Integral Difference Ratio Functions on Integers

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To Jozef, on his 80th birthday, with our gratitude for sharing with us his prophetic vision of « Informatique »

Abstract. Various problems lead to the same class of functions from integers to integers: functions having integral difference ratio, i.e. verifying $f(a) - f(b) \equiv 0 \pmod{(a-b)}$ for all $a > b$. In this paper we characterize this class of functions from $\mathbb{Z}$ to $\mathbb{Z}$ via their *à la Newton* series expansions on a suitably chosen basis of polynomials (with rational coefficients). We also exhibit an example of such a function which is not polynomial but Bessel like.

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1 Introduction

We deal with the following class of functions which appears in Pin & Silva, 2011 (see §4.2 and §5.3 in [10]), as a characterization of a special strong notion of uniform continuity.

Definition 1. *Let $X \subseteq \mathbb{Z}$ (where $\mathbb{Z}$ denotes the set of integers). A map $f \colon X \to \mathbb{Z}$ has* integral difference ratio *if* $\dfrac{f(i) - f(j)}{i - j} \in \mathbb{Z}$ *, for all distinct $i, j \in X$.*

Observe the following simple properties about these maps.

Proposition 2. *1. The set of maps $f \colon X \to \mathbb{Z}$ having integral difference ratio is closed under addition and multiplication. In particular, it contains all polynomials with integral coefficients.*
2. The set of maps $f \colon X \to \mathbb{Z}$ having integral difference ratio is closed under composition.

Proof. For multiplication, use the identity $f(i)g(i) - f(j)g(j) = f(i)(g(i) - g(j)) + g(j)(f(i) - f(j))$. □

Which non-polynomial maps have integral difference ratio? This is the question we deal with.

In [1] we related the integral difference ratio property to functions $f \colon \mathbb{N} \to \mathbb{N}$ (where $\mathbb{N}$ is the set of nonnegative integers) such that any lattice of finite subsets of $\mathbb{N}$ closed under decrement is also closed under inverse image by f (Theorem 4). In §2 we extend this result to functions $\mathbb{Z} \to \mathbb{Z}$ (Theorem 6).

In our paper [2] we characterized the functions $f \colon \mathbb{N} \to \mathbb{Z}$ having integral difference ratio in terms of their Newton expansions over the "binomial polynomials". In §3 we give a similar characterization for functions $f \colon \mathbb{Z} \to \mathbb{Z}$ (Theorem 14). This is the main result of the paper, its proof runs through §4 and §5. Though both characterizations rely on analogous ideas, the $\mathbb{Z}$ case is not reducible to the $\mathbb{N}$ case:

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we have to consider à la Newton expansions over a different family of polynomials. Even though these polynomials have rational (non integer) coefficients, they map $\mathbb{Z}$ into $\mathbb{Z}$.

The characterization we give (Theorem 14) insures that there are uncountably many non-polynomial functions having integral difference ratio. In [2] we explicited non polynomial maps $f : \mathbb{N} \to \mathbb{Z}$ having integral difference ratio; the map $g : \mathbb{Z} \to \mathbb{Z}$ such that $g(x) = f(x^2)$ also has integral difference ratio and is non polynomial. In §6 we exhibit a non-polynomial example related to Bessel functions which does not so reduce to a map $\mathbb{N} \to \mathbb{Z}$.

2 Integral difference ratio functions and lattices

In this section, we extend Theorem 4 of our paper [1] to functions $\mathbb{Z} \to \mathbb{Z}$.

A *lattice* of subsets of a set X is a family of subsets of X such that $L \cap M$ and $L \cup M$ are in $\mathcal{L}$ whenever $L, M \in \mathcal{L}$. Let $f : X \to X$. A lattice $\mathcal{L}$ of subsets of X is closed under f^{-1} if $f^{-1}(L) \in \mathcal{L}$ whenever $L \in \mathcal{L}$. *Closure under decrement* means closure under Suc^{-1}, where Suc is the successor function.

We let $\mathcal{P}_{<\omega}(X)$ denote the class of finite subsets of X. For $L \subseteq \mathbb{Z}$ and $t \in \mathbb{Z}$ we let $L - t = \{x - t \mid x \in L\}$.

Proposition 3. *Let X be $\mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{N}_\alpha = \{x \in \mathbb{Z} \mid x \geq \alpha\}$ with $\alpha \in \mathbb{Z}$. For L a finite subset of X let $\mathcal{L}_X(L)$ be the family of sets of the form $\bigcup_{j \in J} \bigcap_{i \in I_j} X \cap (L - i)$ where J and the I_j's are finite non empty subsets of $\mathbb{N}$. Then $\mathcal{L}_X(L)$ is the smallest sublattice of $\mathcal{P}_{<\omega}(X)$ containing L and closed under decrement.*

The following characterization is proved in [1]:

Theorem 4. *Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a non decreasing function. The following conditions are equivalent:*

$(1)_\mathbb{N}$ *For every finite subset L of $\mathbb{N}$, the lattice $\mathcal{L}_\mathbb{N}(L)$ is closed under f^{-1}.*
$(2)_\mathbb{N}$ *The function f has integral difference ratio and $f(a) \geq a$ for all $a \in \mathbb{N}$.*

In order to extend Theorem 4 to functions $\mathbb{Z} \to \mathbb{Z}$, we need the $\mathbb{Z}$-version of Lemma 3.1 in [1].

Lemma 5. *Let $f : \mathbb{Z} \to \mathbb{Z}$ be a nondecreasing function such that $f(x) - f(y) \equiv 0 \mod (x - y)$ for every $x > y \in \mathbb{Z}$. Then, for any set $L \subseteq \mathbb{Z}$, we have $f^{-1}(L) = \bigcup_{a \in f^{-1}(L)} \bigcap_{t \in L - a} (L - t)$.*

Proof. Let $a \in f^{-1}(L)$. As $t \in L - a \Leftrightarrow a \in L - t$, we have $a \in \bigcap_{t \in L - a} L - t$, proving inclusion $\subseteq$.

For the other inclusion, let $b \in \bigcap_{t \in L - a} L - t$ with $a \in f^{-1}(L)$. To prove that $f(b) \in L$, we argue by way of contradiction. Suppose $f(b) \notin L$. Since $f(a) \in L$ we have $a \neq b$. The condition on f insures the existence of $k \in \mathbb{Z}$ such that $f(b) - f(a) = k(b - a)$. In fact, $k \in \mathbb{N}$ since f is nondecreasing.

Suppose first that $a < b$. Since $k \in \mathbb{N}$ and $f(a) + k(b - a) = f(b) \notin L$ there exists a least $r \in \mathbb{N}$ such that $f(a) + r(b - a) \notin L$. Moreover, $r \geq 1$ since $f(a) \in L$. Let $t = f(a) - a + (r - 1)(b - a)$. By minimality of r, we get $t + a = f(a) + (r - 1)(a - b) \in L$. Now $t + a \in L$ implies $t + b \in L$. But $t + b = f(a) + r(b - a) \notin L$, this contradicts the definition of r.

Suppose next that $a > b$. Since $k \in \mathbb{N}$ and $f(b) + k(a - b) = f(a) \in L$ there exists a least $r \in \mathbb{N}$ such that $f(b) + r(a - b) \in L$. Moreover, $r \geq 1$ since $f(b) \notin L$. Let $t = f(b) - b + (r - 1)(a - b)$. By minimality of r, we get $t + b = f(b) + (r - 1)(a - b) \notin L$. Now $t + a \in L$ implies $t + b \in L$, contradiction. □

We can now extend Theorem 4 to functions $\mathbb{Z} \to \mathbb{Z}$.

Theorem 6. *Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a non decreasing function. The following conditions are equivalent:*

$(1)_{\mathbb{Z}}$ *For every finite subset L of $\mathbb{Z}$, the lattice $\mathcal{L}_{\mathbb{Z}}(L)$ is closed under f^{-1}.*
$(2)_{\mathbb{Z}}$ *The function f has integral difference ratio and $f(a) \geq a$ for all $a \in \mathbb{Z}$.*

Proof. • $(1)_{\mathbb{Z}} \Rightarrow (2)_{\mathbb{Z}}$. Assume $(1)_{\mathbb{Z}}$ holds. We first prove inequality $f(x) \geq x$ for all $x \in \mathbb{Z}$. Observe that (by Proposition 3) $\mathcal{L}_{\mathbb{Z}}(\{z\}) = \{X \in \mathcal{P}_{<\omega}(\mathbb{Z}) \mid X = \emptyset \text{ or } \max X \leq z\}$. In particular, letting $z = f(x)$ and applying $(1)_{\mathbb{Z}}$ with $\mathcal{L}(\{f(x)\})$, we get $f^{-1}(\{f(x)\} \in \mathcal{L}_{\mathbb{Z}}(\{f(x)\})$ hence $x \leq \max(f^{-1}(\{f(x)\})) \leq f(x)$.

To show that f has integral difference ratio, we reduce to the $\mathbb{N}$ case.

For $\alpha \in \mathbb{Z}$, let $Suc_\alpha : \mathbb{N}_\alpha \to \mathbb{N}_\alpha$ be the successor function on $\mathbb{N}_\alpha = \{z \in \mathbb{Z} \mid z \geq \alpha\}$. The structures $\langle \mathbb{N}, Suc\rangle$ and $\langle \mathbb{N}_\alpha, Suc_\alpha\rangle$ are isomorphic. Since $f(x) \geq x$ for all $x \in \mathbb{Z}$, the restriction $f \restriction \mathbb{N}_\alpha$ maps $\mathbb{N}_\alpha$ into $\mathbb{N}_\alpha$. In particular, using Theorem 4, conditions $(1)_{\mathbb{N}_\alpha}$ and $(2)_{\mathbb{N}_\alpha}$ (relative to $f \restriction \mathbb{N}_\alpha$) are equivalent.

We show that condition $(2)_{\mathbb{N}_\alpha}$ holds. Let $L \subseteq \mathbb{N}_\alpha$ be finite. Condition $(1)_{\mathbb{Z}}$ insures that $\mathcal{L}_{\mathbb{Z}}(L)$ is closed under f^{-1}. In particular, $f^{-1}(L) \in \mathcal{L}_{\mathbb{Z}}(L)$. Using Proposition 3, we get $f^{-1}(L) = \bigcup_{j \in J} \bigcap_{i \in I_j} (L - i)$ for finite J, I_j's included in $\mathbb{N}$ hence $(f \restriction \mathbb{N}_\alpha)^{-1}(L) = f^{-1}(L) \cap \mathbb{N}_\alpha = \bigcup_{j \in J} \bigcap_{i \in I_j} (\mathbb{N}_\alpha \cap (L - i)) \in \mathcal{L}_{\mathbb{N}_\alpha}(L)$. This proves condition $(1)_{\mathbb{N}_\alpha}$. Since $(1)_{\mathbb{N}_\alpha} \Rightarrow (2)_{\mathbb{N}_\alpha}$ we see that $f \restriction \mathbb{N}_\alpha$ has integral difference ratio Now, α is arbitrary in $\mathbb{Z}$ and the integral difference ratio property of $f \restriction \mathbb{N}_\alpha$ for all $\alpha \in \mathbb{Z}$ yields the integral difference ratio property for f. Thus, condition $(2)_{\mathbb{Z}}$ holds.

• $(2)_{\mathbb{Z}} \Rightarrow (1)_{\mathbb{Z}}$. Assume $(2)_{\mathbb{Z}}$. Then f is not constant since $f(x) \geq x$ for all $x \in \mathbb{Z}$. Also, $f^{-1}(\alpha)$ is finite for all α : let b be such that $f(b) = \beta \neq \alpha$, by the integral difference ratio property the nonzero integer $\alpha - \beta$ is divided by $a - b$ for all $a \in f^{-1}(\alpha)$ hence $f^{-1}(\alpha)$ is finite.

To prove $(1)_{\mathbb{Z}}$ it suffices to prove that $f^{-1}(L) \in \mathcal{L}_{\mathbb{Z}}(L)$ whenever $L \subseteq \mathbb{Z}$ is finite. By Lemma 5 we have $f^{-1}(L) = \bigcup_{a \in f^{-1}(F)} \bigcap_{n \in L - a} (L - n)$. Observe that $f^{-1}(F)$ is finite since F is finite and so is each $f^{-1}(a)$. Also, for each $a \in f^{-1}(F)$, the set $L - a$ is finite (as is L) . Thus, the above formula expresses $f^{-1}(L)$ as a finite combination of unions and intersections of decrements of L. This yields $f^{-1}(F) \in \mathcal{L}_{\mathbb{Z}}(L)$. □

3 Newton series expansions of functions having integral difference ratio

Elementary algebra shows that all polynomials have integral difference ratio. To obtain non polynomial function having integral difference ratio functions, we need a precise characterization via Newton series.

3.1 Newton basis for functions $\mathbb{N} \to \mathbb{Z}$

Definition 7. *Let $X = \mathbb{N}$ or $X = \mathbb{Z}$. A sequence of one-variable polynomials $(P_k)_{k \in \mathbb{N}}$ with rational coefficients is a* Newton basis *for maps $X \to \mathbb{Z}$ if the following conditions are satisfied:*

(1) For every $x \in X$ and $k \in \mathbb{N}$, $P_k(x)$ is in $\mathbb{Z}$.
(2) For every $x \in X$, the set $\{k \in \mathbb{N} \mid P_k(x) \neq 0\}$ is finite.
(3) The correspondence which associates to a sequence $(a_k)_{k \in \mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$ the map $f : X \to \mathbb{Z}$ such that

$$f(x) = \sum_{k \in \mathbb{N}} a_k \, P_k(x) \tag{1}$$

is a bijection between sequences in $\mathbb{Z}^{\mathbb{N}}$ and maps $X \to \mathbb{Z}$.
The right side of equation (1) *is called the* Newton series expansion *of f.*

The following result (cf. [2]) dates back to Newton.

Proposition 8. *The binomial polynomials $\binom{x}{k} = \dfrac{\prod_{i=0}^{k-1}(x-i)}{k!}$, $k \in \mathbb{N}$ (with $\binom{x}{0} = 1$), constitute a Newton basis for maps $\mathbb{N} \to \mathbb{Z}$.*

3.2 Characterization of functions $\mathbb{N} \to \mathbb{Z}$ having integral difference ratio

Definition 9. *For $k \in \mathbb{N}$, $k \geq 1$, $lcm(k)$ is the least common multiple of all positive integers less than or equal to k. By convention, $lcm(0) = 1$.*

We proved in [2] the following characterization of functions $\mathbb{N} \to \mathbb{Z}$ having integral difference ratio:

Theorem 10. *Let $f : \mathbb{N} \to \mathbb{Z}$ be a function with Newton expansion $\sum_{k\in\mathbb{N}} a_k \binom{x}{k}$. The following conditions are equivalent:*

(1) f has integral difference ratio.
(2) $lcm(k)$ divides a_k for all $k \in \mathbb{N}$.

3.3 A Newton basis for functions $\mathbb{Z} \to \mathbb{Z}$

The polynomials $\binom{x}{k}$ are not a Newton basis for maps $\mathbb{Z} \to \mathbb{Z}$ since condition (2) of Definition 7 fails for all negative x and all $k \in \mathbb{N}$. We design another sequence of polynomials tailored for $\mathbb{Z} \to \mathbb{Z}$ maps.

Definition 11. *The $\mathbb{Z}$-Newtonian polynomials are defined as follows:*

$$P_0(x) = 1 \quad , \quad P_{2k}(x) = \frac{1}{(2k)!} \prod_{i=-k+1}^{i=k} (x - i) \quad , \quad P_{2k+1}(x) = \frac{1}{(2k+1)!} \prod_{i=-k}^{i=k} (x - i)$$

Let us explicit the first polynomials in the above sequence:

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{x\ (x-1)}{2!} \quad P_3(x) = \frac{(x+1)x(x-1)}{3!} \quad P_4(x) = \frac{(x+1)x(x-1)\ (x-2)}{4!}$$
$$P_5(x) = \frac{(x+2)(x+1)x(x-1)(x-2)}{5!} \quad P_6(x) = \frac{(x+2)(x+1)x(x-1)(x-2)(x-3)}{6!} \quad \ldots$$

Proposition 12. *The $\mathbb{Z}$-Newtonian polynomials define maps on $\mathbb{Z}$ which take values in $\mathbb{Z}$ and satisfy the following equations for $k, n \in \mathbb{N}$,*

$$P_{2k+1}(n) = \begin{cases} \binom{k+n}{2k+1} & \text{if } n > k \\ 0 & \text{if } 0 \leq n \leq k \end{cases} \qquad P_{2k}(n) = \begin{cases} \binom{k+n-1}{2k} & \text{if } n > k \\ 0 & \text{if } 0 \leq n \leq k \end{cases} \tag{2}$$

$$P_{2k+1}(-n) \ = \ -P_{2k+1}(n) \qquad P_{2k}(-n) = \begin{cases} \binom{k+n}{2k} & \text{if } n \geq k \\ 0 & \text{if } 0 \leq n < k \end{cases} \tag{3}$$

Proof. Observe that, for any $a, b, x \in \mathbb{Z}$ such that $a < 0 \leq b$, we have

$$\frac{1}{(b-a+1)!} \prod_{i=a}^{i=b} (x - i) = \begin{cases} \binom{x-a}{b-a+1} & \text{if } x > b \\ 0 & \text{if } a \leq x \leq b \\ (-1)^{b-a+1} \binom{|x|+b}{b-a+1} & \text{if } x < a \end{cases}$$

Thus, the P_n's map $\mathbb{Z}$ into $\mathbb{Z}$ and satisfy conditions (2) and (3). □

Proposition 13. *The $\mathbb{Z}$-Newtonian polynomials are a Newton basis for maps $\mathbb{Z} \to \mathbb{Z}$.*

Proof. Conditions (2), (3) in Proposition 12 insure that equation (1) of Definition 7 reduces to

$$f(x) \;=\; \sum_{n \in \{0,\ldots,2|x|+1\}} a_n \; P_n(x) \tag{4}$$

which involves a finite sum. Moreover, all terms of this sum are in $\mathbb{Z}$ when the a_n's are in $\mathbb{Z}$. Thus, for any sequence $(a_n)_{n\in\mathbb{N}}$ of integers in $\mathbb{Z}$, equation (4) defines a map $f : \mathbb{Z} \to \mathbb{Z}$.

To prove the converse, observe that the instances of equation (4) can be written

$$\begin{array}{llll} f(0) = a_0 & f(1) = a_0 + a_1 & f(2) = a_0 + 2a_1 + a_2 + a_3 & \ldots \\ & f(-1) = a_0 - a_1 + a_2 & f(-2) = a_0 - 2a_1 + 3a_2 - a_3 + a_4 & \ldots \end{array}$$

In general, for $k \geq 1$, Proposition 12 yields

$$\begin{array}{rr} f(2k) = L_{2k}(a_0,\ldots,a_{4k-2}) + a_{4k-1} & f(-2k) = L_{-2k}(a_0,\ldots,a_{4k-1}) + a_{4k} \\ f(2k+1) = L_{2k+1}(a_0,\ldots,a_{4k}) + a_{4k+1} & f(-2k-1) = L_{-2k-1}(a_0,\ldots,a_{4k+1}) + a_{4k+2} \end{array}$$

where $L_n(a_0,\ldots,a_{2n-2})$ and $L_{-n}(a_0,\ldots,a_{2n-1})$ are linear combinations of the a_i's with coefficients in $\mathbb{Z}$. This shows that, given any $f : \mathbb{Z} \to \mathbb{Z}$, there is a unique sequence of coefficients $(a_n)_{n\in\mathbb{N}}$ making equation (1) of Definition 7 true, and all these coefficients are in $\mathbb{Z}$. □

3.4 Functions $\mathbb{Z} \to \mathbb{Z}$ having integral difference ratio

We can now state the main result of the paper which characterizes the functions $f : \mathbb{Z} \to \mathbb{Z}$ having integral difference ratio,

Theorem 14. *Let $\sum_{k\in\mathbb{N}} a_k P_k(x)$ be the $\mathbb{Z}$-Newtonian expansion of a function $f : \mathbb{Z} \to \mathbb{Z}$. Then the following conditions are equivalent:*

(1) f has integral difference ratio,
(2) $lcm(k)$ divides a_k for all k.

The next two sections are devoted to the proof of Theorem 14.

4 Some properties involving the unary least common multiple function *lcm* and binomial coefficients

The unary function *lcm* (cf. Definition 9) has many interesting properties and recently regained interest, cf. [9,8,4,5,3]. In this section, we prove three lemmas used in the proof of Theorem 14. They link the *lcm* function and binomial coefficients. Lemma 15 already appears in [2]: for the sake of selfcontainment we repeat its short proof. Lemma 17 is a variation tailored for the $\mathbb{Z}$ case of results in [2]. Lemma 16 is a crucial specific result with a very long proof.

Lemma 15. *If $0 \leq n-k < p \leq n$ then p divides $lcm(k)\binom{n}{k}$.*

Proof. By induction on $n \geq 1$. The initial case $n = 1$ is trivial since condition $0 \leq n - k < p \leq n$ yields $p = k = 1$. Induction step: assuming the result for n, we prove it for $n+1$. Suppose $0 \leq n+1-k < p \leq n+1$. *Case* $p \leq n$. Then $0 \leq n-k < p \leq n$ and $0 \leq n-(k-1) < p \leq n$, so that, by induction hypothesis, p divides $lcm(k)\binom{n}{k}$ and $lcm(k-1)\binom{n}{k-1}$. A fortiori, p divides $lcm(k)\binom{n}{k}+lcm(k)\binom{n}{k-1} = lcm(k)\binom{n+1}{k}$. *Case* $p = n + 1$. Then $k \geq 1$ and $lcm(k)\binom{n+1}{k} = (n+1)\frac{lcm(k)}{k}\binom{n}{k-1}$ is a multiple of $n+1$. □

Lemma 16. *If $p \geq 0$ then $2(p+k)$ divides $lcm(2k)\binom{p+2k-1}{2k-1}$.*

Proof. For $x \geq 1$, let $Val(x)$ denote the 2-valuation of x, i.e. the largest i such that 2^i divides x.

The Lemma is proved through a series of claims. Throughout the proof, B will denote $\binom{p+2k-1}{2k-1}$.

Claim 1. The number $p + k$ divides $B\, lcm(2k-1)$.

Proof. Let $p' = p + k$, $n' = p + 2k - 1$ and $k' = 2k - 1$ and apply Lemma 15. □

Claim 2. If k is a power of 2 then $2(p + k)$ divides $B\, lcm(2k)$.

Proof. Observe that $lcm(2k) = 2\, lcm(2k-1)$ if k is a power of 2 and apply Claim 1. □

Claim 3. The number $2^{Val(k)+1}$ divides $lcm(2k)$.

Proof. Since $2^{Val(k)}$ divides k it also divides $lcm(k)$. To conclude, observe that $Val(2k) = Val(k)+1$. □

Claim 4. The 2-valuation of B is the number of carries when adding $2k - 1$ and p in base 2.

Proof. This is an instance of Kummer's theorem (1852, cf. [6]) for base $s = 2$: if s is prime and $b \leq a$, the largest i such that s^i divides $\binom{a}{b}$ is the number of carries when adding b and $a - b$ in base s. □

In the next claims we consider binary expansions with possibly non significant zeros ahead to get some prescribed large enough length.

Claim 5. Let $t \geq 1$ be the 2-valuation of $(2k'+1)+(2p'+1)$, i.e. $(2k'+1)+(2p'+1) = 2^t\,(2q+1)$ for some q. For n large enough (e.g., $2^n \geq 2^t\,(2q+1)$), let $k_n \ldots k_1 1$ and $p_n \ldots p_1 1$ be the length $n+1$ binary expansions of $2k'+1$ and $2p'+1$. Then $k_i + p_i = 1$ for $1 \leq i \leq t-1$ and $p_t = k_t$.

Proof. Let $q_n \ldots q_{t+1} 10 \ldots 0$ (with a tail of t zeros) be the length $n+1$ binary expansion of $2^t\,(2q+1)$. The addition of $2k'+1$ and $2p'+1$ in base 2 is depicted below

digit rank	:	r	$\cdots$	$t+1$	t	$t-1$	$\cdots$	3	2	1	0
$(2k'+1)$	:	$\cdots$	$\cdots$	k_{t+1}	k_t	k_{t-1}	$\cdots$	k_3	k_2	k_1	1
$+\ (2p'+1)$	:	$\cdots$	$\cdots$	p_{t+1}	p_t	p_{t-1}	$\cdots$	p_3	p_2	p_1	1
$=\ 2^t\,(2q+1)$	:	q_r	$\cdots$	q_{t+1}	1	0	$\cdots$	0	0	0	0

Observe that adding the digits $k_0 = 1$ and $p_0 = 1$ leads to $q_0 = 0$ and creates a carry. An easy induction on $i = 1, \ldots, t-1$ shows that, in order to get the tail of t zeros in the sum, the incoming carry has to propagate from rank i to rank $i+1$ and equality $k_i + p_i = 1$ holds. Finally, since $q_t = 1$ and there is an incoming carry at rank t, we have $p_t = k_t$. □

Claim 6. Let p, k have the same 2-valuation ℓ, i.e. $p = 2^\ell(2p'+1)$ and $k = 2^\ell(2k'+1)$. Let t be the 2-valuation of $(2k'+1)+(2p'+1)$. For n large enough (say $2^n \geq p+2k-1$), let $p_n \cdots p_1 1$ be the length $n+1$ binary expansion of $2p'+1$. Let N be the number of 1's in $p_t \cdots p_1 1$. Then 2^N divides $B = \binom{p+2k-1}{2k-1}$.

Proof. Let $k_n \cdots k_1 1$ be the length $n+1$ binary expansion of $2k'+1$. Applying Claim 5 to $2k'+1$ and $2p'+1$ we see that $k_i + p_i = 1$ for $1 \leq i \leq t-1$ and $k_t = p_t$.

By Claim 4, to show that 2^N divides B we reduce to prove that the number of carries when adding p and $2k-1$ is at least N. The binary expansions of k, $2k-1$ and p are as follows:

rank :	$\cdots$	$t+\ell+1$	$t+\ell$	$t+\ell-1$	$\cdots$	$\ell+2$	$\ell+1$	ℓ	$\ell-1$	$\cdots$	0
$k = 2^\ell(2k'+1)$:	$\cdots$	k_{t+1}	k_t	k_{t-1}	$\cdots$	k_2	k_1	1	0	$\ldots$	0
$2k$–1 :	$\cdots$	k_t	k_{t-1}	k_{t-2}	$\cdots$	k_1	0	1	1	$\ldots$	1
$p = 2^\ell(2p'+1)$:	$\cdots$	p_{t+1}	p_t	p_{t-1}	$\cdots$	p_2	p_1	1	0	$\ldots$	0

In the addition of $2k-1$ and p the first carry occurs at rank ℓ. Hence, the number of carries in this addition is equal to the number of carries in the addition of the integers obtained by deleting the ℓ last digits, i.e. the numbers $\lambda = 2^{-\ell}\big((2k-1)-(2^\ell-1)\big)$ and $2p'+1$. We thus reduce to show that there are at least N carries in the addition of λ and $2p'+1$. The binary expansions of λ and $2p'+1$ are

rank :	$n+1$	n	$\cdots$	$t+2$	$t+1$	t	$t-1$	$\cdots$	3	2	1	0
λ :	k_n	k_{n-1}	$\cdots$	k_{t+1}	k_t	k_{t-1}	k_{t-2}	$\cdots$	k_2	k_1	0	1
$2p'+1$:	$\cdots$	$\cdots$	$\cdots$	p_{t+2}	p_{t+1}	p_t	p_{t-1}	$\cdots$	p_3	p_2	p_1	1

with $k_i = 1 - p_i$ for $i = 1, \ldots, t-1$ and $k_t = p_t$. We prove by induction on the rank $i = 0, \ldots, t$ that, in the addition of λ and $2p'+1$, for all $0 \leq i \leq t$, if $p_i = 1$ then there is a carry at rank i.
Case $i = 0$. Since the added digits at rank 0 are both equal to 1, there is a carry.
Case $1 \leq i \leq t$ *and* $p_i = 0$. There is nothing to prove.
Case $i = 1$ *and* $p_1 = 1$. The added digits at rank 1 are 0 and 1 (since $p_1 = 1$). Since there is an incoming carry (that from rank 0) a carry is created at rank 1.
Case $1 \leq i \leq t$ *and* $p_i = 1$ *and* $p_{i-1} = 0$. Since $i-1 < t$ we have $k_{i-1} + p_{i-1} = 1$ hence $k_{i-1} = 1$. Thus, the added digits at rank i (namely k_{i-1} and p_i) are both equal to 1 hence there is a carry.
Case $1 \leq i \leq t$ *and* $p_i = 1$ *and* $p_{i-1} = 1$. By the induction hypothesis, a carry occurs at rank $i-1$. Thus, at rank i there is an incoming carry (the one from rank $i-1$) and the digit p_i is 1, hence (whatever be the digit of λ at rank i) there is a carry at rank i.

This shows that there are at least N carries in the addition of λ and $2p+1$. □

Claim 7. Let p, k, ℓ, t, N be as in Claim 6. Then $2^{\ell+t+1-N}$ divides $lcm(2k)$.

Proof. There are N ones in $p_t p_{t-1} \cdots p_1 1$, hence there are at most $N-1$ ones and at least $t+1-N$ zeros in $p_{t-1} \cdots p_1$. By Claim 5, $k_{t-1} \cdots k_1$ contains at least $t+1-N$ ones. Thus, the number of significant digits of $k_{t-1} \cdots k_1 01$ is at least $t+3-N$. The binary expansion of $2k-1$ is $k_n \cdots k_t k_{t-1} \cdots k_1 01$ followed by ℓ ones hence $2k-1$ has at least $\ell+t+3-N$ significant digits. Consequently, $2k-1 \geq 2^{\ell+t+2-N}$ and $2^{\ell+t+2-N}$ divides $lcm(2k-1)$ and, a fortiori, $lcm(2k)$. □

Recall that integers a, b are coprime if 1 is their unique positive common divisor, i.e. $gcd(a, b) = 1$. The last claim is elementary number theory.

Claim 8. Let a, b, c be integers. If a, b are coprime and divide c, then ab also divides c.

We can now proceed with the proof of Lemma 16. We argue by cases.

• *Case* $Val(p) \neq Val(k)$.

Let $m = \inf(Val(p), Val(k))$. Exactly one of the two integers $p\,2^{-m}$ and $k\,2^{-m}$ is odd so that $p + k = 2^m(2^{-m}p + 2^{-m}k) = 2^m(2q+1)$ for some q. Now,
- Since $m \leq Val(k)$, Claim 3 insures that 2^{m+1} divides $lcm(2k)$.
- Claim 1 insures that $p + k = 2^m(2q+1)$ divides $B\,lcm(2k-1)$. A fortiori $(2q+1)$ divides $B\,lcm(2k)$.

As 2^{m+1} and $(2q+1)$ are coprime, Claim 8 implies that $2(p+k) = 2^{m+1}(2q+1)$ divides $B\,lcm(2k)$.

• *Case* $Val(p) = Val(k) = \ell$.

Then $p + k = 2^\ell(2k'+1) + 2^\ell(2p'+1) = 2^{\ell+t}(2q+1)$ with $t \geq 1$. There are three subcases.

- *Subcase* $k \geq 2^{\ell+t}$. Then $2^{\ell+t+1} \leq 2k$, hence $2^{\ell+t+1}$ divides $lcm(2k)$. Claim 1 insures that $p + k = 2^{\ell+t}(2q+1)$ divides $B\,lcm(2k-1)$, A fortiori $2q+1$ divides $B\,lcm(2k)$. Finally, by Claim 8 we conclude that $2(p+k) = 2^{\ell+t+1}(2q+1)$ divides $B\,lcm(2k)$.

- *Subcase* k *is a power of* 2 . Apply Claim 2.

- *Subcase* $k = 2^\ell(2k'+1) < 2^{\ell+t}$ *for some* $k' \neq 0$. Claim 6 insures that 2^N divides B. Claim 7 insures that $2^{\ell+t+1-N}$ divides $lcm(2k)$. As a consequence $2^{\ell+t+1}$ divides $B\,lcm(2k)$. By Claim 1, $p+k = 2^{\ell+t}(2q+1)$ divides $B\,lcm(2k-1)$. A fortiori $(2q+1)$ divides $B\,lcm(2k)$. Finally by Claim 8, $2(p+k) = 2^{\ell+t+1}(2q+1)$ divides $B\,lcm(2k)$. □

Lemma 17. *If* $n, k, b \in \mathbb{N}$ *and* $b \geq k$ *then* n *divides* $A^n_{k,b} = lcm(k)\left(\binom{b+n}{k} - \binom{b}{k}\right)$.

Proof. We argue by double induction on k and b with the conditions

$$(\mathcal{P}_{k,b}) \quad \forall n \in \mathbb{N},\ \ n \text{ divides } A^n_{k,b} \qquad , \qquad (\mathcal{P}_k) \quad \forall b \geq k,\ \forall n \in \mathbb{N},\ \ n \text{ divides } A^n_{k,b}\ .$$

Conditions $(\mathcal{P}_0)$ and $(\mathcal{P}_1)$ are trivial since $A^n_{0,b} = 0$ and $A^n_{1,b} = n$.
Suppose $k \geq 1$ and $(\mathcal{P}_k)$ is true. To prove $(\mathcal{P}_{k+1})$, we prove by induction on $b \geq k+1$ that $(\mathcal{P}_{k+1,b})$ holds.

In the base case $b = k+1$, applying Pascal's rule, we have

$$\begin{aligned} A^n_{k+1,k+1} &= lcm(k+1)\left(\binom{k+1+n}{k+1} - \binom{k+1}{k+1}\right) = lcm(k+1)\left(\binom{k+n}{k} + \binom{k+n}{k+1} - 1\right) \\ &= lcm(k+1)\left(\binom{k+n}{k} - \binom{k}{k}\right) + lcm(k+1)\binom{k+n}{k+1} = \frac{lcm(k+1)}{lcm(k)}\ A^n_{k,k} + lcm(k+1)\binom{k+n}{k+1} \end{aligned}$$

Since $(\mathcal{P}_{k,k})$ holds (induction hypothesis on k), n divides $A^n_{k,k}$ hence n divides the first term. If $n \leq k+1$ then n divides $lcm(k+1)$ hence n also divides the second term. If $n > k+1$, applying Lemma 15 with $n' = k+n$, $p' = n$ and $k' = k+1$, we see that $n = p'$ divides the second term. Thus, in both cases n divides $A^n_{k+1,k+1}$ and $(\mathcal{P}_{k+1,k+1})$ holds.

Suppose now that $(\mathcal{P}_{k+1,c})$ holds for $k+1 \leq c \leq b$. We prove $(\mathcal{P}_{k+1,b+1})$. Using Pascal's rule, we get

$$\begin{aligned} A^n_{k+1,b+1} &= lcm(k+1)\ \left(\binom{b+1+n}{k+1} - \binom{b+1}{k+1}\right) = lcm(k+1)\ \left(\binom{b+n}{k} + \binom{b+n}{k+1} - \binom{b}{k} - \binom{b}{k+1}\right) \\ &= lcm(k+1)\ \left(\left(\binom{b+n}{k} - \binom{b}{k}\right) + \left(\binom{b+n}{k+1} - \binom{b}{k+1}\right)\right) = \left(\frac{lcm(k+1)}{lcm(k)}\ A^n_{k,b}\right) + A^n_{k+1,b} \end{aligned}$$

Since $(\mathcal{P}_{k,b})$ and $(\mathcal{P}_{k+1,b})$ hold, n divides both terms of the above sum, hence n divides $A^n_{k+1,b+1}$ and $(\mathcal{P}_{k+1,b+1})$ holds. □

Lemma 18. *Let* $B(n,k,i) = \binom{n+k-1}{2k} - \binom{i+k}{2k}$ *and* $C(n,k,i) = \binom{n+k}{2k+1} + \binom{i+k}{2k+1}$.
For all $n \geq 2$, *and* $1 \leq i \leq n-1$, *the following hold*

$$n+i \text{ divides } lcm(2k)\, B(n,k,i) \qquad \text{for } 1 \leq k \leq i \tag{5}$$
$$n+i \text{ divides } lcm(2k+1)\, C(n,k,i) \qquad \text{for } 0 \leq k \leq i \tag{6}$$

Proof. By induction on $n \geq 2$. Base case : $n = 2$ clear as $n-1 = 1 = i$. Induction : assuming that (5) and (6) hold for n, we first prove that (5) holds for $n+1$, and we then prove that (6) holds for $n+1$.

• Proof that (5) holds for $n+1$.
Let $1 \leq i \leq n$ and $1 \leq k \leq i$. Then, applying Pascal's rule,

$$\begin{aligned} B(n+1,k,i) &= \binom{n+k}{2k} - \binom{i+k}{2k} = \left[\binom{n+k-1}{2k} + \binom{n+k-1}{2k-1}\right] - \left[\binom{i+k+1}{2k} - \binom{i+k}{2k-1}\right] \\ &= \left[\binom{n+k-1}{2k} - \binom{i+1+k}{2k}\right] + \left[\binom{n+k-1}{2k-1} + \binom{(i+1)+(k-1)}{2k-1}\right] \\ B(n+1,k,i) &= B(n,k,i+1) + C(n,k-1,i+1) \end{aligned} \tag{7}$$

By the induction hypothesis, applied for n and $i+1$, provided that $i+1 \leq n-1$, i.e. $i \leq n-2$:
– (5) holds for n hence $n+i+1$ divides $lcm(2k)\, B(n,k,i)$
– (6) holds for n hence $n+i+1$ divides $lcm(2k-1)\, C(n,k-1,i+1)$
Since $lcm(2k-1)$ divides $lcm(2k)$, we see that $n+i+1$ divides $lcm(2k)\, C(n,k-1,i+1)$ for $i \leq n-2$.
Summing and using (7), we obtain that $n+i+1$ divides $lcm(2k)\, B(n+1,k,i)$ for $i \leq n-2$.
It remains to prove the same result for $i = n-1$ and $i = n$.

For $i = n$, this is clear since $B(n+1,k,n) = \left(\binom{n+1+k-1}{2k} - \binom{i+k}{2k}\right) = 0$.

For $i = n-1$, Pascal's rule yields $B(n+1,k,n-1) = \binom{n+k}{2k} - \binom{n-1+k}{2k} = \binom{n-1+k}{2k-1}$.
As $k \leq n$, $n-k \geq 0$ and we can apply Lemma 16 with $p = n-k$, hence: $2n = 2(p+k)$ divides $lcm(2k)\binom{p+2k-1}{2k-1} = lcm(2k)\binom{n-1+k}{2k-1} = lcm(2k)\, B(n+1,k,n-1)$. To conclude, observe that $n+1+i = n+1+n-1 = 2n$.

• Proof that (6) holds for $n+1$.
Assume that (5) and (6) hold for n. Let $1 \leq i \leq n$, by Pascal's rule

$$\begin{aligned} C(n+1,k,i) &= \binom{n+k+1}{2k+1} + \binom{i+k}{2k+1} = \left[\binom{n+k}{2k+1} + \binom{n+k}{2k}\right] + \left[\binom{i+1+k}{2k+1} - \binom{i+k}{2k}\right] \\ &= \left[\binom{n+k}{2k+1} + \binom{i+1+k}{2k+1}\right] + \left[\binom{n+k}{2k} - \binom{i+k}{2k}\right] \\ C(n+1,k,i) &= C(n,k,i+1) + B(n+1,k,i) \end{aligned} \tag{8}$$

We know that (5) holds for $n+1$. Thus, for $1 \leq i \leq n$ and $1 \leq k \leq i$, $n+1+i$ divides $lcm(2k)B(n+1,k,i)$ hence also $lcm(2k+1)B(n+1,k,i)$. This also trivially holds for $k = 0$ as $B(n+1,0,i) = 0$. By the induction hypothesis (6) holds for n. Thus, $n+(i+1)$ divides $lcm(2k+1)C(n,k,i+1)$ for $1 \leq i+1 \leq n-1$, i.e. $0 \leq i \leq n-2$, and $0 \leq k \leq i+1$. Summing and using (8), we obtain that $n+i+1$ divides $lcm(2k+1)\, C(n+1,k,i)$ for $1 \leq i \leq n-2$ and $0 \leq k \leq i$.
It remains to prove the same result for $i = n-1$ and $i = n$.

For $i = n-1$, we have $n+1+i = 2n$ and $C(n+1,k,n-1) = 2\binom{n+k}{2k+1}$. Lemma 15, applied with $p' = n$, $n' = n+k$ and $k' = 2k+1$, shows that n divides $lcm(2k+1)\binom{n+k}{2k+1}$, hence $2n$ divides $lcm(2k+1)C(n+1,k,n-1)$.

For $i = n$ we have $n+1+i = 2n+1$ and

$$\begin{aligned} C(n+1,k,n) &= \binom{n+k+1}{2k+1} + \binom{n+k}{2k+1} = \frac{(n+k+1)!}{(2k+1)!\,(n-k)!} + \frac{(n+k)!}{(2k+1)!\,(n-k-1)!} \\ &= \frac{(n+k)!}{(2k+1)!\,(n-k-1)!}\left(\frac{n+k+1}{n-k}+1\right) = \frac{(n+k)!}{(2k)!\,(n-k)!}\,\frac{2n+1}{2k+1} \\ lcm(2k+1)C(n+1,k,n) &= \frac{lcm(2k+1)}{2k+1} \times \binom{n+k}{2k} \times (2n+1) \end{aligned}$$

The first two factors are integers, hence $2n+1$ divides $lcm(2k+1)\,C(n+1,k,n)$. □

5 Proof of Theorem 14

5.1 Proof of implication (1) ⇒ (2) in Theorem 14

In this subsection we assume that $f : \mathbb{Z} \to \mathbb{Z}$ has integral difference ratio and that $f(x) = \sum_{k\in\mathbb{N}} a_k P_k(x)$ is its $\mathbb{Z}$-Newtonian expansion. To prove that $lcm(n)$ divides a_n we have to prove that i divides a_n for all $i \le n$. To give the flavor of the proof, we look at the first values of n. We have:

$f(0) = a_0,\ f(1) = a_0+a_1,\ f(-1) = a_0-a_1+a_2,\ f(2) = a_0+2a_1+a_2+a_3,\ f(-2) = a_0-2a_1+3a_2-a_3+a_4,$
$f(3) = a_0+3a_1+3a_2+4a_3+a_4+a_5,\quad f(-3) = a_0-3a_1+6a_2-4a_3+5a_4-a_5+a_6\,,$

Applying the integral difference ratio property, we see that

2 divides	$f(1)-f(-1) = 2a_1 - a_2$	hence 2 divides a_2
2 divides	$f(2)-f(0) = 2a_1+a_2+a_3$	hence 2 divides a_3
3 divides	$f(2)-f(-1) = 3a_1+a_3$	hence 3 divides a_3
2 divides	$f(-2)-f(0) = -2a_1+3a_2-a_3+a_4$	hence 2 divides a_4
3 divides	$f(-2)-f(1) = -3a_1+3a_2-a_3+a_4$	hence 3 divides a_4
4 divides	$f(2)-f(-1) = 4a_1-2a_2+2a_3-a_4$	hence 4 divides a_4

By induction on $n \ge 1$, we prove property $\mathcal{I}(n)$: *$lcm(2n-1)$ divides a_{2n-1} and $lcm(2n)$ divides a_{2n}*. The cases $n = 1, 2$ have just been done. The inductive step is split in four cases corresponding to Lemmas 19 to 22. Assuming $\mathcal{I}(j)$ for all $j < n$ we prove

Lemma 19 (Middle number n)	n divides a_{2n-1} and n divides a_{2n}
Lemma 20 (Below the middle number n)	If $2 \le i < n$ then i divides a_{2n-1} and a_{2n}.
Lemma 21 (Above the middle number n, case a_{2n-1})	If $1 \le i \le n-1$ then $n+i$ divides a_{2n-1}
Lemma 22 (Above the middle number n, case a_{2n})	If $1 \le i \le n$ then $n+i$ divides a_{2n}

We state equations (9) which follows from Proposition 12 (equation (2)),and will be used for proving the lemmas.

$$f(n) = \sum_{j=0}^{2n-1} a_j P_j(n) \text{ with } P_j(n) = \left\{\begin{array}{ll} P_{2k}(n) = \binom{k+n-1}{2k} & \text{for } j = 2k \\ P_{2k+1}(n) = \binom{k+n}{2k+1} & \text{for } j = 2k+1 \end{array}\right\} = \binom{n+\lfloor (j-1)/2 \rfloor}{j} \quad (9)$$

Lemma 19. *If condition $\mathcal{I}(s)$ holds for all $s < n$ then n divides a_{2n-1} and a_{2n}.*

Proof. 1. We first show that n divides $2n-1$. The case $n \leq 2$ has been done above. Suppose $n \geq 2$. By the integral difference ratio property, n divides $f(n) - f(0)$. As $P_j(0) = 0$ for all $j \geq 1$ we have $f(0) = a_0$. Also, $P_{2n-1}(n) = 1$. Thus, $f(n) - f(0) = \Big(\sum_{j=1}^{2n-2} a_j\, P_j(n)\Big) + a_{2n-1}$ where the $P_j(n)$ are given in Equation (9). As $0 \leq n + \lfloor (j-1)/2 \rfloor - j < n \leq n + \lfloor (j-1)/2 \rfloor$, Lemma 15 insures that n divides $lcm(j) P_j(n)$. Now, by the induction hypothesis $\mathcal{I}(s)$ holds for all $s < n$ and thus $lcm(j)$ divides a_j for $j = 1, \ldots, 2n-2$. Therefore n divides all the terms in the sum $\sum_{j=1}^{2n-2} a_j\, P_j(n)$. Hence n divides a_{2n-1}.

2. Similarly, using equation (3) in Proposition 12, we have $f(-n) = \sum_{j=0}^{2n} a_j\, P_j(-n)$. An analogous use of Lemma 15 and the fact that n divides $f(-n) - f(0)$ allows to conclude that n divides a_{2n}. □

Lemma 20. *If condition $\mathcal{I}(s)$ holds for all $s < n$ then i divides a_{2n-1} and a_{2n} for all $2 \leq i < n$.*

Proof. 1. Fix i such that $2 \leq i < n$. We first prove that i divides a_{2n-1}. By the integral difference ratio property, i divides $f(n) - f(n-i)$. Equation (2) yields

$$\begin{aligned} f(n) - f(n-i) &= \sum_{j=0}^{2n-1} a_j P_j(n) - \sum_{j=0}^{2n-2i-1} a_j P_j(n-i) \\ &= \left(\sum_{j=1}^{2n-2i-1} a_j \Big(P_j(n) - P_j(n-i) \Big) \right) + \left(\sum_{j=2n-2i}^{i-1} a_j P_j(n) \right) + \left(\sum_{j=i}^{2n-2} a_j P_j(n) \right) + a_{2n-1} \qquad (10) \end{aligned}$$

- *Third sum.* The induction hypothesis $\mathcal{I}(s)$, for $s < n$, imply that $lcm(j)$ divides a_j $j \leq 2s \leq 2n-2$: a fortiori, for $i \leq j \leq 2n-2$, i divides a_j: hence, i divides $a_i, a_{i+1}, \ldots, a_{2n-2}$ and also all terms in the third sum.
- *Second sum.* Let $n' = n + \lfloor (j-1)/2 \rfloor$, $k' = j$, and $p = i$. For $j \leq i-1$ we have $0 \leq n' - k' < p \leq n'$ hence Lemma 15 applies and insures that i divides $lcm(k') \binom{n'}{k'} = lcm(j) \binom{n + \lfloor (j-1)/2 \rfloor'}{j} = lcm(j) P_j(n)$ (by equation (9)). Again $\mathcal{I}(s)$, $s < n$, insure that $lcm(j)$ divides a_j for $j < 2n$, hence i a fortiori divides all the terms $a_j P_j(n)$ in the second sum.
- *First sum.* The terms are $a_{2k} \left[\binom{k+n-1}{2k} - \binom{k+n-i-1}{2k} \right]$ or $a_{2k+1} \left[\binom{k+n}{2k+1} - \binom{k+n-i}{2k+1} \right]$ with $k < n-i$. Thus the hypothesis of Lemma 17, namely $2k \leq k+n-i-1$ (resp. $2k+1 \leq k+n-i$) hold for each term $\big(P_j(n) - P_j(n-i)\big)$, $j < 2n-2i$ of the first sum, implying that i divides each term $lcm(j)\big(P_j(n) - P_j(n-i)\big)$. Since conditions $\mathcal{I}(s)$, $s < n$, insure that $lcm(j)$ divides a_j for $j \leq 2s < 2n-2$, we see that i divides all terms $a_j\big(P_j(n) - P_j(n-i)\big)$ in the first sum.

Since i divides the left member and all terms of the three sums in equation (10) it must divide a_{2n-1}.

2. The proof for a_{2n} is similar using $f(-n) - f(-n+i)$ and equation (3) of Proposition 12. □

Lemma 21. *If condition $\mathcal{I}(s)$ holds for all $s < n$ then $n+i$ divides a_{2n-1} for all $1 \leq i \leq n-1$.*

Proof. By the integral difference ratio property, $n+i$ divides $D = f(n) - f(-i) = \sum_{j=0}^{2n-1} a_j P_j(n) - \sum_{j=0}^{2i} a_j P_j(-i)$. D can be split into four sums

$$D = \left(\sum_{k=0}^{i-1} a_{2k+1} \left(P_{2k+1}(n) - P_{2k+1}(-i) \right) \right) + \left(\sum_{k=1}^{i} a_{2k} \left(P_{2k}(n) - P_{2k}(-i) \right) \right) + \left(\sum_{j=2i+1}^{2n-2} a_j P_j(n) \right) + a_{2n-1}$$

Using equations (2), (3) and (9), we rewrite D as

$$D = \left(\sum_{k=0}^{i-1} a_{2k+1}(\tbinom{n+k}{2k+1} + \tbinom{i+k}{2k+1})\right) + \left(\sum_{k=1}^{i} a_{2k}(\tbinom{n+k-1}{2k} - \tbinom{i+k}{2k})\right) + \left(\sum_{j=2i+1}^{2n-2} a_j \tbinom{n+\lfloor (j-1)/2 \rfloor}{j}\right) + a_{2n-1}$$

• *First/second sum.* Induction conditions $\mathcal{I}(s)$, for all $s < n$, insure that $lcm(2k)$ divides a_{2k} and $lcm(2k+1)$ divides a_{2k+1} for $k \leq i < n$; Moreover, as $1 \leq i \leq n-1$, Lemma 18 shows that $n+i$ divides $lcm(2k+1)\,(\binom{n+k}{2k+1} + \binom{i+k}{2k+1})$ and $lcm(2k)\,(\binom{n+k-1}{2k} - \binom{i+k}{2k})$. Hence, $n+i$ divides all terms in the first and the second sum.

• *Third sum.* Let $n' = n + \lfloor (j-1)/2 \rfloor$, $k' = j$, and $p = n+i$. Since $i \leq n-1$, for $2i+1 \geq j \leq 2n-2$ we have $0 \leq n' - k' < p \leq n'$ hence we can apply Lemma 15 which insures that $n+i$ divides $lcm(j)\,P_j(n)$; moreover, the induction conditions $\mathcal{I}(j)$ hold for all $j < n$, hence $lcm(j)$ divides a_j for all $j \leq 2n-2$.Thus, $n+i$ divides all terms in the third sum.

Since it divides the left member and all terms in the above three sums, $n+i$ must divide a_{2n-1}. ☐

Lemma 22. *If condition $\mathcal{I}(s)$ holds for all $s < n$ then $n+i$ divides a_{2n} for all $1 \leq i \leq n$.*

Proof. By the integral difference ratio property, $n+i$ divides $f(-n) - f(i)$. Equations (2) (3) yield

$$f(-n) - f(i) = \textstyle\sum_{j=0}^{2n} a_j P_j(-n) - \sum_{j=0}^{2i} a_j P_j(i)$$
$$= \left(\sum_{k=0}^{i-1} a_{2k+1}\,(P_{2k+1}(-n) - P_{2k+1}(i))\right) + \left(\sum_{k=1}^{i} a_{2k}\,(P_{2k}(-n) - P_{2k}(i))\right) + \left(\sum_{j=2i+1}^{2n-1} a_j P_j(-n)\right) + a_{2n}$$

• *First sum.* Since P_{2k+1} is odd, by equation (3) of Proposition 12 the first sum is the opposite of the first sum in the proof of Lemma 21, hence it is divided by $n+i$.

• *Second sum.* Equations (2), (3) insure that $P_{2k}(-n) - P_{2k}(i) = \binom{k+n}{2k} - \binom{k+i-1}{2k}$. In case $2 \leq i \leq n$ we let $n' = n+1$ and $i' = i-1$. We have $1 \leq i' \leq n'-1$, $P_{2k}(-n) - P_{2k}(i) = P_{2k}(n-1) - P_{2k}(i) = P_{2k}(n') - P_{2k}(i'+1)$ and we can apply Lemma 18. Exactly as in the proof of Lemma 21, we deduce that $n+i = n'+i'$ divides each term of the second sum.

Consider now the case $i=1$. The second sum reduces to one term: $a_2(P_2(-n) - P_2(1)) = a_2 n(n+1)/2$. As 2 divides a_2, we see that $n+1$ divides this term.

• *Third sum.* Let $j = 2k$ or $j = 2k+1$, and $2i+1 \leq j \leq 2n-1$, equation (3) shows that $P_{2k+1}(-n) = -\binom{k+n}{2k+1}$ and $P_{2k}(-n) = \binom{k+n}{2k}$. Let $n' = k+n$, $k' = 2k$, and $p = n+i$, as $i \leq n$ and $2k \leq 2n-1$ we have $0 \leq n' - k' < p \leq n'$, hence we can apply Lemma 15 which insures that $n+i$ divides $lcm(2k)\,P_{2k}(-n)$. Then, as $\mathcal{I}(j)$ hold for all $j < n$, $lcm(2k)$ divides a_{2k} for $2k < 2n$ and $n+i$ divides $a_{2k}P_{2k}(-n)$. The case $k'' = 2k+1$ is similar. Thus, $n+i$ divides all terms in the third sum.

Since $n+i$ divides the left member and all terms in the above three sums, it must divide a_{2n}. ☐

Lemmas 19, 20, 21, 22 together with the base cases complete the proof of Theorem 14.

5.2 Proof of implication (2) ⇒ (1) in Theorem 14

We assume that the $\mathbb{Z}$-Newton expansion $\sum_{n \in \mathbb{N}} a_k\,P_k(x)$ of $f : \mathbb{Z} \to \mathbb{Z}$ is such that $lcm(n)$ divides a_n for all n. We want to prove that f has integral difference ratio. As for given $i, j \in \mathbb{Z}$, $f(i) - f(j)$ is a sum of finitely many $a_n P_n(i) - a_n P_n(j)$, it suffices to prove that each function $x \mapsto lcm(n) P_n(x)$ has integral difference ratio. Let $j < i$, $i, j \in \mathbb{Z}$. To prove that $i - j$ divides $lcm(n)(P_n(i) - P_n(j))$, we argue by disjunction of cases on the parity of n and the signs of i, j, i.e. relative to the positions of i, j with respect to the intervals $]-\infty, -k]$, $[-k, k]$, $[k, +\infty[$ for $k = \lfloor n/2 \rfloor$. We rely on conditions 2, 3 in Proposition 12.

1. *Case $n = 2k$ and $i, j \in]-\infty, -k]$.* Then $P_{2k}(i) - P_{2k}(j) = \begin{pmatrix} k+|i| \\ 2k \end{pmatrix} - \begin{pmatrix} k+|j| \\ 2k \end{pmatrix}$ and Lemma 17

applied with $b = k + |i| \geq 2k$, $n = |j| - |i|$ insures that $|j| - |i| = i - j$ divides $lcm(2k)(P_{2k}(j) - P_{2k}(i))$.

2. *Case $n = 2k$ and $j \in]-\infty, -k]$ and $i \in]-k, k]$.* Then $P_{2k}(i) - P_{2k}(j) = -\binom{k+|j|}{2k}$. Let $n' = k + |j|$, $k' = 2k$ and $p' = i - j = i + |j|$. Then $0 \leq n' - k' < p' \leq n'$, and Lemma 15 insures that $i - j$ divides $lcm(k')\binom{n'}{k'} = lcm(2k)(P_{2k}(j) - P_{2k}(i))$.

3. *Case $n = 2k$ and $j \in]-\infty, -k]$ and $i \in]k, +\infty[$.* Then $P_{2k}(i) - P_{2k}(j) = \binom{k+i-1}{2k} - \binom{k+|j|}{2k}$.
– *subcase* $|j| \leq i - 1$ Let $n' = i$ and $i' = |j|$. As $i' \leq n' - 1$ Lemma 18 (5) applies and insures that $n' + i' = i + |j| = i - j$ divides $lcm(2k)B(n', k, i') = lcm(2k)(P_{2k}(i) - P_{2k}(j))$.
– *subcase* $|j| \geq i$ Let $n' = |j| + 1$ and $i' = i - 1$. Again by Lemma 18 (5), $n' + i' = i + |j| = i - j$ divides $lcm(2k)B(n', k, i') = lcm(2k)(P_{2k}(j) - P_{2k}(i))$.

4. *Case $n = 2k$ and $i, j \in]-k, -k]$.* Clear as $P_{2k}(i) = P_{2k}(j) = 0$.

5. *Case $n = 2k$ and $j \in]-k, -k]$ and $i \in]k, +\infty[$.* Then $P_{2k}(i) - P_{2k}(j) = \binom{k+i}{2k}$. Let $n' = k + i$, $k' = 2k$ and $p' = i - j$. We have $0 \leq n' - k' < p' \leq n'$, hence by Lemma 15, $p' = i - j$ divides $lcm(2k)\binom{k+i}{2k}$.

6. *Case $n = 2k$ and $i, j \in]k, +\infty[$.* Then $P_{2k}(i) - P_{2k}(j) = \binom{k+i-1}{2k} - \binom{k+j-1}{2k}$ with $2k \leq k + j - 1$, we can thus conclude using Lemma 17.

7. *Case $n = 2k + 1$ and $i, j \in]-\infty, -k[$.* Then $P_{2k+1}(i) - P_{2k+1}(j) = -\binom{k+|i|}{2k+1} + \binom{k+|j|}{2k+1}$: applying Lemma 17 with $b = |i|$, $n = |j| - |i|$ we conclude that $n = i - j$ divides $lcm(2k+1)\big(P_{2k+1}(i) - P_{2k+1}(j)\big)$.

8. *Case $n = 2k + 1$ and $j \in]-\infty, -k[$ and $i \in [-k, k]$.* Then $P_{2k+1}(i) - P_{2k+1}(j) = \binom{k+|j|}{2k+1}$. We conclude as in case 2. above, with Lemma 15.

9. *Case $n = 2k+1$ and $j \in]-\infty, -k[$ and $i \in]k, +\infty[$.* Then $P_{2k+1}(i) - P_{2k+1}(j) = \binom{k+i}{2k+1} + \binom{k+|j|}{2k+1}$.
– *subcase* $|j| \leq i - 1$: let $n' = i$, $i' = |j|$ and apply Lemma 18 (6).
– *subcase* $i \leq |j| - 1$: let $n' = |j|$, $i' = i$ and apply Lemma 18 (6).
– *subcase* $i = |j|$: then $P_{2k+1}(i) - P_{2k+1}(j) = 2\binom{k+i}{2k+1}$; Lemma 15, applied with $n' = k + i$, $k' = 2k + 1$ and $p' = i$ ($0 \leq n' - k' < p' \leq n'$ hold), implies that i divides $lcm(2k+1)\binom{k+i}{2k+1}$, hence $2i = i - |j|$ divides $lcm(2k+1)\big(P_{2k+1}(i) - P_{2k+1}(j)\big)$.

10. *Case $n = 2k + 1$ and $i, j \in [-k, -k]$.* Trivial since then $P_{2k+1}(i) = P_{2k+1}(j) = 0$.

11. *Case $n = 2k+1$ and $j \in [-k, k]$ and $i \in]k, +\infty[$.* Then $P_{2k+1}(i) - P_{2k+1}(j) = \binom{k+i}{2k+1}$. Let $n' = k + i$, $k' = 2k + 1$, and $p' = i - j$: as $0 \leq n' - k' = i - k - 1$, as $|j| \leq k$ and $i > k$, $i - k - 1 < p' = i - j \leq n'$, the hypothesis of Lemma 15 hold and Lemma 15 yields $i - j$ divides $lcm(2k+1)\big(P_{2k+1}(i) - P_{2k+1}(j)\big)$.

12. *Case $n = 2k + 1$ and $i, j \in]k, +\infty[$.* Similar to Case 7. □

6 Non polynomial functions having integral difference ratio

Let us mention a straightforward consequence of Theorem 14.

Corollary 23. *There are non polynomial functions $\mathbb{Z} \to \mathbb{Z}$ having integral difference ratio.*

Proof. In fact there are uncountably many such functions: let a_n be any element of $lcm(n)\mathbb{N}$. □

We now explicit some non polynomial functions having integral difference ratio. We first briefly recall such examples $\mathbb{N} \to \mathbb{Z}$ (Theorem 25) obtained in [2] and then explicit functions $\mathbb{Z} \to \mathbb{Z}$ (Theorem 28).

Lemma 24. *For all k, we have $lcm(k)$ divides $\dfrac{(2k)!}{k!}$.*

Proof. We have $lcm(2k) = \prod_{p \text{ prime}} p^{N(p)}$ with $N(p) = \sup\{i \mid p^i \leq 2k\}$. For p prime, let $M(p)$ be the largest integer divided by $p^{N(p)}$ and $\leq 2k$. Then $2M(p) > 2k$ hence $M(p) > k$. In particular, $M(p)$ hence $p^{N(p)}$ divides $(2k)!/k!$. As a product of pairwise coprime integers, $lcm(2k) = \prod_{p \text{ prime}} p^{N(p)}$ also divides $(2k)!/k!$. □

Theorem 25. *Let e be the Neper constant. The following functions $\mathbb{N} \to \mathbb{Z}$ have integral difference ratio:*

$$f : x \mapsto \begin{cases} 1 & \text{if } x = 0 \\ \lfloor e\ x! \rfloor & \text{if } x \in \mathbb{N} \setminus \{0\} \end{cases} \qquad f_h : x \mapsto \begin{cases} \lfloor \sinh(1)\ x! \rfloor & \text{if } x \text{ odd} \\ \lfloor \cosh(1)\ x! \rfloor & \text{if } x \text{ even} \end{cases}$$

Remark 26. Function $\lfloor e\ x! \rfloor$ does *not* have integral difference ratio (cf. [2]).

Proof. Recall Taylor-Lagrange formula applied to the real function $t \mapsto e^t$: for all $t \in \mathbb{R}$,

$$e^t = \left(\frac{1}{0!} + \frac{t}{1!} + \frac{t^2}{2!} + \cdots + \frac{t^{k-1}}{(k-1)!} + \frac{t^k}{k!} \right) + e^{\theta\, t} \frac{t^{k+1}}{(k+1)!} \tag{11}$$

for some $0 < \theta < 1$ depending on k and t.

Let $f : \mathbb{N} \to \mathbb{Z}$ be the function associated with the Newton series $f(x) = \sum_{n \in \mathbb{N}} n! \binom{x}{n}$. Theorem 10 insures that f has integral difference ratio. By (11) above, there exists θ, with $0 < \theta < 1$ such that

$$f(x) = \sum_{n \in \mathbb{N}} n! \binom{x}{n} = \sum_{n=0}^{x} \frac{x!}{(x-n)!} = x! \left(\frac{1}{x!} + \frac{1}{(x-1)!} + \cdots + \frac{1}{1!} + \frac{1}{0!} \right) = x! \left(e - e^{\theta} \frac{1}{(x+1)!} \right)$$

Thus, $ex! = f_a(x) + \dfrac{e^{\theta}}{(x+1)}$ For $x \in \mathbb{N}$, $x \geq 2$, we have $0 < e^{\theta}/(x+1) < e/3 < 1$ and the last equality yields $f(x) = \lfloor e\ x! \rfloor$. Also, $f(0) = 1 < 2 = \lfloor e\ 0! \rfloor$, $f_1(1) = 2 = \lfloor e\ 1! \rfloor$.

Similarly, Lemma 24 and Theorem 10 insure that $f_h(x) = \sum_{n \in \mathbb{N}} (2n)! \binom{x}{2n}$ has integral difference ratio and a similar computation yields

$$f_h(x) = \sum_{k \in \mathbb{N}} (2k)! \binom{x}{2k} = \sum_{k=0}^{\lfloor x/2 \rfloor} \frac{x!}{(x-2k)!} = \begin{cases} x! \sum_{k=0}^{\frac{x-1}{2}} \dfrac{1}{(2k+1)!} & \text{if } x \text{ odd} \\ x! \sum_{k=0}^{\frac{x}{2}} \frac{1}{(2k)!} & \text{if } x \text{ even} \end{cases}$$

Applying Taylor–Lagrange formula, we get θ_o, θ_e in $]0,1[$ such that $f_h(x) = x! \left(\sinh(1) - \dfrac{\sinh(\theta_o)}{(x+1)!} \right)$ if x is odd and $f_h(x) = x! \left(\cosh(1) - \frac{\sinh(\theta_e)}{(x+1)!} \right)$ if x is even. Whence the result as in the previous case. □

It is easy to lift the integral difference ratio property from functions $\mathbb{N} \to \mathbb{Z}$ to functions $\mathbb{Z} \to \mathbb{Z}$.

Proposition 27. *Suppose $f : \mathbb{N} \to \mathbb{Z}$ has integral difference ratio and let $g : \mathbb{Z} \to \mathbb{Z}$ be such that $g(x) = f(x^2)$. Then g has integral difference ratio. In particular, there is a function $g : \mathbb{Z} \to \mathbb{Z}$ having integral difference ratio and such that $g(x) \in \{\lfloor e\ (x^2)! \rfloor, \lfloor e\ (x^2)! \rfloor - 1\}$.*

Proof. Since $a^2 - b^2 =$ divides $f(a^2) - f(b^2) = g(a) - g(b)$ so does $a - b$. $\square$

Here is an example of a non polynomial function $\mathbb{Z} \to \mathbb{Z}$ having integral difference ratio and which is not relevant to Proposition 27.

Theorem 28. *The function defined by* $n \mapsto \sqrt{\frac{e}{\pi}} \times \frac{\Gamma(1/2)}{2 \times 4^n \times n!} \int_1^\infty e^{-t/2}(t^2-1)^n dt$ *for* $n \geq 0$ *and by* $f(n) = -f(|n|-1)$ *for* $n < 0$ *maps* $\mathbb{Z}$ *into* $\mathbb{Z}$ *and has integral difference ratio.*

Proof. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function with $\mathbb{Z}$-Newton expansion $f(x) = \sum_{k \in \mathbb{N}} \frac{2k!}{k!} P_{2k}(x)$, i.e. $a_{2k} = (2k)!/k!$ and $a_{2k+1} = 0$. It is clearly nonpolynomial and, by Theorem 14, it has integral difference ratio.

For $n \geq 0$ we have, by [7], page 2, formula 0.126, and page 917 formulas 8.432 1 & 3,

$$
\begin{aligned}
f(n) &= \sum_{k=0}^{n} \frac{2k!}{k!} \frac{(n+k)(n+k-1)\cdots(n-k+2)(n-k+1)}{(2k)!} = \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!} \\
&= \sqrt{\frac{e}{\pi}} \times K_{n+\frac{1}{2}}\left(\frac{1}{2}\right) = \sqrt{\frac{e}{\pi}} \times \frac{\Gamma(\frac{1}{2})}{2 \times 4^n \times n!} \int_1^\infty e^{-\frac{t}{2}}(t^2-1)^n dt \\
f(-n) &= \sum_{k=0}^{n} \frac{2k!}{k!} \frac{(-n+k)(-n+k-1)\cdots(-n-k+2)(-n-k+1)}{(2k)!} \\
&= \sum_{k=0}^{n} (-1)^{2k} \frac{(n+k-1)\cdots(n-k)}{k!} = \sum_{k=0}^{n} \frac{(n+k-1)!}{k!\,(n-k-1)!} = f(n-1)
\end{aligned}
$$

where $K_\nu(x) = \int_0^\infty e^{-x\cosh t} \cosh(\nu t) dt$ is associated with the Bessel function of the third kind. $\square$

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