

CHARACTERIZING CONGRUENCE PRESERVING FUNCTIONS
 $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ VIA RATIONAL POLYNOMIALS

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Abstract

We introduce a basis of rational polynomial-like functions P_0, \dots, P_{n-1} for the free module of functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$. We then characterize the subfamily of congruence preserving functions as the set of linear combinations of the functions $lcm(k) P_k$ where $lcm(k)$ is the least common multiple of $2, \dots, k$ (viewed in $\mathbb{Z}/m\mathbb{Z}$). As a consequence, when $n \geq m$, the number of such functions is independent of n .

1. Introduction

The notion of congruence preserving function on rings of residue classes was introduced in Chen [3] and studied in Bhargava [1].

Definition 1.1. *Let $m, n \geq 1$. A function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is said to be congruence preserving if for all d dividing m*

$$\forall a, b \in \{0, \dots, n-1\} \quad a \equiv b \pmod{d} \implies f(a) \equiv f(b) \pmod{d} \quad (1)$$

Remark 1.2. 1. If $n \in \{1, 2\}$ or $m = 1$ then every function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is trivially congruence preserving.

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2. Observe that since d is assumed to divide m , equivalence modulo d is a congruence on $(\mathbb{Z}/m\mathbb{Z}, +, \times)$. However, since d is not supposed to divide n , equivalence modulo d may not be a congruence on $(\mathbb{Z}/n\mathbb{Z}, +, \times)$.

Example 1.3. 1. For functions $\mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$, condition (1) reduces to the conditions $f(3) \equiv f(0) \pmod{3}$, $f(4) \equiv f(1) \pmod{3}$, $f(5) \equiv f(2) \pmod{3}$.
 2. For functions $\mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}/8\mathbb{Z}$, condition (1) reduces to $f(2) \equiv f(0) \pmod{2}$, $f(3) \equiv f(1) \pmod{2}$, $f(4) \equiv f(0) \pmod{4}$, $f(5) \equiv f(1) \pmod{4}$.

In this paper, we characterize congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ using the following ingredients. We denote by \mathbb{Z} the set of integers and by \mathbb{N} that of nonnegative integers (including zero).

Definition 1.4. *The unary lcm function $\mathbb{N} \rightarrow \mathbb{N}$ maps 0 to 1 and $k \geq 1$ to the least common multiple of $1, 2, \dots, k$.*

A natural way to associate to each map from \mathbb{N} to \mathbb{Z} a map from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ is to restrict F to $\{0, \dots, n-1\}$ and take its values modulo m .

Definition 1.5. *To each map $F : \mathbb{N} \rightarrow \mathbb{Z}$ we associate the map $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ defined by $f = \pi_m \circ F \circ \iota_n$ where $\pi_m(x) = x \pmod{m}$ and $\iota_n(z)$ is the least element of $\pi_n^{-1}(z)$ (belonging to $\{0, \dots, n-1\}$). Thus diagram (2) commutes*

$$\begin{array}{ccc}
 \mathbb{N} & \xrightarrow{F} & \mathbb{Z} \\
 \iota_n \uparrow & & \downarrow \pi_m \\
 \mathbb{Z}/n\mathbb{Z} & \xrightarrow{f} & \mathbb{Z}/m\mathbb{Z}
 \end{array} \tag{2}$$

Applying Definition 1.5 to binomial coefficients $\binom{x}{k}$ we get a basis of the $\mathbb{Z}/m\mathbb{Z}$ -module of functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$.

Proposition 1.6. *Let $P_k : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ be associated to the $\mathbb{N} \rightarrow \mathbb{N}$ binomial function $x \mapsto \binom{x}{k}$. For every function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ there is a unique sequence (a_0, \dots, a_{n-1}) of elements of $\mathbb{Z}/m\mathbb{Z}$ such that*

$$f = \sum_{k=0}^{k=n-1} a_k P_k \tag{3}$$

The family $\{P_0, \dots, P_{n-1}\}$ is thus a basis of the $\mathbb{Z}/m\mathbb{Z}$ -module of functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$.

Our main result (Theorem 1.7) can be stated as

Theorem 1.7. *A function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is congruence preserving if and only if, for each $k = 0, \dots, n-1$, in equation (3) the coefficient a_k is a multiple of the residue of $\text{lcm}(k)$ in $\mathbb{Z}/m\mathbb{Z}$.*

The paper is organized as follows.

Proposition 1.6 is proved in Section 2 where, after recalling Chen’s notion of polynomial function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ (cf. [3, 4]), we extend it to a notion of rational polynomial function.

The proof of our main result Theorem 1.7 is given in Section 3. We adapt the techniques of our paper [2], exploiting similarities between Definition 1.1 and the condition studied in [2] for functions $f : \mathbb{N} \rightarrow \mathbb{Z}$ (namely, $x-y$ divides $f(x)-f(y)$ for all $x, y \in \mathbb{N}$). As a consequence of Theorem 1.7 the number of congruence preserving functions is independent of n for $n \geq m$ and even for $n \geq gpp(m)$ (the greatest prime power dividing m). Also, every congruence preserving function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is rational polynomial for a polynomial of degree strictly less than the minimum between n and $gpp(m)$.

In Section 4 we use our main theorem to count the congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$. We thus get an expression equivalent to that obtained by Bhargava in [1] and which makes apparent the fact that, for $n \geq gpp(m)$ (hence for $n \geq m$), this number depends only on m and is independent of n .

2. Representing functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ by rational polynomials

In [3, 1], congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ are introduced and studied together with an original notion of polynomial function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$.

Definition 2.1 (Chen [3]). *A function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is polynomial if it is associated (in the sense of Definition 1.5) to a function $F : \mathbb{N} \rightarrow \mathbb{Z}$ given by a polynomial in $\mathbb{Z}[X]$.*

Polynomial functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ are obviously congruence preserving. Are all congruence preserving functions polynomial? Chen [3] observe that it is not the case for some values of n, m , for instance $n = 6, m = 8$. He also proves that a stronger identity holds for infinitely many pairs (n, m) : *every function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is polynomial if and only n is not greater than the first prime factor of m .* Using counting arguments, Bhargava [1] characterizes the pairs (n, m) such that every congruence preserving function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is polynomial.

Some polynomials in $\mathbb{Q}[X]$ (i.e. polynomials with rational coefficients) happen to map integers into integers.

Definition 2.2. *For $k \in \mathbb{N}$, let $P_k \in \mathbb{Q}[X]$ be the following polynomial:*

$$P_k(x) = \binom{x}{k} = \frac{\prod_{i=0}^{k-1} (x - i)}{k!}.$$

We will use the following examples later on:

$$P_0(x) = 1 \quad , \quad P_1(x) = x, \quad P_2(x) = x(x-1)/2 \quad , \quad P_3(x) = x(x-1)(x-2)/6 \quad , \\ P_4(x) = x(x-1)(x-2)(x-3)/24 \quad , \quad P_5(x) = x(x-1)(x-2)(x-3)(x-4)/120.$$

In [5] (1915), Pólya used the P_k to give the following very elegant and elementary characterization of polynomials in $\mathbb{Q}[X]$ mapping integers to integers.

Theorem 2.3 (Pólya). *A polynomial in $\mathbb{Q}[X]$ is integer-valued on \mathbb{Z} if and only if it can be written as a \mathbb{Z} -linear combination of the polynomials P_k .*

It turns out that the representation of functions $\mathbb{N} \rightarrow \mathbb{Z}$ as \mathbb{Z} -linear combinations of the P_k 's used in [2] also fits in the case of functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$: every such function is a $(\mathbb{Z}/m\mathbb{Z})$ -linear combination of the P_k 's.

Definition 2.4. *A function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is rat-polynomial if is associated in the sense of Definition 1.5 with some polynomial in $\mathbb{Q}[X]$. The degree of f is the smallest among the degrees of such polynomials.*

We denote by $P_k^{n,m}$ the rat-polynomial function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ associated with the polynomial P_k of Definition 2.2 in the sense of Definition 1.5. When there is no ambiguity, $P_k^{n,m}$ will be denoted simply as P_k .

Remark 2.5. In Definition 2.4, the polynomial *crucially depends* on the choice of representatives of elements of $\mathbb{Z}/n\mathbb{Z}$: e.g. for $n = m = 6$, $0 \equiv 6 \pmod{6}$ but $0 = P_2(0) \not\equiv P_2(6) = 3 \pmod{6}$. The chosen representatives for elements of $\mathbb{Z}/n\mathbb{Z}$ will always be $\{0, \dots, n-1\}$.

We now prove the representation result by the P_k 's.

Proof of Proposition 1.6. Let us start with uniqueness. We have $f(0) = a_0$ hence the first coefficient a_0 is $f(0)$. We have $f(1) = a_0 + a_1$, hence $a_1 = f(1) - f(0)$. By induction, and noting that $P_k(k) = 1$, we have $f(k) = Q(k) + a_k \cdot P_k(k) = Q(k) + a_k$, hence we are able determine a_k .

For existence, argue backwards to see that this sequence suits. □

Remark 2.6. The evaluation of $a_k P_k(x)$ in $\mathbb{Z}/m\mathbb{Z}$ has to be done as follows: for x an element of $\mathbb{Z}/n\mathbb{Z}$, we consider it as an element of $\{0, \dots, n-1\} \subseteq \mathbb{N}$ and we evaluate $P_k(x) = \frac{1}{k!} \prod_{i=0}^{k-1} (x-i)$ as an element of \mathbb{N} , then we consider the remainder modulo m , and finally we multiply the result by a_k in $\mathbb{Z}/m\mathbb{Z}$. For instance, for $n = m = 8$, $4 P_2(3) = 4 \times \frac{3 \times 2}{2} = 4 \times 3 = 4$, but we might be tempted to evaluate it as $4 P_2(3) = \frac{4 \times 3 \times 2}{2} = \frac{0}{2} = 0$, which does **not** correspond to our definition. However, dividing a_k by a factor of the denominator is allowed.

Corollary 2.7. (1) *Every function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is rat-polynomial with degree less than n .*

(2) *The family of rat-polynomial functions $(P_k)_{k=0, \dots, n-1}$ is a basis of the $(\mathbb{Z}/m\mathbb{Z})$ -module of functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$.*

Example 2.8. The function $f: \mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}/6\mathbb{Z}$ defined by

$$0 \mapsto 0 \quad 1 \mapsto 3 \quad 2 \mapsto 4 \quad 3 \mapsto 3 \quad 4 \mapsto 0 \quad 5 \mapsto 1$$

is represented by the rational polynomial $P_f(x) = 3x + 4\frac{x(x-1)}{2}$ which can be simplified into $P_f(x) = 3x - x(x-1)$ on $\mathbb{Z}/6\mathbb{Z}$.

Example 2.9. The function $f: \mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/8\mathbb{Z}$ given by Chen [3] as a non polynomial congruence preserving function, namely the function defined by $f(0) = 0$, $f(1) = 3$, $f(2) = 4$, $f(3) = 1$, $f(4) = 4$, $f(5) = 7$, is represented by the rational polynomial with coefficients $a_0 = 0$, $a_1 = 3$, $a_2 = 6$, $a_3 = 2$, $a_4 = 4$, $a_5 = 4$, i.e.

$$\begin{aligned} f(x) &= 3x + 6\frac{x(x-1)}{2} + 2\frac{x(x-1)(x-2)}{2} + 4\frac{x(x-1)(x-2)(x-3)}{8} \\ &\quad + 4\frac{x(x-1)(x-2)(x-3)(x-4)}{8} \\ &= 3x + 3x(x-1) + x(x-1)(x-2) + \frac{x(x-1)(x-2)(x-3)}{2} \\ &\quad + \frac{x(x-1)(x-2)(x-3)(x-4)}{2}. \end{aligned}$$

3. Characterizing congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$

Congruence preserving functions $f: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ can be characterized by a simple condition on the coefficients of the rat-polynomial representation of f given in Proposition 1.6.

3.1. Proof of Theorem 1.7

For proving Theorem 1.7 we will need some relations involving binomial coefficients and the unary *lcm* function; these relations are stated in the next three lemmata. The proofs are elementary but technical and can be found in our paper [2].

Lemma 3.1. *If $0 \leq n - k < p \leq n < m$ then p divides $\text{lcm}(k) \binom{n}{k}$.*

Lemma 3.2. *If $n, k, b \in \{0, 1, \dots, m - 1\}$ and $k \leq b$ then n divides $A_{k,b}^n = \text{lcm}(k) \left(\binom{b+n}{k} - \binom{b}{k} \right)$.*

The following is an immediate consequence of Lemma 3.2 (set $a = b + n$).

Lemma 3.3. *If $m > a \geq b$ then $a - b$ divides $\text{lcm}(k) \left(\binom{a}{k} - \binom{b}{k} \right)$ for all $k \leq b$.*

Besides these lemmata, we shall use a classical result in $\mathbb{Z}/m\mathbb{Z}$. For $x, y \in \mathbb{Z}$ we say x divides y in $\mathbb{Z}/m\mathbb{Z}$ if and only if the residue class of x divides the residue class of y in $\mathbb{Z}/m\mathbb{Z}$.

Lemma 3.4. *Let $a_1, \dots, a_k \geq 1$ and c be their least common multiple. If a_1, \dots, a_k all divide x in $\mathbb{Z}/m\mathbb{Z}$ then so does c .*

Proof. It suffices to consider the case $k = 2$ since the passage to any k is done via a straightforward induction. Let $c = a_1b_1 = a_2b_2$ with b_1, b_2 coprime. Let t, u be such that $x = a_1t = a_2u$ in $\mathbb{Z}/m\mathbb{Z}$. Then $x \equiv a_1t \equiv a_2u \pmod{m}$. Using Bézout identity, let $\alpha, \beta \in \mathbb{Z}$ be such that $\alpha b_1 + \beta b_2 = 1$. Then

$$c(t\alpha + u\beta) = a_1b_1t\alpha + a_2b_2u\beta \equiv^{\text{mod } m} x\alpha b_1 + x\beta b_2 = x$$

hence $c(t\alpha + u\beta) = x$, proving that c divides x in $\mathbb{Z}/m\mathbb{Z}$. □

Proof of Theorem 1.7. “Only if” part. Assume $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is congruence preserving and consider its decomposition $f(x) = \sum_{k=0}^{n-1} a_k P_k^{n,m}(x)$ given by Proposition 1.6. We show that $\text{lcm}(k)$ divides a_k in $\mathbb{Z}/m\mathbb{Z}$ for all $k < n$.

Claim 1. *For all $m > k \geq 1$, k divides a_k .*

Proof. By induction on k . Recall that $f(k) = a_i \sum_{i=0}^{n-1} \binom{k}{i} = a_i \sum_{i=0}^k \binom{k}{i}$ by noting that $\binom{k}{i} = 0$ for $i > k$.

Induction Basis: The case $k = 1$ is trivial. For $k = 2$, if 2 does not divide m then 2 is invertible in $\mathbb{Z}/m\mathbb{Z}$, hence 2 divides a_2 . Otherwise, observe that, as 2 divides $2 - 0$, and f is congruence preserving, 2 divides $f(2) - f(0) = 2a_1 + a_2$ hence 2 divides a_2 .

Induction: assuming that ℓ divides a_ℓ for every $\ell \leq k$, we prove that $k + 1$ divides a_{k+1} . Assume first that $k + 1$ divides m , then

$$\begin{aligned} f(k+1) - f(0) &= (k+1)a_1 + \left(\sum_{i=2}^k \binom{k+1}{i} a_i \right) + a_{k+1} \\ &= (k+1)a_1 + \left(\sum_{i=2}^k (k+1) \frac{a_i}{i} \binom{k}{i-1} \right) + a_{k+1}. \end{aligned} \tag{4}$$

By the induction hypothesis, $\frac{a_i}{i}$ is an integer for $i \leq k$. Since f is congruence preserving, $k + 1$ divides $f(k + 1) - f(0)$ hence $k + 1$ divides the last term a_{k+1} of the sum.

Assume now that $k + 1$ does not divide m , then $k + 1 = a \times b$ with b dividing m and a coprime with m . Hence a is invertible in $\mathbb{Z}/m\mathbb{Z}$ and, by the congruence preservation property of f , b divides $f(k + 1) - f(0)$; as b divides $k + 1$, equation (4) implies that b divides a_{k+1} , and $a \times b$ also divides a_{k+1} (by invertibility of a and Lemma 3.4). □

Claim 2. For all $1 \leq p \leq k$, p divides a_k . Thus, $\text{lcm}(k)$ divides a_k in $\mathbb{Z}/m\mathbb{Z}$.

Proof. The last assertion of Claim 2 is a direct application of Lemma 3.4 to the first assertion which we now prove. The case $p = 1$ is trivial. We prove the case $p \geq 2$ by induction on p .

- *Basic case $p = 2$:* 2 divides a_k for all $k \geq 2$. If 2 does not divide m , then 2 is invertible and divides all numbers in $\mathbb{Z}/m\mathbb{Z}$; assume that 2 divides m . We argue by induction on $k \geq 2$.

- *Basis.* Apply Claim 1: 2 divides a_2 .

- *Induction.* Assuming that 2 divides a_i for all $2 \leq i \leq k$ we prove that 2 divides a_{k+1} . Two cases can occur.

- *Subcase 1: $k+1$ is odd.* Then, k is even, 2 divides k and, by congruence preservation, 2 divides $f(k+1) - f(1)$. We have

$$f(k+1) - f(1) = ka_1 + \left(\sum_{i=2}^k a_i \binom{k+1}{i} \right) + a_{k+1},$$

2 divides the a_i for $2 \leq i \leq k$ by the induction hypothesis, 2 also divides k , hence, 2 divides a_{k+1} .

- *Subcase 2: $k+1$ is even.* Then 2 divides $f(k+1) - f(0)$. Now,

$$f(k+1) - f(0) = (k+1)a_1 + \left(\sum_{i=2}^k a_i \binom{k+1}{i} \right) + a_{k+1},$$

$k+1$ is even and 2 divides the a_i for $2 \leq i \leq k$ by the induction hypothesis, thus, 2 divides a_{k+1} .

- *Induction step: $p \geq 2$ and $p+1 < n$.* Assume that

$$\text{for all } q \leq p, q \text{ divides } a_\ell \text{ for all } \ell \text{ such that } q \leq \ell < n \tag{5}$$

and prove that $p+1$ divides a_k for all k such that $p+1 \leq k < n$. Again, we use induction on $k \geq p+1$ and we assume that k divides m in order to use congruence preservation. When k does not divide m we factorize it as $k = ab$ with b dividing m and a coprime with m and a similar proof will show that b divides a_k and k divides a_k (cf. the proof of Induction in Claim 1).

- *Basis $k = p+1$.* Follows from Claim 1: $p+1$ divides a_{p+1} .

- *Induction.* Assuming that $p+1$ divides a_i for all i such that $p+1 \leq i \leq k$, we prove that $p+1$ divides a_{k+1} . As $p+1$ divides $k+1 - (k-p)$ and f is congruence preserving, $p+1$ divides $f(k+1) - f(k-p)$ which is given by

$$f(k+1) - f(k-p) = \sum_{i=1}^{k-p} a_i \left(\binom{k+1}{i} - \binom{k-p}{i} \right) + \left(\sum_{i=k+1-p}^k a_i \binom{k+1}{i} \right) + a_{k+1}. \tag{6}$$

First look at the terms of the first sum corresponding to $1 \leq i \leq p$. The induction

hypothesis (5) on p implies that q divides a_k for all $q \leq p$ and $k \geq q$. In particular, letting $k = i$ and using Lemma 3.4, we see that $lcm(i)$ divides a_i in $\mathbb{Z}/m\mathbb{Z}$. As $(k+1) - (k-p) = p+1$, by Lemma 3.2 we have: $p+1$ divides $lcm(i) \left(\binom{k+1}{i} - \binom{k-p}{i} \right)$.

A fortiori, $p+1$ divides $a_i \left(\binom{k+1}{i} - \binom{k-p}{i} \right)$.

We now turn to the terms of the first sum corresponding to $p+1 \leq i \leq k-p$ (if there are any). Again by the induction hypothesis (on k), $p+1$ divides a_i for all $p+1 \leq i \leq k$. Thus, each term of the first sum is divisible by $p+1$.

Consider now the terms of the second sum. By the induction hypothesis (on k), $p+1$ divides a_i for all $p+1 \leq i \leq k$. It remains to look at the terms associated with the i 's such that $k+1-p \leq i \leq p$ (there are such i 's in case $k+1-p < p+1$). For such i 's we have $0 \leq (k+1) - i \leq (k+1) - p < p+1 \leq k+1$ and Lemma 3.1 (used with $k+1, i$ and $p+1$ in place of n, k and p) insures that $p+1$ divides $lcm(i) \binom{k+1}{i}$. Now, for such i 's, the induction hypothesis (5) on p shows that $lcm(i)$ divides a_i . Thus, $p+1$ divides each $a_i \binom{k+1}{i}$.

As $p+1$ divides the k first terms of the right-hand side of (6) and also divides the left-hand side, it must divide the last term a_{k+1} of the right-hand side. This ends the proof of the induction in the inductive step hence also the proof of Claim 2, and of the “only if” part of the Theorem. \square

“If” part of Theorem 1.7. Assuming all the a_k 's in equation (3) are divisible by $lcm(k)$ in $\mathbb{Z}/m\mathbb{Z}$ we prove that f is congruence preserving, i.e. that, for all $a, b \in \{0, \dots, n-1\}$, if $a - b$ divides n then $a - b$ divides $f(a) - f(b)$ in $\mathbb{Z}/m\mathbb{Z}$.

If all the a_k 's in equation (3) are divisible by $lcm(k)$ then f can be written in the form $f(n) = \sum_{k=0}^n b_k lcm(k) \binom{n}{k}$. Consequently,

$$f(a) - f(b) = \left(\sum_{k=0}^b b_k lcm(k) \left(\binom{a}{k} - \binom{b}{k} \right) \right) + \sum_{k=b+1}^a b_k lcm(k) \binom{a}{k}.$$

By Lemma 3.3, $a - b$ divides each term of the first sum.

Consider the terms of the second sum. For $b+1 \leq k \leq a$, we have $0 \leq a - k < a - b \leq a$ and Lemma 3.1 (used with a, k and $a - b$ in place of n, k and p) insures that $a - b$ divides $lcm(k) \binom{a}{k}$. Hence, $a - b$ divides each term of the second sum. \square

3.2. On a family of generators

We now sharpen the degree of the rat-polynomial representing a congruence preserving function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$. We need first some properties of the lcm function and a definition.

Lemma 3.5. In $\mathbb{Z}/m\mathbb{Z}$ we have $lcm(k) = u \times \prod p_i^{\alpha_i^k}$ with u invertible in $\mathbb{Z}/m\mathbb{Z}$, p_i prime and dividing m , and $\alpha_i^k = \max\{\beta_i | p_i^{\beta_i} \leq k\}$.

Definition 3.6. For $m \geq 1$, with prime factorization $m = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$, let $gpp(m) = \max_{i=1, \dots, \ell} p_i^{\alpha_i}$ be the greatest power of prime dividing m .

Example 3.7. In $\mathbb{Z}/m\mathbb{Z}$ the element $lcm(k)$ is zero for k large enough.

In $\mathbb{Z}/8\mathbb{Z}$	k	1	2	3	4	5	6	7	8	$gpp(8) = 8$			
	$lcm(k)$	1	2	2	4	4	4	4	0				
In $\mathbb{Z}/12\mathbb{Z}$	k	1	2	3	4	5	6	7	8	9	10	11	$gpp(12) = 4$
	$lcm(k)$	1	2	6	0	0	0	0	0	0	0	0	

Lemma 3.8. The number $gpp(m)$ is the least integer k such that $lcm(k)$ is zero in $\mathbb{Z}/m\mathbb{Z}$. Moreover for all $\ell \geq gpp(m)$, $lcm(\ell)$ is zero in $\mathbb{Z}/m\mathbb{Z}$.

Remark 3.9. (1) Either $gpp(m) = m$ or $gpp(m) \leq m/2$.
 (2) In general, $gpp(m)$ is greater than $\lambda(m)$, the least k such that m divides $k!$ considered in [3]: for $m = 8$, $gpp(m) = 8$ whilst $\lambda(m) = 4$.

Using Lemma 3.8, we can get a better version of Theorem 1.7.

Theorem 3.10. A function $f: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is congruence preserving if and only if it is associated in the sense of Definition 1.5 with a rational polynomial $P = \sum_{k=0}^{d-1} a_k \binom{x}{k}$ where $d = \min(n, gpp(m))$ and such that $lcm(k)$ divides a_k in $\mathbb{Z}/m\mathbb{Z}$ for all $k < d$.

Proof. For $k \geq gpp(m)$, m divides $lcm(k)$ hence the coefficient a_k is 0. □

Theorem 3.11. (1) Every congruence preserving function $f: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is rat-polynomial with degree less than $gpp(m)$.

(2) The family of rat-polynomial functions

$$\mathcal{F} = \{lcm(k)(P_k^{n,m}) | 0 \leq k < \min(n, gpp(m))\}$$

generates the set of congruence preserving functions.

(3) \mathcal{F} is a basis of the set of congruence preserving functions if and only if m has no prime divisor $p < \min(n, m)$ (in case $n \geq m$ this means that m is prime).

Proof. (1) and (2) are restatements of Theorem 3.10. We prove (3).

“Only IF” part. Assuming m has a prime divisor $p < \min(n, m)$, let p be the least one. Then \mathcal{F} is not linearly independent. In $\mathbb{Z}/m\mathbb{Z}$, $lcm(p) \neq 0$ hence $lcm(p) P_p^{n,m}$ is not the null function since $P_p^{n,m}(p) = 1$. However $(m/p) lcm(p) = 0$ hence $(m/p) lcm(p) P_p^{n,m}$ is the null function. As $(m/p) \neq 0$, we see that \mathcal{F} cannot be a basis.

“IF” part. Assume that m has no prime divisor $p < \min(n, m)$. We prove that \mathcal{F}

is $\mathbb{Z}/m\mathbb{Z}$ -linearly independent. Suppose that the $\mathbb{Z}/m\mathbb{Z}$ -linear combination $L = \sum_{k=0}^{\min(n, gpp(m))-1} a_k lcm(k) P_k^{n,m}$ is the null function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$. By induction on $k = 0, \dots, \min(n, gpp(m)) - 1$ we prove that $a_k = 0$.

- *Basic cases* $k = 0, 1$. Since $L(0) = a_0$ we get $a_0 = 0$. Since $L(1) = a_0 + a_1 \cdot 1$ we get $a_1 = 0$.

- *Induction step.* Assuming that $k \geq 2$ and $a_i = 0$ for $i = 0, \dots, k-1$, we prove that $a_k = 0$. Note that $P_\ell^{n,m}(k) = \binom{k}{\ell}$ for $k < \ell < n$. Since $a_i = 0$ for $i = 0, \dots, k-1$, and $P_k^{n,m}(k) = 1$ we get $L(k) = a_k lcm(k)$. Since $k < \min(n, gpp(m))$ and m has no prime divisor $p < \min(n, m)$, the numbers $lcm(k)$ and m are coprime hence $lcm(k)$ is invertible in $\mathbb{Z}/m\mathbb{Z}$ and equality $L(k) = a_k lcm(k) = 0$ implies $a_k = 0$. \square

4. Counting congruence preserving functions

We are now interested in the number of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$. As two different rational polynomials correspond to different functions by Proposition 1.6 (uniqueness of the representation by a rational polynomial), the number of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is equal to the number of polynomials representing them.

Proposition 4.1. *Let $CP(n, m)$ be the number of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$. For $m = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$, we have*

$$\begin{aligned}
 CP(n, m) &= p_1^{p_1+p_1^2+\dots+p_1^{e_1}} \times \dots \times p_\ell^{p_\ell+p_\ell^2+\dots+p_\ell^{e_\ell}} \quad \text{if } n \geq gpp(m), \text{ and} \\
 CP(n, m) &= \prod_{\{i|p_i^{e_i} < gpp(m)\}} p_i^{p_i+p_i^2+\dots+p_i^{e_i}} \times \prod_{\{i|p_i^{e_i} \geq gpp(m)\}} p_i^{p_i+p_i^2+\dots+p_i^{\lfloor \log_p n \rfloor + n(e - \lfloor \log_p n \rfloor)}} \\
 &\quad \text{if } n < gpp(m).
 \end{aligned}$$

Equivalently, using an à la Vinogradov’s notation for better readability and writing $E(p, \alpha)$ in place of p^α we have

$$\begin{aligned}
 CP(n, m) &= \prod_{i=1}^{\ell} E(p_i, \sum_{k=1}^{e_i} p_i^k) \quad \text{if } n \geq gpp(m), \text{ and} \\
 CP(n, m) &= \prod_{\{i|p_i^{e_i} < gpp(m)\}} E(p_i, \sum_{k=1}^{e_i} p_i^k) \times \prod_{\{i|p_i^{e_i} \geq gpp(m)\}} E(p_i, \sum_{k=1}^{\lfloor \log_p n \rfloor} p_i^k + n(e - \lfloor \log_p n \rfloor)) \\
 &\quad \text{if } n < gpp(m).
 \end{aligned}$$

Corollary 4.2. *For $n \geq gpp(m)$, $CP(n, m)$ does not depend on n .*

Proof of Proposition 4.1. By Theorem 3.10, we must count the number of n -tuples of coefficients (a_0, \dots, a_{n-1}) , with a_k a multiple of $lcm(k)$ in $\mathbb{Z}/m\mathbb{Z}$.

Claim 1. For $m = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$, for all n , $CP(n, m) = \prod_{i=1}^{\ell} CP(n, p_i^{e_i})$.

Proof. Let $\lambda(m, k)$ be the number of multiples of $lcm(k)$ in $\mathbb{Z}/m\mathbb{Z}$, i.e. order of the subgroup generated by $lcm(k)$ in $\mathbb{Z}/m\mathbb{Z}$.

Since $\mathbb{Z}/m\mathbb{Z}$ is isomorphic to $\prod_{i=1}^{\ell} \mathbb{Z}/p_i^{e_i}\mathbb{Z}$, we have $\lambda(m, k) = \prod_{i=1}^{\ell} \lambda(p_i^{e_i}, k)$ for each k . Thus, the number of n -tuples (a_0, \dots, a_{n-1}) such that $lcm(k)$ divides a_k is equal to

$$\prod_{k < n} \lambda(m, k) = \prod_{k < n} \prod_{i=1}^{\ell} \lambda(p_i^{e_i}, k) = \prod_{i=1}^{\ell} \prod_{k < n} \lambda(p_i^{e_i}, k).$$

The trick in the proof is the permutation of the two products; hence the Claim by using Theorem 1.7. □

Claim 1 reduces the problem to counting the congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/p_i^{e_i}\mathbb{Z}$. We will now use Proposition 3.10 for this counting.

Claim 2.

$$CP(n, p^e) = \begin{cases} p^{p+p^2+\dots+p^e} & \text{if } n \geq p^e \\ p^{p+p^2+\dots+p^l+(e-l)n} & \text{if } p^l \leq n < p^e \text{ with } l = \lfloor \log_p n \rfloor. \end{cases}$$

Proof. By Theorem 3.10, as $gpp(p^e) = p^e$, letting $\nu = \inf(n, p^e)$, $CP(n, p^e) = CP(\nu, p^e) = \prod_{k < \nu} \lambda(p^e, k)$. For $p^j \leq k < p^{j+1}$ the order $\lambda(p^e, k)$ of the subgroup generated by $lcm(k)$ in $\mathbb{Z}/p^e\mathbb{Z}$ is p^{e-j} and there are $p^{j+1} - p^j$ such k 's.

- Assume first $n \geq p^e$, then $CP(n, p^e) = CP(p^e, p^e) = p^M$ with

$$\begin{aligned} M &= ep + (e-1)(p^2 - p) + \dots + (e-j)(p^{j+1} - p^j) + \dots + p^e - p^{e-1} \\ &= ep + \sum_{j=1}^{e-1} (e-j)(p^{j+1} - p^j) = p + p^2 + \dots + p^e. \end{aligned}$$

- Assume then $p^l \leq n < p^e$, with $l = \lfloor \log_p n \rfloor$; then $CP(n, p^e) = p^M$ with

$$\begin{aligned} M &= ep + \sum_{j=1}^{l-1} (e-j)(p^{j+1} - p^j) + (e-l)(n - p^l) \\ &= p + p^2 + \dots + p^l + n(e-l). \end{aligned}$$

□

This finishes the proof of Proposition 4.1. □

Remark 4.3. In [1] the number of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/p^e\mathbb{Z}$ is shown to be equal to $p^{en - \sum_{k=1}^{n-1} \min\{e, \lfloor \log_p k \rfloor\}}$. For $p^i \leq k < p^{i+1}$, $\lfloor \log_p k \rfloor = i$, hence: for $k \leq p^e$, $\min\{e, \lfloor \log_p k \rfloor\} = \lfloor \log_p k \rfloor$ and for $k \geq p^e$, $\min\{e, \lfloor \log_p k \rfloor\} = e$.

We thus have

- if $n \geq p^e$,

$$\begin{aligned} \sum_{k=1}^{n-1} \min\{e, \lfloor \log_p k \rfloor\} &= \sum_{k=1}^{p^e-1} \lfloor \log_p k \rfloor + \sum_{k=p^e}^{n-1} e \\ &= \sum_{j=0}^{e-1} j(p^{j+1} - p^j) + e(n - p^e) \\ &= -(p + \dots + p^e) + ep^e + e(n - p^e) \end{aligned}$$

hence $en - \sum_{k=1}^{n-1} \min\{e, \lfloor \log_p k \rfloor\} = p + \dots + p^e$

and $p^{en - \sum_{k=1}^{n-1} \min\{e, \lfloor \log_p k \rfloor\}} = p^{p+p^2+\dots+p^e}$

which coincides with our counting in Claim 2.

- if $n < p^e$, and $l = \lfloor \log_p n \rfloor$, then similarly

$$\begin{aligned} \sum_{k=1}^{n-1} \lfloor \log_p k \rfloor &= \sum_{k=1}^{l-1} \lfloor \log_p k \rfloor + \sum_{k=l}^{n-1} \lfloor \log_p k \rfloor \\ &= \sum_{j=0}^{l-1} j(p^{j+1} - p^j) + l(n - p^l) = -(p + \dots + p^l) + nl \end{aligned}$$

and $en - \sum_{k=1}^{n-1} \lfloor \log_p k \rfloor = p + \dots + p^l + (e - l)n$, which again coincides with our counting in Claim 2.

5. Conclusion

We proved that the rational polynomials $lcm(k) P_k^{n,m}$ generate the $(\mathbb{Z}/m\mathbb{Z})$ -submodule of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$. When n is larger than the greatest prime power dividing m , the number of functions in this submodule is independent of n . An open problem is the existence of a basis of this submodule.

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References

[1] M. BHARGAVA, *Congruence preservation and polynomial functions from \mathbb{Z}_n to \mathbb{Z}_m* , Discrete Mathematics 173 (1997), p. 15 – 21.
 [2] P. CÉGIELSKI, S. GRIGORIEFF, I. GUESSARIAN, *Newton expansion of functions over natural integers having integral difference ratios*, Int. J. Number Theory, Vol. 11 No 7 (2015), p. 2109-2139.
 [3] Z. CHEN, *On polynomial functions from \mathbb{Z}_n to \mathbb{Z}_m* , Discrete Math. 137 (1995), p. 137–145.

- [4] Z. CHEN, *On polynomial functions from $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$ to \mathbb{Z}_m* , Discrete Math. 162 (1996), p. 67–76.
- [5] G. PÓLYA, *Über ganzwertige ganze Funktionen*, Rend. Circ. Mat. Palermo 40 (1915), p. 1–16.