

Leaves are

or

or

$S$
Bounding for $\text{scol}_r$

**Case 2**

At step $i$

- Split
- $\leq s$ vertices

Red BFS-tree

- $X_p$
- $\leq d$
- $\leq d(d-1)$
- $\leq d(d-1)^{r-2}$

Leaves are

- Green
- $\text{or}$
- Red

$|\text{Sreach}[G, L, v]| \leq s + 2s + (1 + d + \cdots + d(d - 1)^{r-2} - 1)s + (d(d - 1)^{r-1})s$
Upper bound of $\text{scol}_r$.

Thus

$$\text{scol}_r(G) \leq \left(3 + d \sum_{i=0}^{r-1} (d - 1)^i\right)s \leq (d^r + 3)s$$

Corollary

For every graph $G$ and every positive integer $r$ we have

$$\text{scol}_r(G) \leq \begin{cases} 
2 \omega(G) & \text{if } \text{tww}(G) = 0, \\
3 \omega(G) & \text{if } \text{tww}(G) = 1, \\
5 \omega(G) & \text{if } \text{tww}(G) = 2, \\
3(\text{tww}(G) - 1)^r \omega(G) & \text{if } \text{tww}(G) \geq 3.
\end{cases}$$
Theorem - [Grohe, Kreutzer, Rabinovich, Siebertz, and Stavropoulos]

Let $G$ be a $d$-regular graph of girth at least $4g + 1$, where $d \geq 7$. Then for every $r \leq g$,

$$
\text{scol}_r(G) \geq \frac{d}{2} \left( \frac{d - 2}{4} \right)^{2^\lfloor \log_2 r \rfloor - 1}.
$$
Lower bounds of \(scol_r\) - 2-lifts

A 2-lift of \(G\) is a graph with vertex set \(V(G) \times \{0, 1\}\) such that each edge \(xy\) of \(G\) gives a matching between \((x, 0), (x, 1)\) and \((y, 0), (y, 1)\).
Lower bounds of $\text{scol}_r$ - 2-lifts

A 2-lift of $G$ is a graph with vertex set $V(G) \times \{0, 1\}$ such that each edge $xy$ of $G$ gives a matching between $(x, 0)$, $(x, 1)$ and $(y, 0)$, $(y, 1)$.

Lemma

If $G'$ is obtained from $G$ by a sequence of 2-lifts then

$$\text{tww}(G') \leq 2 \text{tww}(G).$$
Lower bounds of $scol_r$ - High girth by 2-lifts

Let $G = (V, E)$ be an $r$-regular graph with $n$ vertices and $m = nr/2$ edges, and let $G'$ be the $r$-regular graph with $V(G') = V \times 2^E$ matching $(x, X)$ with $(y, Y)$ if $xy \in E$ and $X \Delta Y = \{xy\}$.

Let $G = (V, E)$ be an $r$-regular graph with $n$ vertices and $m = nr/2$ edges, and let $G'$ be the $r$-regular graph with $V(G') = V \times 2^E$ matching $(x, X)$ with $(y, Y)$ if $xy \in E$ and $X \Delta Y = \{xy\}$.

Then $G'$ is obtained from $G$ by a sequence of 2-lifts, $girth(G') \geq 2 \cdot girth(G)$.

Repeat this and use the bound on $scol_r$ for regular graphs with high girth.
Lower bounds of $\text{scol}_r$ - High girth by 2-lifts

Let $G = (V, E)$ be an $r$-regular graph with $n$ vertices and $m = nr/2$ edges, and let $G'$ be the $r$-regular graph with $V(G') = V \times 2^E$ matching $(x, X)$ with $(y, Y)$ if $xy \in E$ and $X \Delta Y = \{xy\}$.

Then
- $G'$ is obtained from $G$ by a sequence of 2-lifts,
- $\text{girth}(G') \geq 2 \text{girth}(G)$.

Repeat this and use the bound on $\text{scol}_r$ for regular graphs with high girth.
**Lemma**

For every integer $\Delta \geq 7$ and every integers $r$ and $g \geq 4r + 1$ there exists a $\Delta$-regular graph $G$ with girth at least $g$, $2\Delta - 1 \leq \text{tww}(G) \leq 2\Delta$, and

$$\text{scol}_r(G) \geq \frac{\Delta}{2} \left( \frac{\Delta - 2}{4} \right)^{2^\lfloor \log_2 r \rfloor - 1} \geq \frac{\text{tww}(G)}{4} \left( \frac{\text{tww}(G) - 4}{8} \right)^{2^\lfloor \log_2 r \rfloor - 1}.$$ 

**Corollary**

For every integer $d \geq 14$, every positive integer $s$, and every integer $r$ of the form $2^k$, there exists a graph $G$ with $\text{tww}(G) \leq d$, $b\omega(G) = s$, and

$$\text{scol}_r(G) \geq \frac{ds}{4} \left( \frac{d - 4}{8} \right)^{r-1} \geq 2 \left( \frac{\text{tww}(G) - 4}{8} \right)^r b\omega(G).$$
THANK YOU FOR YOUR ATTENTION!