On the varieties of languages associated to some varieties of finite monoids with commuting idempotents

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Abstract

Eilenberg has shown that there is a one-to-one correspondence between varieties of finite monoids and varieties of recognizable languages. In this paper, we give a description of a variety of languages close to the class of piecewise testable languages considered by I. Simon. The corresponding variety of monoids is the variety of $\mathcal{J}$-trivial monoids with commuting idempotents. This result is then generalized to the case of finite monoids with commuting idempotents whose regular $\mathcal{D}$-classes are groups from a given variety of groups.

1 Introduction

The study of recognizable (or regular) languages was initiated by Kleene in 1954, when he showed that a language that can be recognized by a system possessing only a finite memory - a finite automaton, in the current terminology - can be expressed from the letters by using three natural operations only, now known as Kleene's operations : union, concatenation and star. This famous theorem was the first of a series of fundamental results obtained by Brzozowski, Mc Naughton, I. Simon, Schtzenberger, and others. The theory of varieties, introduced by Eilenberg twelve years ago, gave a unified presentation of these apparently isolated results. It was known for a long time that one could associate a finite monoid, called the syntactic monoid, with any recognizable language. Eilenberg used this tool to show that there is a one-to-one correspondence between certain classes of recognizable languages (the varieties of languages) and certain classes of finite monoids (the varieties of finite monoids). The term "variety" is borrowed from universal algebra and is indeed very close to the classical definition, a variety of finite monoids being a class of finite monoids closed under submonoids, quotients and *finite* direct products. Eilenberg's theorem gave a new impetus to the theory of recognizable languages (and even to the theory of finite monoids) and a number of articles have been devoted to the detailed study of the correspondence between languages and monoids [2, 3, 6]. We also point out the nice connection between logic and varieties of languages developed

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by W. Thomas and the recent applications of this theory to the complexity of Boolean circuits discovered by Barrington, Straubing and Thérien.

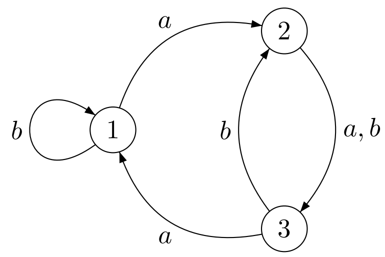
In this paper, we are especially interested by the languages corresponding to the intersection of two important varieties of finite monoids. The first variety is the variety **J** of "$\mathcal{J}$-trivial" monoids. Although the reader may be unfamiliar with the term "$\mathcal{J}$-trivial", it corresponds to a simple algebraic property (two elements generating the same ideal are equal), and **J** is generated by a very natural class of finite monoids : the *ordered* monoids in which the identity is the greatest element. The corresponding languages are the piecewise testable languages [8] and form the first level of Straubing's concatenation hierarchy [9]. The second variety is the variety of finite monoids with commuting idempotents. Again, this a very naturally defined variety of monoids, which is generated by all the monoids of partial one-to-one maps on a finite set [1]. The corresponding variety of languages was described in [4, 5].

However, the question remain open to characterize the languages corresponding to the intersection of these two varieties. Surprisingly, the previous known results of [4, 5, 8] do not seem to help much to solve this problem. In fact the solution given in this paper is based on an argument of [1] and gives a slightly more general result. Given a variety of finite groups **H**, let **V**(**H**) be the variety of finite monoids with commuting idempotents whose regular $\mathcal{J}$-classes are groups of the variety **H**. We give a description of the languages corresponding to these varieties of finite monoids.

2 Some automata

Recall that a (partial, deterministic) automaton $\mathcal{A} = (Q, A, \cdot\,)$ is defined by a set of states Q, an alphabet A and a partial action $(q, a) \to q \cdot a$ of A on Q. This action is extended to an action of A^* on Q by setting $q \cdot 1 = q$ for every $q \in Q$, and, for $u \in A^*$ and $a \in A$, $q \cdot (ua) = (q \cdot u) \cdot a$ whenever $(q \cdot u)$ and $(q \cdot u) \cdot a$ are defined. If each letter induces a permutation of Q, then $\mathcal{A}$ is called a group-automaton.

Example 2.1 Let $\mathcal{A} = (\{1, 2, 3\}, \{a, b\}, \cdot\,)$ where $1 \cdot a = 2$, $2 \cdot a = 3$, $3 \cdot a = 1$, $1 \cdot b = 1$, $2 \cdot b = 3$, $3 \cdot b = 2$. Then $\mathcal{A}$ is a group-automaton.



In this paper, we shall consider a special class of automata. An automaton $\mathcal{A} = (Q, A, \cdot\,)$ is "good" if there exists a total pre-order (i.e., reflexive and transitive relation) on Q such that, for every letter $a \in A$ and for every q, $q_1, q_2 \in Q$, the following conditions are satisfied.

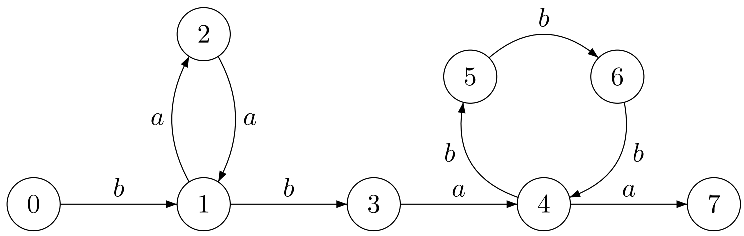
(1) If $q_1 \leqslant q_2$, and $q_1 \cdot a$ and $q_2 \cdot a$ are defined, then $q_1 \cdot a \leqslant q_2 \cdot a$.

(2) If $q \cdot a$ is defined, then $q \leqslant q \cdot a$.
(3) If $q_1 \cdot a = q_2 \cdot a$ then $q_1 = q_2$.
(4) Let $\sim$ be the equivalence associated to $\leqslant$ ($q \sim q'$ if and only if $q \leqslant q'$ and $q' \leqslant q$). If $q \cdot a \sim q$, then a induces a permutation on the $\sim$-classes of q.

Conditions (1) to (3) state that each letter induces an increasing, extensive and injective partial function on Q. Note that if C is an equivalence class for $\sim$, condition (4) states that if a letter a maps at least an element of C to some other element of C, then a induces a permutation on C. Denote by A_C the set of all letters inducing a permutation on C. Then the automaton $\mathcal{A}_C = (C, A_C, \cdot\,)$ is a group-automaton and defines a permutation group G_C.

Let $\mathbf{H}$ be a variety of finite groups. If $G_C \in \mathbf{H}$ for every equivalence class C of $\sim$, we say that $\mathcal{A}$ is "$\mathbf{H}$-good".

Example 2.2 Let $\mathcal{A}$ be the automaton represented by the following diagram.



Then $\mathcal{A}$ is good for the total pre-order $0 \leqslant 1 \sim 2 \leqslant 3 \leqslant 4 \sim 5 \sim 6 \leqslant 7$.

Conditions (1)–(4) have some interesting algebraic counterparts.

Proposition 2.1 *Let M be the transition monoid of an automaton satisfying condition (3). Then the idempotents commute in M.*

Proof. Let $\mathcal{A} = (Q, A, \cdot\,)$ be an automaton satisfying (3), and let e be an idempotent of the transition monoid M of $\mathcal{A}$. Then for every $q \in Q$ such that $q \cdot e$ is defined, we have $q \cdot e = (q \cdot e) \cdot e$ and thus $q = q \cdot e$ by (3). Therefore e is a subidentity and it follows immediately that idempotents commute in M. $\square$

Proposition 2.2 *Let M be the transition monoid of an automaton satisfying conditions (2),(3) and (4). Then the regular $\mathcal{D}$-classes of M are groups.*

Proof. We first observe that conditions (2), (3) and (4) are satisfied by every element u of M. For instance assume that $u = a_1 \ \cdots \ a_k$ and that $q \cdot u \sim q$. Then $q \leqslant q \cdot a_1 \leqslant q \cdot a_1 a_2 \leqslant \cdots \leqslant q \cdot u \sim q$ and thus $q \sim q \cdot a_1 \sim q \cdot a_1 a_2 \sim \cdots \sim q \cdot u$. It follows by (4) that the letters $a_1, \ldots, a_k$ induce a permutation on the $\sim$-class $C(q)$ of q. Therefore u itself induces a permutation on $C(q)$.

Assume now $u \leqslant_{\mathcal{R}} v$. Then $ux = v$ for some $x \in M$. Thus if $q \cdot v$ is defined, $q \cdot v = q \cdot ux$ and thus $q \cdot u$ is also defined and $q \cdot u \leqslant q \cdot ux = q \cdot v$ by (2). It follows that if $u \mathrel{\mathcal{R}} v$, then $q \cdot u$ and $q \cdot v$ are simultaneously defined and $q \cdot u \sim q \cdot v$.

Suppose now that u is idempotent. Then $q \cdot u = q$ if $q \cdot u$ is defined, and thus by (4), u induces the identity on $C(q)$. Furthermore $q \sim q \cdot v$ and by (4), v induces a permutation on $C(q)$. It follows that $uv = vu = v$ and thus $u \mathcal{H} v$. Thus the $\mathcal{R}$-class containing u is a group.

Finally Proposition 2.1 shows that the idempotents commute in M. It follows that two $\mathcal{L}$-equivalent idempotents are equal. Therefore the $\mathcal{D}$-class containing the idempotent u is a group. □

Corollary 2.3 *The transition monoid of an* $\mathbf{H}$*-good automaton belongs to* $\mathbf{V}(\mathbf{H})$.

Proof. In fact, condition (1) is not required to get the conclusion. By Propositions 2.1 and 2.2, the only thing still to be proved is that every group in the transition monoid belongs to $\mathbf{H}$. But, as shown above, every element of a group G induces a permutation on every equivalence class of $\sim$.

Let $C_1, \ldots, C_r$ be the set of $\sim$-classes and let $\pi : G_{C_1} \times \cdots \times G_{C_r}$ be the map defined by $g\pi = (g_1, \ldots, g_r)$ where g_k is the restriction of g to C_k. Then π is an injective group morphism, and since $G_{C_1}, \ldots, G_{C_r} \in \mathbf{H}$, we also have $G \in \mathbf{H}$. □

Let $\mathcal{A} = (Q, A, \cdot\,)$ be an $\mathbf{H}$-good automaton. Given two states $q_1, q_2 \in Q$, we set $S(q_1, q_2) = \{q \in Q \mid q_1 \leqslant q \leqslant q_2\}$. Then the restriction of the action of $\mathcal{A}$ to the set $S(q_1, q_2)$ defines an automaton $\mathcal{A}(q_1, q_2)$.

Proposition 2.4 *If* $\mathcal{A}$ *is* $\mathbf{H}$*-good, then* $\mathcal{A}(q_1, q_2)$ *is* $\mathbf{H}$*-good for every pair of states* (q_1, q_2).

Proof. This follows from the definition of an $\mathbf{H}$-good automaton. □

To conclude this section, we give a description of the languages recognized by $\mathbf{H}$-good automata. Let $\mathbf{H}$ be a variety of finite groups, and let $\mathcal{X}$ be the corresponding variety of languages. We call $\mathbf{H}$-elementary a language of the form

$$L_0 a_1 L_1 a_2 \ \cdots \ a_k L_k,$$

where $k \geqslant 0$, $a_1, \ldots, a_k \in A$, and, for $0 \leqslant i \leqslant k$, $L_i \in B_i^* \mathcal{X}$, where B_i is a subset of A not containing a_i or a_{i+1}.

We can now state

Proposition 2.5 *Every language recognized by an* $\mathbf{H}$*-good automaton is a finite union of* $\mathbf{H}$*-elementary languages.*

Proof. Let $\mathcal{A} = (Q, A, \cdot\,)$ be an $\mathbf{H}$-good automaton. It suffices to show that for every $q, q' \in Q$, the set $L(q, q') = \{u \in A^* \mid q \cdot u = q'\}$ is a finite union of $\mathbf{H}$-elementary languages. Set

$$A(q, q') = \{a \in A \mid q \cdot a = q'\}.$$

Then we have

$$L(q, q') = \bigcup L(q_0, q_0') a_1 L(q_1, q_1') \ \cdots \ a_k L(q_k, q_k'),$$

where the union runs over the letters such that $a_1 \in A(q'_0, q_1)$, ..., $a_k \in A(q'_{k-1}, q_k)$, and over the sequences such that $q = q_0 \sim q'_0 < q_1 \sim q'_1 < \dots < q_k \sim q'_k = q'$ where, as usual, $q < q'$ means $(q \leqslant q')$ and not $(q' \leqslant q)$.

Now since $q_i \sim q'_i$, each $L(q_i, q'_i)$ is recognized by a group-automaton $\mathcal{B}_i = (Q_i, B_i, \cdot)$ such that $a_i \notin B_i$ and $a_{i+1} \notin B_i$. Thus $L(q, q')$ is a finite union of $\mathbf{H}$-elementary languages. $\square$

3 The main result

The aim of this section is to prove the following result, which provides a description of the variety of languages $\mathcal{V}(\mathbf{H})$ corresponding to $\mathbf{V}(\mathbf{H})$.

Theorem 3.1 *For every alphabet A, $A^*\mathcal{V}(\mathbf{H})$ is the Boolean algebra generated by all $\mathbf{H}$-elementary languages.*

We first prove

Proposition 3.2 *Every $\mathbf{H}$-elementary language of A^* belongs to $A^*\mathcal{V}(\mathbf{H})$.*

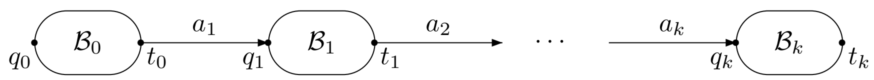
Proof. Let $L = L_0 a_1 L_1 a_2 \ \cdots \ a_k L_k$ be an $\mathbf{H}$-elementary language, where each $L_i \in B_i^* \mathcal{X}$. Then each L_i is recognized by a group automaton $\mathcal{A}_i = (Q_i, B_i, \cdot)$ with initial state q_i and set of final states $F_i \subseteq Q_i$. Set, for each $t_i \in Q_i$,

$$L_{t_i} = \{u \in A^* \mid q_i \cdot u = t_i\}.$$

Then $L_i = \bigcup_{t_i \in F_i} L_{t_i}$ and L is a finite union of languages of the form

$$K = L_{t_0} a_1 L_{t_1} a_2 \ \cdots \ a_k L_{t_k}.$$

Now the automaton $\mathcal{A}$ represented in the diagram



is an $\mathbf{H}$-good automaton that recognizes K with q_0 as initial state and t_k as (the only) final state. Thus K is recognized by the transition monoid M of $\mathcal{A}$, and by Corollary 2.3, $M \in \mathcal{V}(\mathbf{H})$. Therefore $K \in A^*\mathcal{V}(\mathbf{H})$ and hence $L \in A^*\mathcal{V}(\mathbf{H})$, since a variety of languages is closed under union. $\square$

Let $A^*\mathcal{B}(\mathbf{H})$ be the Boolean algebra generated by all $\mathbf{H}$-elementary languages. Proposition 3.2 shows that $A^*\mathcal{B}(\mathbf{H})$ is contained in $A^*\mathcal{V}(\mathbf{H})$. To prove the opposite inclusion, it suffices to show that if a language $L \subseteq A^*$ is recognized by a monoid M of $\mathbf{V}(\mathbf{H})$, then $L \in A^*\mathcal{B}(\mathbf{H})$.

Let $\eta : A^* \to M$ be a monoid morphism that saturates L (that is, $L = L\eta\eta^{-1}$). We first recall a result of Ash [1].

Proposition 3.3 *There exists an integer $N > 0$ such that every word $w \in A^*$ can be factorized as $w = u_0 v_1 u_1 \ \cdots \ v_k u_k$ where*

(a) *$v_1\eta, \dots, v_k\eta$ are regular,*

(b) *if b_{i-1} is the last letter of u_{i-1} and if a_i is the first letter of u_i, then $(b_{i-1}v_i)\eta$ and $(v_ia_i)\eta$ are not regular,*

(c) $|u_0 \cdots u_k| \leqslant N$.

Note that, since the regular $\mathcal{D}$-classes of M are groups, "regular" means "element of a group". We denote by $u\alpha$ the set of all letters occurring in a word u. Proposition 3.3 can be made more precise as follows.

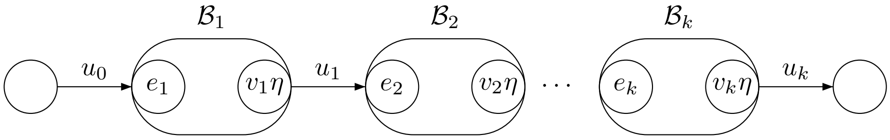
Proposition 3.4 *Let $w = u_0v_1u_1 \ \cdots \ v_ku_k$ be a factorization of w satisfying conditions (a) and (b) of Proposition 3.3. Then*

(d) *for $1 \leqslant i \leqslant k$, the last letter of u_{i-1} and the first letter of u_i do not belong to the set $v_i\alpha$.*

Proof. Let a_i be the first letter of u_i. If $a_i \in v_i\alpha$, then $v_i = v_i'a_iv_i''$. Now $v_i\eta$ is in a group H_i of M and thus $v_i\eta \ \mathcal{R} \ (v_iv_i')\eta \ \mathcal{R} \ (v_iv_i'a_i)\eta \ \mathcal{R} \ v_i^2\eta$. Therefore, $v_i\eta, (v_iv_i')\eta, (v_iv_i'a_i)\eta \in H_i$. Thus, by Green's lemma, the right translation $x \to x(a_i\eta)$ maps G onto itself. In particular $(v_ia_i)\eta \in H_i$, and this contradicts condition (b). Thus $a_i \notin v_i\alpha$. The proof for the last letter of u_{i-1} is dual. $\square$

We now associate with each factorization $w = u_0v_1u_1 \cdots v_ku_k$ satisfying conditions (a) and (b) an automaton constructed as follows. First, each v_i belongs to a group H_i of M, whose identity is an idempotent e_i. Now each letter a_i of $v_i\alpha$ acts by right multiplication on H_i (more precisely, if $h \in H_i$, then $h \cdot a = h(a\eta)$): this defines a group automaton $\mathcal{B}_i = (H_i, v_i\alpha, \cdot\,)$.

We consider also the minimal automaton of the word $u = u_0u_1 \cdots u_k$ defined as follows. The set of states is the set of left factors of u and, for each letter $a \in A$ and for each left factor x of u, $x \cdot a = xa$ if xa is a left factor of u and is undefined otherwise. We now "sew" the automata $\mathcal{B}$ and $\mathcal{B}_i$'s together, according to the following diagram.



More formally, the set of states is now the disjoint union of the H_i's and of the set S of left factors of u different from u_0, u_0u_1, $\ldots$, $u_0 \cdots u_{k-1}$. The action of a letter a is given by the following rules :

(i) if $h_i \in H_i$, then $h_i \cdot a = h_i(a\eta)$ if $a \in v_i\alpha$ (same action as in $\mathcal{B}_i$),

(ii) if $s \in S$, then $s \cdot a = sa$ if $sa \in S$ (same action as in $\mathcal{B}$),

(iii) if a is the first letter of u_i, $(v_i\eta) \cdot a = u_0 \cdot u_{i-1}a$,

(iv) if a is the last letter of u_{i-1}, and if $u_{i-1} = u_{i-1}'a$, then $(u_0 \cdots u_{i-2}u_{i-1}') \cdot a = e_i$.

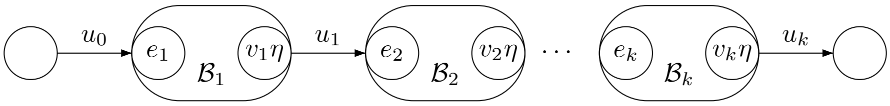
Now Proposition 3.4 shows that the automaton defined in this way is **H**-good.

Let $\mathcal{A} = (Q, A, \cdot\,)$ be an automaton. We denote by $\sim_{\mathcal{A}}$ the equivalence on A^* defined by $u \sim_{\mathcal{A}} v$ if and only if for every $q \in Q$, $q \cdot u = q \cdot v$. It is not difficult to see that $\sim_{\mathcal{A}}$ is in fact a congruence on A^*. The main step in the proof of Theorem 3.1 is the following proposition.

Proposition 3.5 *Let k be the maximal size of the groups in M. Then if $w \sim_{\mathcal{A}} w'$ for every* **H**-*good automaton on the alphabet $w\alpha \cup w'\alpha$ having at most $(|w\alpha| + |w'\alpha|)k(N+1)$ states, then $w\eta = w'\eta$.*

Proof. First assume that $w\eta$ and $w'\eta$ are regular. Then there exist two groups H and H' with identity e and e', respectively, such that $w\eta \in H$ and $w'\eta \in H'$. Since $|H| \leqslant k$ and $|H'| \leqslant k$, w and w' have the same action on the two group automata $(H, w\alpha)$ and $(H', w'\alpha)$. In particular $e(w\eta) = e(w'\eta) = w\eta$ and $e'(w'\eta) = e'(w\eta) = w'\eta$. Thus $e \mathrel{\mathcal{J}} w\eta \mathrel{\mathcal{J}} w'\eta \mathrel{\mathcal{J}} e'$ and hence $e = e'$ since regular $\mathcal{J}$-classes are groups. Thus $w\eta = w'\eta$ as required.

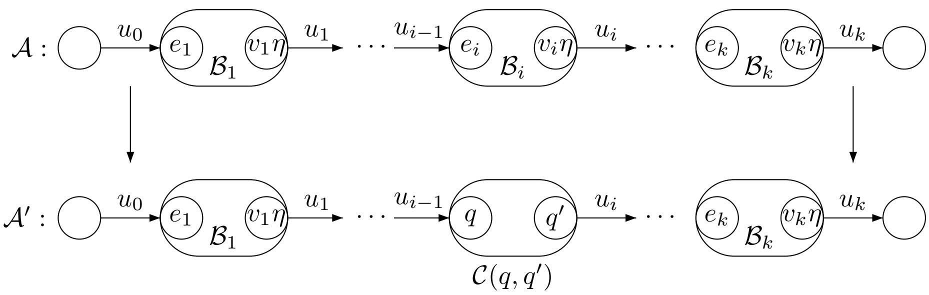
We now prove the proposition by induction on $|w\alpha| + |w'\alpha|$. If $|w\alpha| + |w'\alpha| = 0$ then $w = w' = 1$ and the result is trivial. Suppose now $|w\alpha| + |w'\alpha| > 0$. We may also assume that one of $w\eta$ or $w'\eta$, say $w\eta$, is not regular. In particular $w \neq 1$ and $|w\alpha| \geqslant 1$. Let $w = u_0 v_1 u_1 \cdots v_k u_k$ be a factorization of w given by Proposition 3.3. Since $w\eta$ is not regular, Proposition 3.4 shows that, for $1 \leqslant i \leqslant k$, $v_i\alpha$ is strictly included in $w\alpha$. Let $\mathcal{A}$ be the automaton associated with this factorization and represented in the diagram



The number of states of $\mathcal{A}$ is bounded by $k(N+1)$. Thus $w \sim_{\mathcal{A}} w'$. Therefore w' admits a factorization of the form $w' = u_0 v'_1 u_1 \cdots v'_k u_k$.

Assume $w\eta \neq w'\eta$. Then there exists an index i such that $v_i\eta \neq v'_i\eta$. Since $|v_i\alpha| + |v'_i\alpha| < |w\alpha| + |w'\alpha|$ the induction hypothesis may be applied: there exists an **H**-good automaton $\mathcal{C}$ on the alphabet $v_i\alpha \cup v'_i\alpha$, having at most $(|v_i\alpha| + |v'_i\alpha|)k(N+1)$ states such that $v_i \not\sim_{\mathcal{C}} v'_i$. Let $\mathcal{C} = (Q, v_i\alpha \cup v'_i\alpha, \cdot)$ and let $q \in Q$ be such that $q \cdot v_i \neq q \cdot v'_i$. We may assume that $q \cdot v_i = q'$ is defined (the case when $q \cdot v'_i$ is defined is dual). By Proposition 2.4, the automaton $\mathcal{C}(q, q')$ is also **H**-good.

We now proceed to the "surgical operation" on $\mathcal{A}$ consisting of replacing the subautomaton $\mathcal{B}_i$ by $\mathcal{C}(q, q')$:



The new automaton $\mathcal{A}'$ is still **H**-good, and contains at most $(|v_i\alpha| + |v'\alpha|)k(N+1) \leqslant (|w_i\alpha| + |w'\alpha|)k(N+1)$ states. On the other hand, $w \not\sim_{\mathcal{A}'} w'$, a contradiction. Therefore $w\eta = w'\eta$ and this concludes the proof of Proposition 3.5. $\square$

Let $\sim$ be the congruence on A^* defined by $u \sim v$ if and only if $u \sim_{\mathcal{A}} v$ for every **H**-good automaton having at most $2|A|k(N+1)$ states. Proposition 3.5 immediately implies

Proposition 3.6 *If $w \sim w'$, then $w\eta = w'\eta$.*

We can now conclude the proof of Theorem 3.1. By Proposition 3.6, every language L recognized by η is a finite union of $\sim$-classes. Now every $\sim$-class is the intersection of some $\sim_{\mathcal{A}}$-classes where $\mathcal{A} = (Q, A, \cdot\,)$ is an **H**-good automaton. But $u \sim_{\mathcal{A}} v$ if and only if for every $q \in Q$, $q \cdot u = q \cdot v$. Set, for $q, q' \in Q$,

$$L(q, q') = \{u \in A^* \mid q \cdot u = q'\}.$$

Then we have, for every $u \in A^*$,

$$\{v \in A^* \mid v \sim_{\mathcal{A}} u\} = \bigcap_{u \in L(q,q')} L(q, q') \setminus \bigcup_{u \notin L(q,q')} L(q, q')$$

and by Proposition 2.5, every $L(q, q')$ is a finite union of **H**-elementary languages. Therefore each class is a Boolean combination of **H**-elementary languages and so is L. $\square$

4 The $\mathcal{J}$-trivial case

In this section we consider the variety **V** of finite $\mathcal{J}$-trivial monoids with commuting idempotents. This corresponds to the variety **V**(**H**) when **H** is the trivial variety of groups. Thus Theorem 3.1 can be restated as follows.

Theorem 4.1 *Let $L \subseteq A^*$ be a recognizable language and let M be its syntactic monoid. The following conditions are equivalent:*

(1) *M is $\mathcal{J}$-trivial with commuting idempotents,*

(2) *L is a Boolean combination of languages of the form $A_0^* a_1 A_1^* a_2 \cdots a_k A_k^*$ where $k \geqslant 0$, $a_1, \ldots, a_k \in A$, $A_0, \ldots, A_k \subseteq A$ and, for $1 \leqslant i \leqslant k$, $a_i \notin A_{i-1} \cup A_i$.*

The variety **V** also plays a role in the study of power monoids. Recall that the power monoid $\mathcal{P}(M)$ of a monoid M is the set of all subsets of M with multiplication defined, for all $X, Y \subseteq M$, by

$$XY = \{xy \mid x \in X, y \in Y\}.$$

Given a variety of monoids **W**, we denote by **PW** the variety of finite monoids generated by all monoids $\mathcal{P}(M)$ where $M \in \mathbf{W}$.

Denote by **J**, **R**, and $\mathbf{R^r}$ the varieties of $\mathcal{J}$-trivial, $\mathcal{R}$-trivial and $\mathcal{L}$-trivial monoids, respectively, and by **DA** the variety of monoids whose regular $\mathcal{D}$-classes are Aperiodic semigroups (in fact, rectangular bands!).

It was proved in [7] that $\mathbf{PJ} = \mathbf{PR} = \mathbf{PR^r} = \mathbf{PDA}$. We slightly improve this result by showing

Theorem 4.2 *For any variety* **W** *such that* $\mathbf{V} \subseteq \mathbf{W} \subseteq \mathbf{DA}$, $\mathbf{PW} = \mathbf{PJ}$. *In particular,* $\mathbf{PV} = \mathbf{PJ} = \mathbf{PR} = \mathbf{PR^r} = \mathbf{PDA}$.

Proof. Since $\mathbf{V} \subseteq \mathbf{W} = \mathbf{DA}$, we have $\mathbf{PV} \subseteq \mathbf{PW} = \mathbf{PDA}$. Thus it suffices to show that $\mathbf{PDA}$ is contained in $\mathbf{PV}$. We denote by $\mathcal{V}$, $\mathcal{V}_1$ and $\mathcal{V}_2$ the varieties of languages corresponding to $\mathbf{V}$, $\mathbf{PV}$ and $\mathbf{PDA}$, respectively. By Eilenberg's theorem, it suffices to show that $\mathcal{V}_2$ is contained in $\mathcal{V}_1$. Let A be an alphabet. It was shown in [7] that $A^*\mathcal{V}_2$ is the Boolean algebra generated by the languages of the form $K = A_0^* a_1 A_1^* a_2 \cdots a_k A_k^*$, where $k \geqslant 0$, $a_1, \ldots, a_k \in A$ and $A_0, \ldots, A_k \subseteq A$.

Since $A^*\mathcal{V}_1$ is also a Boolean algebra, it suffices to show that K belongs to $A^*\mathcal{V}_2$. Let B be the disjoint union of $A_0, \ldots, A_k$ and $\{a_1\}, \ldots, \{a_k\}$. Thus

$$B = \bigcup_{1 \leqslant i \leqslant k} B_i \cup \{b_1, \ldots, b_k\}$$

where each B_i is a copy of A_i. There is a natural map φ from B to A which maps each B_i onto A_i and each b_i onto a_i. Further φ extends to a length-preserving morphism $\varphi : B^* \to A^*$. Now let

$$L = B_0^* a_1 B_1^* a_2 \cdots a_k B_k^*.$$

Since $b_{i-1} \notin B_{i-1}$ for $1 \leqslant i \leqslant k$, $L \in B^*\mathcal{V}$ by Theorem 3.1. Now it is known that if $\varphi : B^* \to A^*$ is a length-preserving morphism and if $L \subseteq B^*$ is recognized by a monoid M, then $L\varphi$ is recognized by $\mathcal{P}(M)$. Thus $K = L\varphi \in A^*\mathcal{V}_1$ as required. $\square$

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