Twin-width and generalized coloring numbers

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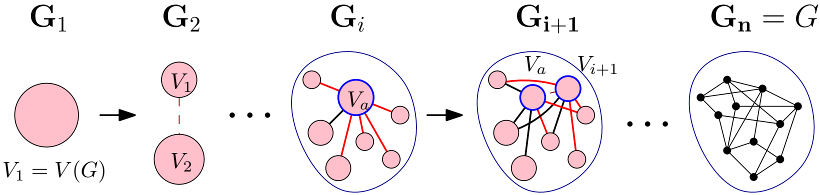
HOSIGRA 2021

Twin-width

The *twin-width* of G is the minimum integer d such that G admits a d-contraction sequence $\mathbf{G}_n, \ldots, \mathbf{G}_1$, which we dually see as a d-construction sequence:

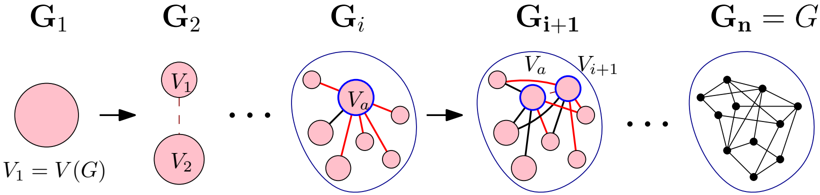
- $\mathbf{G}_i$ with nodes $V_1, \ldots, V_i$, red and black edges;
- $\mathbf{G}_i \to \mathbf{G}_{i+1}$ by splitting some V_a into V_a and V_{i+1};
- black edge $V_aX \to$ black edges V_aX and $V_{i+1}X$;
- red edge $V_aX \to$ cannot give both V_aX and $V_{i+1}X$ black or V_aX and $V_{i+1}X$ non-edges;
- between V_a and V_{i+1}: anything;
- maximum degree in red is $\leq d$;
- $\mathbf{G}_n = G$ and has no red edge.

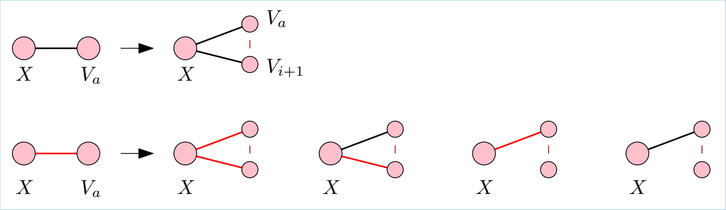
d-construction sequence of G



Twin-width

d-construction sequence of G





Twin-width

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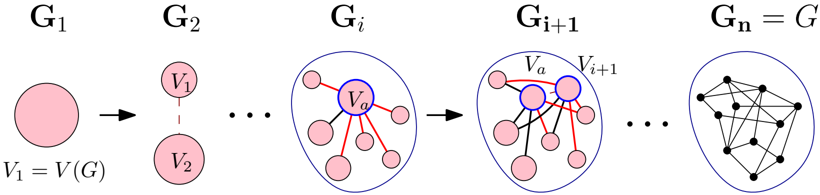
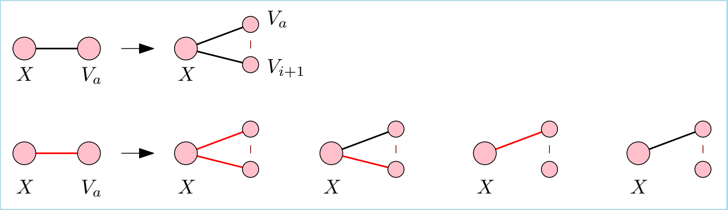
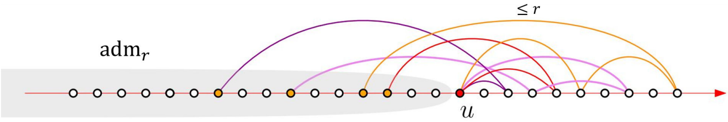


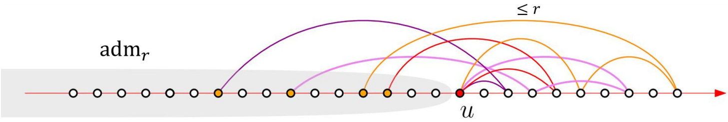


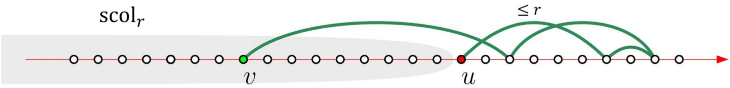
The *twin-width* of G ($\text{tww}(G)$): minimum d.

Generalized coloring number

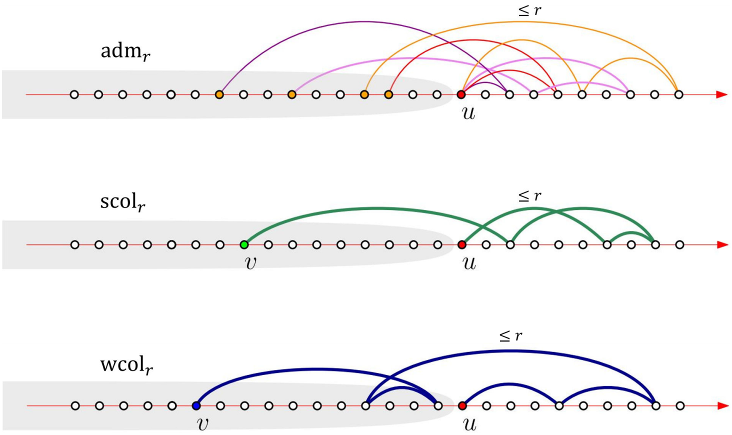


Generalized coloring number

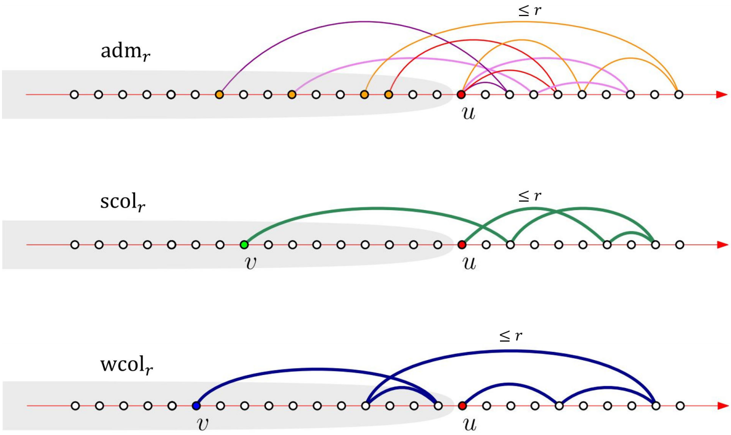




Generalized coloring number



Generalized coloring number



$$\mathrm{adm}_r(G) \leq \mathrm{scol}_r(G) \leq \mathrm{wcol}_r(G)$$

Excluding a biclique

Biclique free classes with bounded twin-width have bounded expansion (Bonnet, Geniet, Kim, Thomassé, Watrigant). These include

- proper minor closed classes of graphs
- some classes of expanders
- but not the class of cubic graphs because every class with bounded twin-width is small.

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$\mathscr{C}$ has bounded expansion.

$\Leftrightarrow$ $\sup\{\mathrm{wcol}_r(G) : G \in \mathscr{C}\} < \infty$ for every integer r. (Zhu)

$\Leftrightarrow$ $\sup\{\mathrm{scol}_r(G) : G \in \mathscr{C}\} < \infty$ for every integer r. (Zhu)

$\Leftrightarrow$ $\sup\{\mathrm{adm}_r(G) : G \in \mathscr{C}\} < \infty$ for every integer r. (Dvořák)

Main Problem

b$\omega(G)$ (*biclique number*): the maximum integer s such that $K_{s,s} \subseteq G$.

Problem

Bound generalized coloring numbers of graphs with biclique number s and twin-width at most d.

Results

Upper bounds:

- exponential upper bound for $\mathrm{scol}_r \lesssim d^r s$ (and adm_r);
- deduce exponential upper bound for wcol_r by

$$\mathrm{wcol}_r(G) \leq 2^{r-1} \max_{1 \leq k \leq r} \mathrm{scol}_k(G)^{r/k}$$

(but now $\mathrm{wcol}_r \lesssim (ds)^r$)

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Lower bounds:

- with high-girth $d/2$-regular graphs, $d \geq 14$, and $r = 2^k$ we get $\mathrm{scol}_r \gtrsim (d/8)^r s$.
- with $K_n^{(r-1)}$ plus blowing we get $\mathrm{adm}_r \gtrsim (\log\log\log d)^{2r} s$;

Upper bound of scol_r

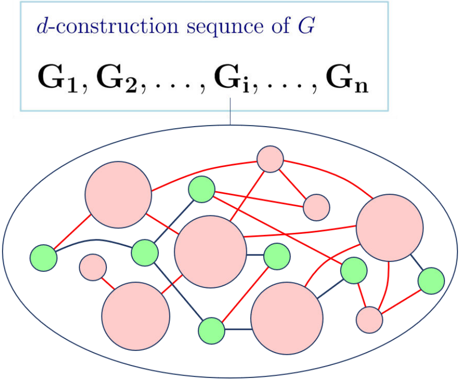
Theorem - [Dreier, Gajarsky, J, Ossona de Mendez, and Raymond]

For every graph G and every positive integer r we have

$$\mathrm{scol}_r(G) \leq \left(3 + d\sum_{i=0}^{r-1}(d-1)^i\right)s \leq (d^r + 3)s$$

where $d = \mathrm{tww}(G)$ and $s = \mathrm{b}\omega(G)$.

Bounding for scol_r - Computing the ordering



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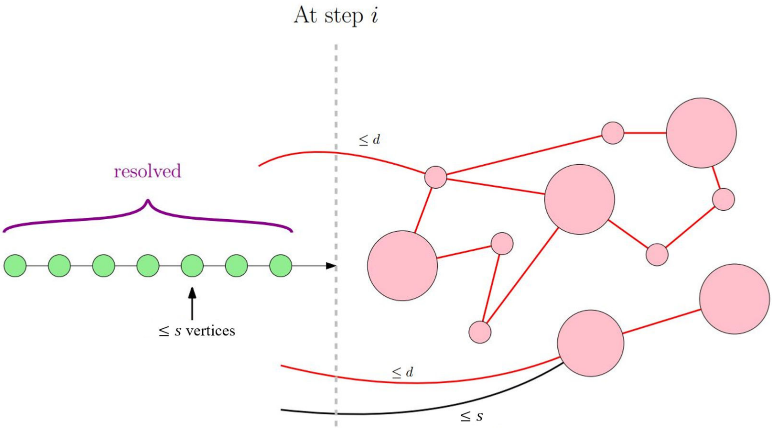

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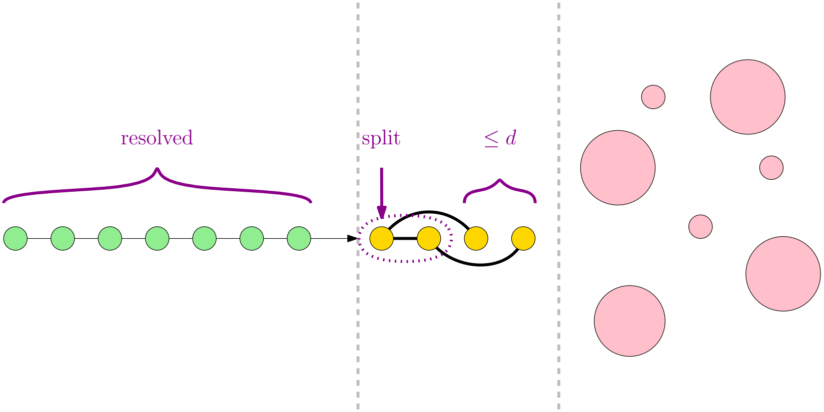

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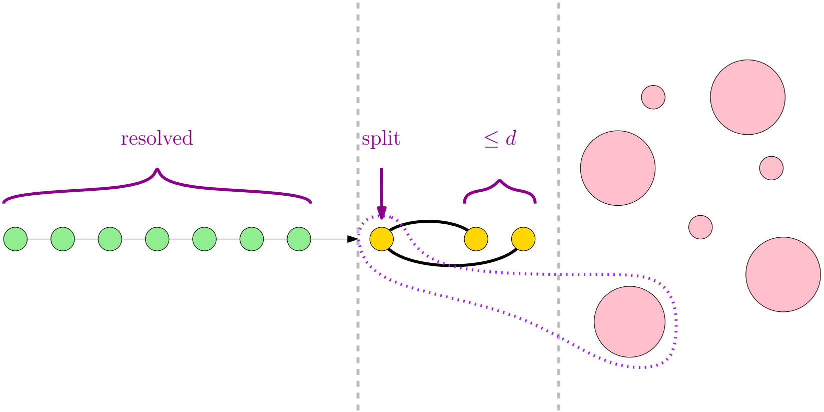

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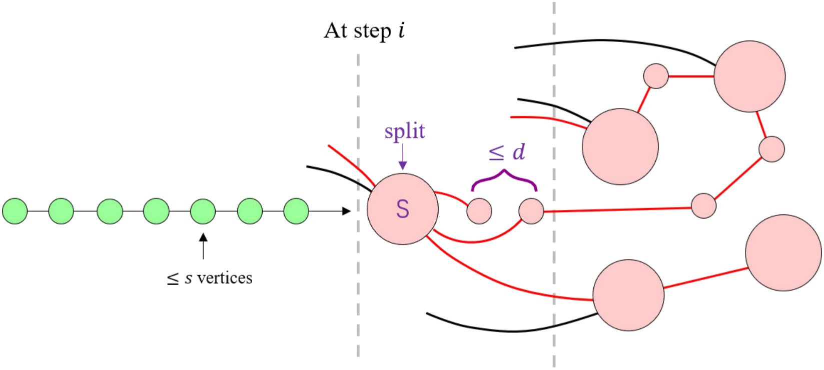
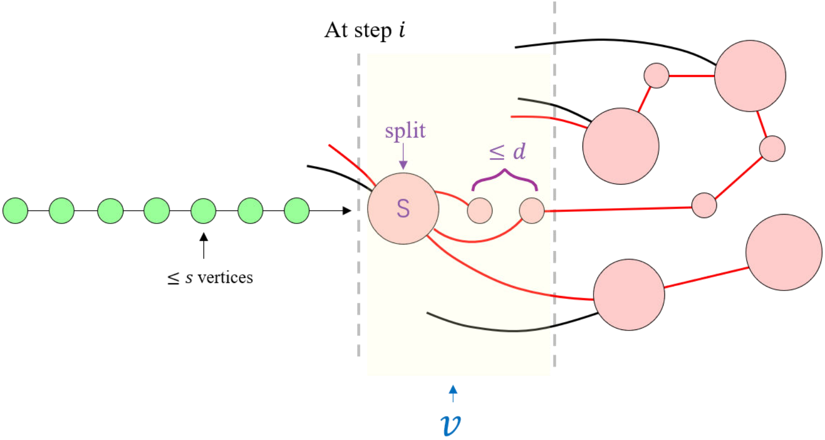
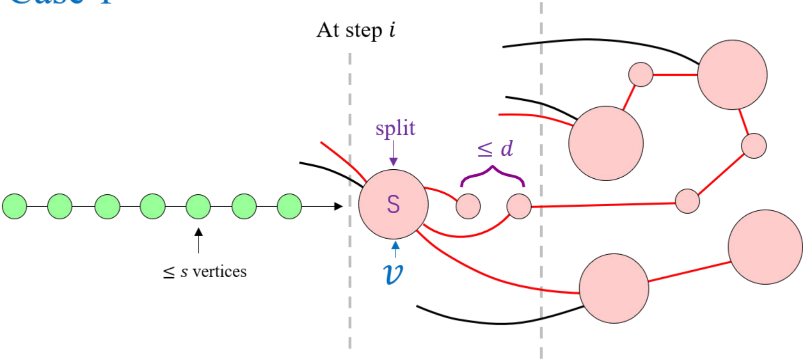


Bounding for scol_r



Bounding for scol_r

Case 1



Bounding for scol_r

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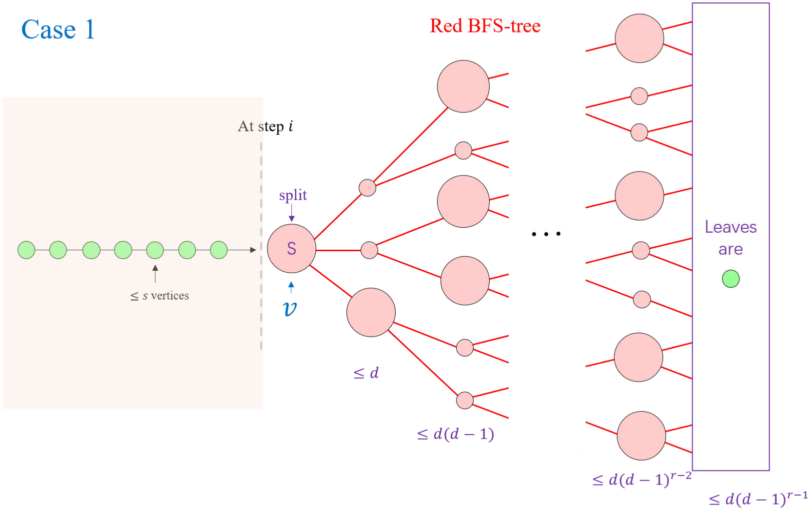

Bounding for scol_r

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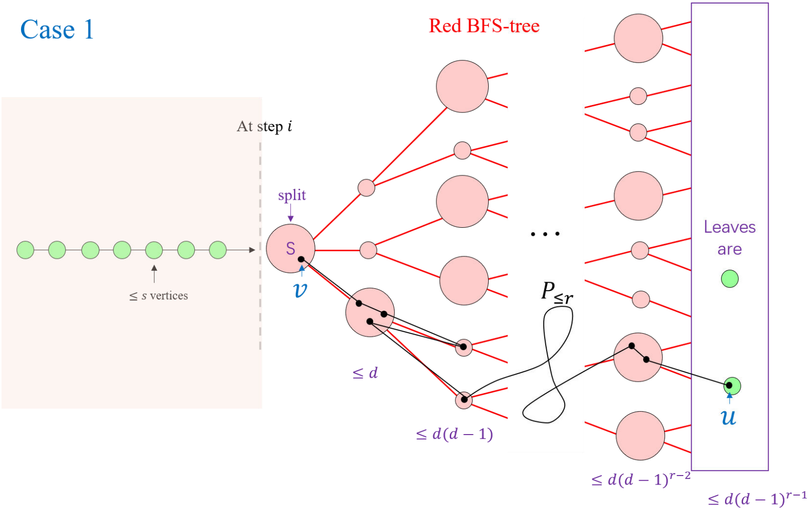
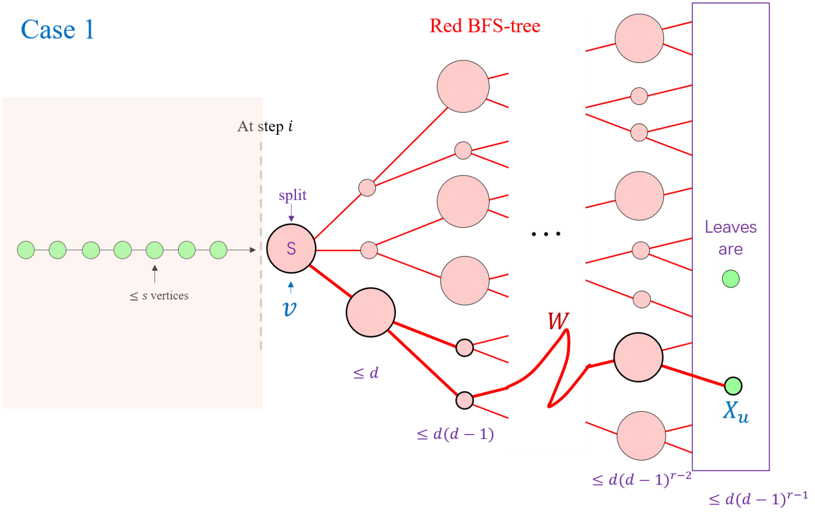
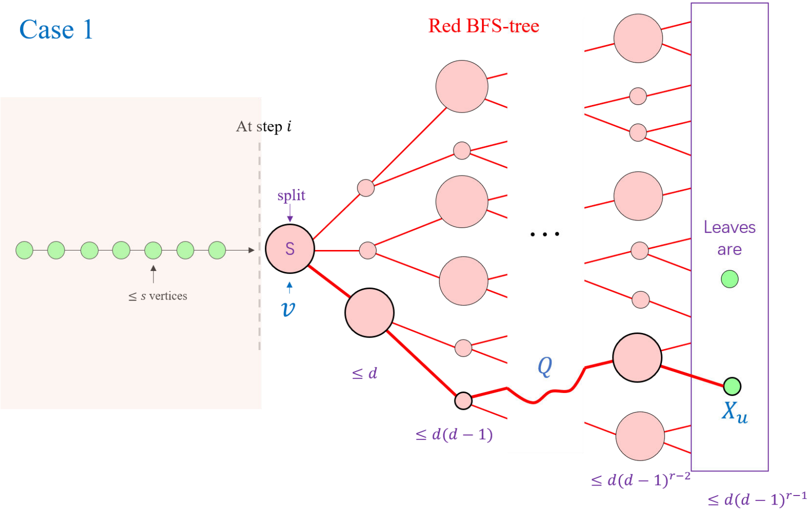



Case 1
Red BFS-tree
At step i
split
S
v
$\leq s$ vertices
$\leq d$
$\leq d(d-1)$
W
$\leq d(d-1)^{r-2}$
Leaves are
X_u
$\leq d(d-1)^{r-1}$

Bounding for scol_r



Bounding for scol_r

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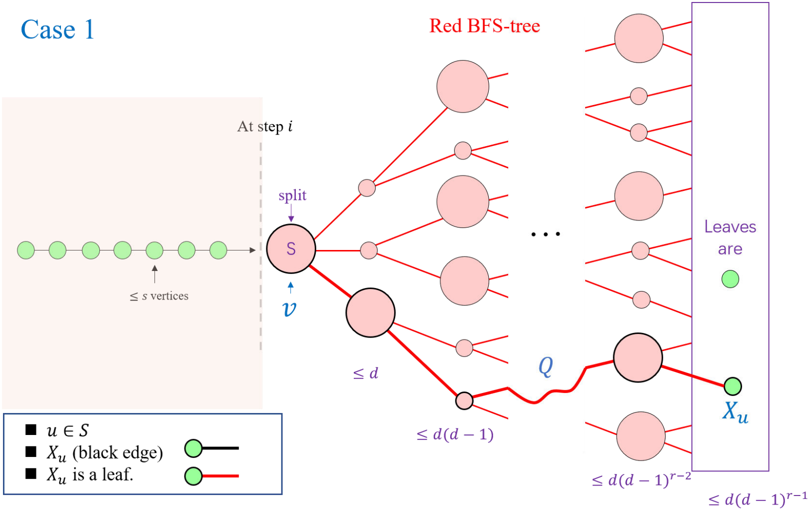
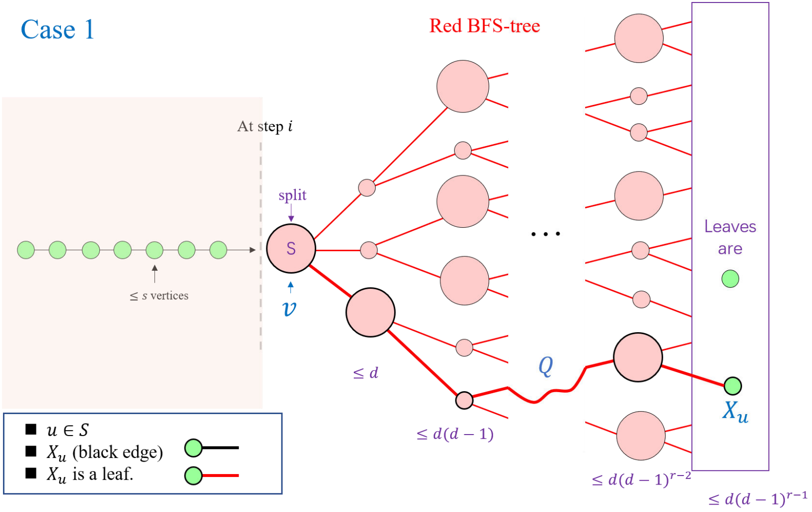


- $u \in S$
- X_u (black edge)
- X_u is a leaf.

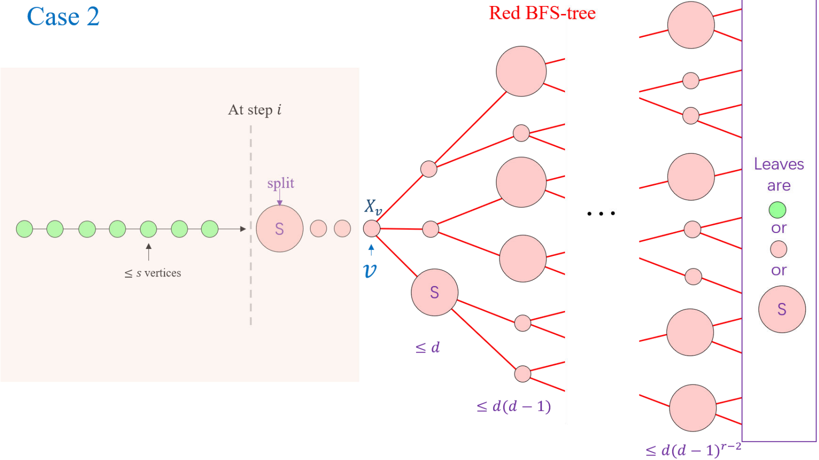
Bounding for scol_r



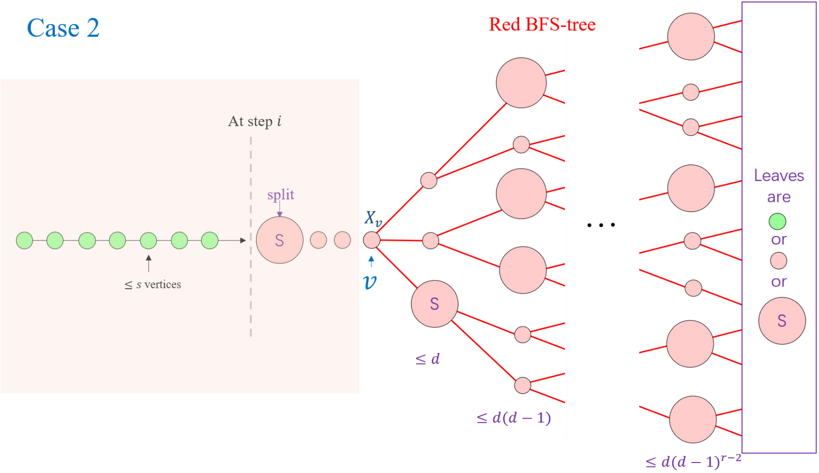
$$|\,\mathrm{Sreach}[G, L, v]| \leq 2s + (1 + d + \cdots + d(d-1)^{r-2})s + (d(d-1)^{r-1})s$$

Bounding for scol_r

Case 2



Bounding for scol_r



$$|\,\mathrm{Sreach}[G,L,v]| \le s + 2s + (1 + d + \cdots + d(d-1)^{r-2} - 1)s + (d(d-1)^{r-1})s$$

Upper bound of scol_r

Thus

$$\mathrm{scol}_r(G) \leq \left(3 + d\sum_{i=0}^{r-1}(d-1)^i\right)s \leq (d^r+3)s$$

Corollary

For every graph G and every positive integer r we have

$$\mathrm{scol}_r(G) \leq \begin{cases} 2\,\mathrm{b}\omega(G) & \text{if } \mathrm{tww}(G) = 0, \\ 3\,\mathrm{b}\omega(G) & \text{if } \mathrm{tww}(G) = 1, \\ 5\,\mathrm{b}\omega(G) & \text{if } \mathrm{tww}(G) = 2, \\ 3(\mathrm{tww}(G)-1)^r\,\mathrm{b}\omega(G) & \text{if } \mathrm{tww}(G) \geq 3. \end{cases}$$

Lower bounds of scol_r

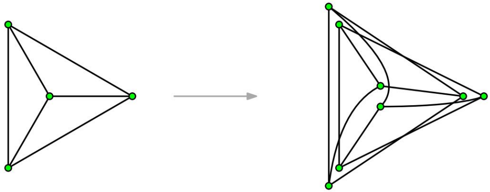
Theorem - [Grohe, Kreutzer, Rabinovich, Siebertz, and Stavropoulos]

Let G be a d-regular graph of girth at least $4g+1$, where $d \geq 7$. Then for every $r \leq g$,

$$\mathrm{scol}_r(G) \geq \frac{d}{2}\left(\frac{d-2}{4}\right)^{2^{\lfloor \log_2 r \rfloor}-1}.$$

Lower bounds of scol_r - 2-lifts

A 2-*lift* of G is a graph with vertex set $V(G) \times \{0,1\}$ such that each edge xy of G gives a matching between $(x,0),(x,1)$ and $(y,0),(y,1)$.



Lower bounds of scol_r - 2-lifts

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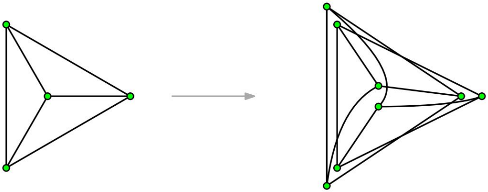


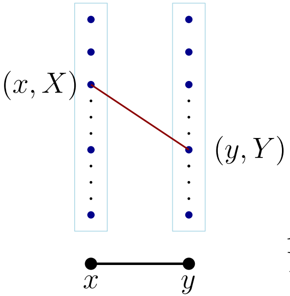
Lemma

If G' is obtained from G by a sequence of 2-lifts then

$$\mathrm{tww}(G') \leq 2\,\mathrm{tww}(G).$$

Lower bounds of scol_r - High girth by 2-lifts

Let $G = (V, E)$ be an r-regular graph with n vertices and $m = nr/2$ edges, and let G' be the r-regular graph with $V(G') = V \times 2^E$ matching (x, X) with (y, Y) if $xy \in E$ and $X \Delta Y = \{xy\}$.



	e_1	e_2	e_3	$\dots$	xy	$\dots$	e_m
$X =$	(0,	1,	0,	$\cdots$	1,	$\cdots$	1)
$Y =$	(0,	1,	0,	$\cdots$	0,	$\cdots$	1)

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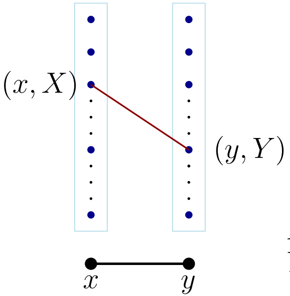

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$Y =$	(0,	1,	0,	$\cdots$	0,	$\cdots$	1)

Then

- G' is obtained from G by a sequence of 2-lifts,
- $\mathrm{girth}(G') \geq 2\,\mathrm{girth}(G)$.

Repeat this and use the bound on scol_r for regular graphs with high girth.

Lower bounds of scol_r

Lemma

For every integer $\Delta \geq 7$ and every integers r and $g \geq 4r + 1$ there exists a Δ-regular graph G with girth at least g, $2\Delta - 1 \leq \mathrm{tww}(G) \leq 2\Delta$, and

$$\mathrm{scol}_r(G)) \geq \frac{\Delta}{2}\left(\frac{\Delta - 2}{4}\right)^{2^{\lfloor \log_2 r \rfloor} - 1} \geq \frac{\mathrm{tww}(G)}{4}\left(\frac{\mathrm{tww}(G) - 4}{8}\right)^{2^{\lfloor \log_2 r \rfloor} - 1}.$$

Corollary

For every integer $d \geq 14$, every positive integer s, and every integer r of the form 2^k, there exists a graph G with $\mathrm{tww}(G) \leq d$, $\mathrm{b}\omega(G) = s$, and

$$\mathrm{scol}_r(G) \geq \frac{ds}{4}\left(\frac{d - 4}{8}\right)^{r-1} \geq 2\left(\frac{\mathrm{tww}(G) - 4}{8}\right)^{r} \mathrm{b}\omega(G).$$

THANK YOU
FOR YOUR ATTENTION!