

The dot-depth hierarchy, 45 years later

Dedicated to Janusz A. Brzozowski for his 80th birthday

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Abstract

In 1970, R. S. Cohen and Janusz A. Brzozowski introduced a hierarchy of star-free languages called the dot-depth hierarchy. This hierarchy and its generalisations, together with the problems attached to them, had a long-lasting influence on the development of automata theory. This survey article reports on the numerous results and conjectures attached to this hierarchy.

This paper is a follow-up of the survey article *Open problems about regular languages, 35 years later* [57]. The *dot-depth hierarchy*, also known as *Brzozowski hierarchy*, is a hierarchy of star-free languages first introduced by Cohen and Brzozowski [25] in 1971. It immediately gave rise to many interesting questions and an account of the early results can be found in Brzozowski's survey [20] from 1976.

1 Terminology, notation and background

Most of the terminology used in this paper was introduced in [57]. We just complete these definitions by giving the ordered versions of the notions of syntactic monoid and variety of finite monoids.

1.1 Syntactic order and positive varieties

An *ordered monoid* is a monoid equipped with an order \leq compatible with the multiplication: $x \leq y$ implies $zx \leq zy$ and $xz \leq yz$.

The *syntactic preorder*¹ of a language L of A^* is the relation \leq_L defined on A^* by $u \leq_L v$ if and only if, for every $x, y \in A^*$,

$$xuy \in L \Rightarrow xvy \in L.$$

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¹Unfortunately, the author used the opposite order in earlier papers (from 1995 to 2011).

The *syntactic congruence* of L is the associated equivalence relation \sim_L , defined by $u \sim_L v$ if and only if $u \leq_L v$ and $v \leq_L u$.

The *syntactic monoid* of L is the quotient $M(L)$ of A^* by \sim_L and the natural morphism $\eta : A^* \rightarrow A^*/\sim_L$ is called the *syntactic morphism* of L . The syntactic preorder \leq_L induces an order on the quotient monoid $M(L)$. The resulting ordered monoid is called the *syntactic ordered monoid* of L .

For instance, the syntactic monoid of the language $\{a, aba\}$ is the monoid $M = \{1, a, b, ab, ba, aba, 0\}$ presented by the relations $a^2 = b^2 = bab = 0$. Its syntactic order is given by the relations $0 < ab < 1$, $0 < ba < 1$, $0 < aba < a$, $0 < b$.

The syntactic ordered monoid of a language was first introduced by Schützenberger [86] in 1956, but thereafter, he apparently only used the syntactic monoid.

A *positive variety of languages* is a class of languages closed under finite unions, finite intersections, left and right quotients and inverses of morphisms. A *variety of languages* is a positive variety closed under complementation.

Similarly, a *variety of finite ordered monoids* is a class of finite ordered monoids closed under taking ordered submonoids, quotients and finite products. Varieties of finite (ordered) *semigroups* are defined analogously. If \mathbf{V} is a variety of ordered monoids, let \mathbf{V}^d denote the dual variety, consisting of all ordered monoids (M, \leq) such that $(M, \geq) \in \mathbf{V}$. We refer the reader to the books [2, 28, 62] for more details.

Eilenberg's variety theorem [28] admits the following ordered version [63]. Let \mathbf{V} be a variety of finite ordered monoids. For each alphabet A , let $\mathcal{V}(A)$ be the set of all languages of A^* whose syntactic ordered monoid is in \mathbf{V} . Then \mathcal{V} is a positive variety of languages. Furthermore, the correspondence $\mathbf{V} \rightarrow \mathcal{V}$ is a bijection between varieties of finite ordered monoids and positive varieties of languages.

By Reiterman's theorem [83], varieties of finite monoids can be defined by a set of profinite identities of the form $u = v$, where u and v are profinite words. Similarly, varieties of finite ordered monoids can be defined by a set of profinite identities of the form $u \leq v$ (see [73]).

1.2 li -varieties versus $+$ -varieties

Let us first recall that a monoid morphism $\varphi : A^* \rightarrow B^*$ is *length-increasing* if for all $u \in A^*$, $|\varphi(u)| \geq |u|$ or equivalently, if $\varphi(A) \subseteq B^+$. A class of languages closed under finite unions, finite intersections, left and right quotients and inverses of length-increasing morphisms is a *positive li -variety of languages*. A positive li -variety of languages closed under complementation is a *li -variety of languages*.

In fact, li -varieties are almost the same thing as $+$ -varieties, a notion due to Eilenberg [28]. A *$+$ -class of languages* \mathcal{C} associates with each finite alphabet A a set $\mathcal{C}(A)$ of regular languages of A^+ , that is, not containing the empty word. A *positive $+$ -variety of languages* is a $+$ -class of languages closed under finite unions, finite intersections, left and right quotients and inverses of semigroup morphisms. A *$+$ -variety of languages* is a positive $+$ -variety closed under complementation.

The precise correspondence between li -varieties and $+$ -varieties is discussed in [105] and [69], pp. 260–261, but we will only need the following result. Let us say that a (positive) li -variety of languages \mathcal{V} is *well suited* if, for each alphabet

A , $\mathcal{V}(A)$ contains the languages $\{1\}$ and A^+ . If \mathcal{V} is a (positive) well-suited li -variety, then the languages of the form $L \cap A^+$, where $L \in \mathcal{V}(A)$, form a (positive) $+$ -variety \mathcal{V}^+ . If \mathcal{W} is a (positive) $+$ -variety of languages, then the languages of the form L or $L \cup \{1\}$, where L is in \mathcal{W} , form a (positive) well-suited li -variety \mathcal{W}' . Moreover the correspondences $\mathcal{V} \rightarrow \mathcal{V}^+$ and $\mathcal{W} \rightarrow \mathcal{W}'$ are inverse bijective correspondences between well-suited li -varieties and $+$ -varieties.

The reader may wonder why two such closely related notions are needed. On the one hand, the notion of li -variety fits perfectly with the more general theory developed in [105] and is also more flexible. For instance, the notion of polynomial closure defined in Section 3 is easier to define (see [69], pp. 260–261 for a discussion). On the other hand, Eilenberg’s variety theorem can be extended to both $+$ -varieties and li -varieties, but it is easier to state for $+$ -varieties: there is a bijective correspondence between $+$ -varieties and varieties of finite semigroups. In other words, languages of a $+$ -variety can be characterized by a property of their syntactic semigroup. By comparison, li -varieties require the use of the syntactic morphism instead of the syntactic semigroup [105]. But since all li -varieties considered in this paper are well-suited, they are also in bijection with varieties of finite semigroups.

2 The dot-depth hierarchy

Let us first come back to the original definition from [25]. Given an alphabet A , the languages \emptyset , $\{1\}$ and $\{a\}$, where $a \in A$, are called *basic languages*. Let \mathcal{E} be the class of basic languages.

Given a class \mathcal{C} of languages, let \mathcal{BC} be its Boolean closure and let \mathcal{MC} be its monoid closure, that is, the smallest class of languages containing \mathcal{C} and the language $\{1\}$ and closed under concatenation product. Star-free languages can be constructed by alternately applying the operators \mathcal{B} and \mathcal{M} to the class \mathcal{E} . This leads to a hierarchy of star-free languages, called the dot-depth hierarchy. The question arises to know whether one should start with the operator \mathcal{B} or \mathcal{M} , but the equality $\mathcal{B.M.B.E} = \mathcal{B.M.B.M.E}$ shows that it just makes a difference for the lower levels.

In his 1976 survey, Brzozowski suggested to start the hierarchy at $\mathcal{B}_0 = \mathcal{B.M.E}$, the class of finite or cofinite² languages. The *dot-depth hierarchy* is the sequence obtained from \mathcal{B}_0 by setting $\mathcal{B}_{n+1} = \mathcal{B.M.B}_n$ for all $n \geq 0$.

It is interesting to quote Brzozowski’s original motivations as reported in [20].

The following motivation led to these concepts. Feedback-free networks of gates, i.e., combinational circuits, constitute the simplest and degenerate forms of sequential circuits. Combinational networks are, of course, characterized by Boolean functions. This suggested that (a) all Boolean operations should be considered together when studying the formation of aperiodic languages from the letters of the alphabet, and (b) since concatenation (or “dot” operator) is linked to the sequential rather than the combinational nature of a language, the number of concatenation levels required to express a given aperiodic language should provide a useful measure of complexity.

²A language is cofinite if its complement is finite.

The term *aperiodic languages* refers to the characterization of star-free languages obtained by Schützenberger [87] in 1965.

Theorem 2.1 *A language is star-free if and only if its syntactic monoid is aperiodic.*

3 Concatenation hierarchies

Further developments lead to a slight change in the definition, motivated by the connection with finite model theory presented in Section 4 and by the algebraic approach discussed in Section 5. The main change consisted in replacing products by marked products. A language L of A^* is a *marked product* of the languages L_0, L_1, \dots, L_n if

$$L = L_0 a_1 L_1 \cdots a_n L_n$$

for some letters a_1, \dots, a_n of A .

Given a set \mathcal{L} of languages, the *polynomial closure* of \mathcal{L} is the set of languages that are finite unions of marked products of languages of \mathcal{L} . The polynomial closure of \mathcal{L} is denoted $\text{Pol } \mathcal{L}$ and the Boolean closure of $\text{Pol } \mathcal{L}$ is denoted $\mathcal{B}\text{Pol } \mathcal{L}$. Finally, let $\text{co-Pol } \mathcal{L}$ denote the set of complements of languages in $\text{Pol } \mathcal{L}$.

Concatenation hierarchies are now defined by alternating Boolean operations and polynomial operations. For historical reasons, they are indexed by half-integers. More precisely, the *concatenation hierarchy* based on \mathcal{L} is the sequence defined inductively as follows: $\mathcal{L}_0 = \mathcal{L}$ and, for each $n \geq 0$,

- (1) $\mathcal{L}_{n+1/2} = \text{Pol } \mathcal{L}_n$ is the polynomial closure of the level n ,
- (2) $\mathcal{L}_{n+1} = \mathcal{B}\mathcal{L}_{n+1/2} = \mathcal{B}\text{Pol } \mathcal{L}_n$ is the Boolean closure of the level $n + 1/2$.

The classes of the form \mathcal{L}_n are called the *full levels* and the classes of the form $\mathcal{L}_{n+1/2}$ are called the *half levels* of the hierarchy.

The dot-depth hierarchy corresponds to the full levels of the concatenation hierarchy based on the class \mathcal{B}_0 of finite or cofinite languages. It should be noted that, apart for level 0, this hierarchy coincides with the concatenation hierarchy starting with the class of languages \mathcal{L}_0 defined by $\mathcal{L}_0(A) = \{\emptyset, \{1\}, A^+, A^*\}$.

Another natural concatenation hierarchy is the *Straubing-Thérien hierarchy*, based on the class of languages \mathcal{V}_0 defined by $\mathcal{V}_0(A) = \{\emptyset, A^*\}$. Other initial classes of languages have been considered in the literature, but we will stick here to these two examples.

It is not clear at first sight whether these hierarchies do not collapse, but this question was solved in 1978 by Brzozowski and Knast [21]. Thomas [114, 115] gave a different proof based on game theory.

Theorem 3.1 *The dot-depth hierarchy is infinite.*

Let D_n be the sequence of languages of $\{a, b\}^*$ defined by $D_0 = \{1\}$ and $D_{n+1} = (aD_n b)^*$. Then one can show that $D_0 \in \mathcal{B}_0$ and for all $n > 0$, $D_n \in \mathcal{B}_n - \mathcal{B}_{n-1}$. The Straubing-Thérien hierarchy is also infinite and the following diagram, in which all inclusions are proper, summarizes the relations between the two hierarchies.

$$\begin{array}{cccccccc}
\mathcal{V}_0 & \subset & \mathcal{V}_{1/2} & \subset & \mathcal{V}_1 & \subset & \mathcal{V}_{3/2} & \subset & \mathcal{V}_2 & \subset & \mathcal{V}_{5/2} & \subset & \dots \\
\cap & & \subset & & \cap & & \subset & & \cap & & \subset & & \\
\mathcal{B}_0 & \subset & \mathcal{B}_{1/2} & \subset & \mathcal{B}_1 & \subset & \mathcal{B}_{3/2} & \subset & \mathcal{B}_2 & \subset & \dots & &
\end{array}
\begin{array}{l}
\text{Star-free} \\
\text{languages}
\end{array}$$

The *dot-depth problem* asks whether the dot-depth hierarchy is decidable.

Problem 1 *Given a half integer n and a regular language L , decide whether L belongs \mathcal{B}_n .*

The corresponding problem for the hierarchy \mathcal{V}_n is also open and the two problems are intimately connected. A particularly appealing aspect of this problem is its close connection with finite model theory.

4 Connection with finite model theory

Let us associate to each word $u = a_0a_1 \dots a_{n-1}$ over the alphabet A a relational structure

$$\mathfrak{M}_u = \{\{0, 1, \dots, n-1\}, <, (\mathbf{a})_{a \in A}\},$$

where $<$ is the usual order on the domain and \mathbf{a} is a predicate giving the positions i such that $a_i = a$. For instance, if $u = abaab$, then $\mathbf{a} = \{0, 2, 3\}$ and $\mathbf{b} = \{1, 4\}$. Given a sentence φ , the language defined by φ is

$$L(\varphi) = \{u \in A^+ \mid \mathfrak{M}_u \text{ satisfies } \varphi\}.$$

The structure associated to the empty word has an empty domain, which leads to potential problems in logic, since some inference rules are not sound when empty structures are allowed. There are two possible solutions to this problem. The first one consists in ignoring the empty word. In this case, one makes the convention that a language L of A^* is defined by φ if $L(\varphi) = L \cap A^+$. The second possibility is to adopt the convention that sentences beginning with a universal quantifier are true and sentences beginning with an existential quantifier are false in the empty model.

For the study of the dot-depth hierarchy, one needs to slightly expand the signature by adding three relational symbols *min*, *max* and *S*, interpreted respectively as the minimal element (0 in our example), the maximal element (4 in our example) and the successor relation *S*, defined by $S(x, y)$ if and only if $y = x + 1$.

First order formulas are now built in the usual way by using these symbols, the equality symbol, (first-order) variables, Boolean connectives and quantifiers. For instance, the sentence

$$\exists x \exists y ((x < y) \wedge (\mathbf{a}x) \wedge (\mathbf{b}y)),$$

intuitively interpreted as *there exist two positions $x < y$ in the word such that the letter in position x is an a and the letter in position y is a b* , defines the language $A^*aA^*bA^*$.

McNaughton and Papert [55] showed that a language is first-order definable if and only if it is star-free. Thomas [113] (see also [56]) refined this result

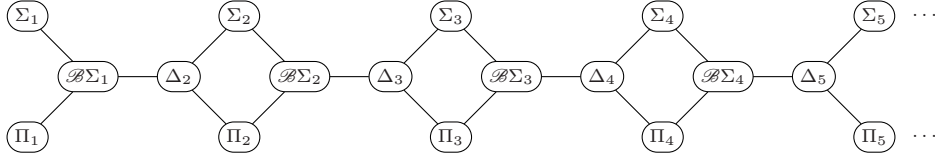
by showing that the dot-depth hierarchy corresponds, level by level, to the quantifier alternation hierarchy of first-order formulas, defined as follows.

A formula is said to be a Σ_n -formula if it is equivalent to a formula of the form $Q(x_1, \dots, x_k)\varphi$ where φ is quantifier free and $Q(x_1, \dots, x_k)$ is a sequence of n blocks of quantifiers such that the first block contains only existential quantifiers (note that this first block may be empty), the second block universal quantifiers, etc. For instance, $\exists x_1 \exists x_2 \forall x_3 \forall x_4 \forall x_5 \exists x_6 \varphi$, where φ is quantifier free, is in Σ_3 . Similarly, if $Q(x_1, \dots, x_k)$ is formed of n alternating blocks of quantifiers beginning with a block of universal quantifiers (which again might be empty), we say that φ is a Π_n -formula.

Denote by Σ_n (resp., Π_n) the class of languages which can be defined by a Σ_n -formula (resp., a Π_n -formula) and by $\mathcal{B}\Sigma_n$ the Boolean closure Σ_n -formulas. Finally, set, for every $n \geq 0$, $\Delta_n = \Sigma_n \cap \Pi_n$. If needed, we use the notation $\Sigma_n[<]$, $\Sigma_n[<, S]$ or $\Sigma_n[<, S, \min, \max]$, depending on the signature. Note that the distinction between the signatures $\{<, S\}$ and $\{<, S, \min, \max\}$ is only useful for the levels Σ_1 , Π_1 and $\mathcal{B}\Sigma_1$. Indeed, for $n \geq 2$, the following equalities hold:

$$\begin{aligned} \Sigma_n[<, S, \min, \max] &= \Sigma_n[<, S], & \Pi_n[<, S, \min, \max] &= \Pi_n[<, S], \\ \Delta_n[<, S, \min, \max] &= \Delta_n[<, S], & \mathcal{B}\Sigma_n[<, S, \min, \max] &= \mathcal{B}\Sigma_n[<, S]. \end{aligned}$$

The resulting hierarchy is depicted in the following diagram:



The next theorem summarizes the results of [55, 113, 56].

Theorem 4.1

- (1) A language is first-order definable if and only if it is star-free.
- (2) A language is in $\Pi_n[<]$ if and only if its complement is in $\Sigma_n[<]$.
- (3) A language is in $\Sigma_n[<]$ if and only if it is in $\mathcal{V}_{n-1/2}$.
- (4) A language is in $\Sigma_n[<, S, \min, \max]$ if and only if it is in $\mathcal{B}_{n-1/2}$.
- (5) A language is in $\mathcal{B}\Sigma_n[<]$ if and only if it is in \mathcal{V}_n .
- (6) A language is in $\mathcal{B}\Sigma_n[<, S, \min, \max]$ if and only if it is in \mathcal{B}_n .

In particular, deciding whether a language has dot-depth n is equivalent to a very natural problem in finite model theory.

The classes Δ_n also have a natural description in terms of unambiguous products. A marked product $L = L_0 a_1 L_1 \cdots a_n L_n$ of n languages L_0, L_1, \dots, L_n is *unambiguous* if every word u of L admits a unique factorization of the form $u_0 a_1 u_1 \cdots a_n u_n$ with $u_0 \in L_0, u_1 \in L_1, \dots, u_n \in L_n$.

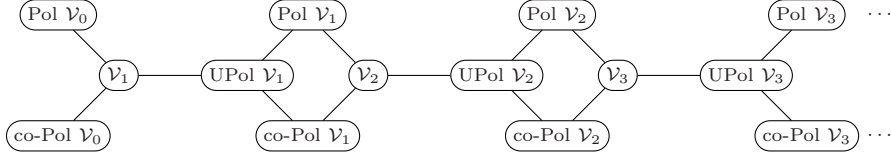
The *unambiguous polynomial closure* $\text{UPol } \mathcal{L}$ of a class of languages \mathcal{L} is the class of languages that are finite disjoint unions of unambiguous products of the form $L_0 a_1 L_1 \cdots a_n L_n$, where the a_i 's are letters and the L_i 's are elements of \mathcal{L} .

The following result was proved by Weil and the author [71] in 1995.

Theorem 4.2

- (1) A language is in $\Delta_{n+1}[\lt]$ if and only if it is in $UPol \mathcal{V}_n$.
- (2) A language is in $\Delta_{n+1}[\lt, S, \min, \max]$ if and only if it is in $UPol \mathcal{B}_n$.

The Straubing-Thérien hierarchy is pictured in the diagram below. A similar diagram for the Brzozowski hierarchy could be obtained by replacing each occurrence of \mathcal{V} by \mathcal{B} .



5 Algebraic approach

The algebraic approach to the study of concatenation hierarchies arises from the following two results [28].

Theorem 5.1 *Each full level \mathcal{V}_n is a variety of languages and every half-level $\mathcal{V}_{n+1/2}$ is a positive variety of languages.*

A similar result holds for the Brzozowski hierarchy.

Theorem 5.2 *Each full level \mathcal{B}_n is a li-variety of languages and every half-level $\mathcal{B}_{n+1/2}$ is a positive li-variety of languages.*

We let \mathbf{V}_n denote the variety of finite monoids corresponding to \mathcal{V}_n and $\mathbf{V}_{n+1/2}$ the variety of ordered monoids corresponding to $\mathcal{V}_{n+1/2}$. Similarly, let \mathbf{B}_n denote the variety of finite semigroups corresponding to \mathcal{B}_n and $\mathbf{B}_{n+1/2}$ the variety of ordered semigroups corresponding to $\mathcal{B}_{n+1/2}$.

The next results involve three operations on varieties: the *semidirect product*, the *Mal'cev product* and the *Schützenberger product*. The semidirect product, denoted $\mathbf{V} * \mathbf{W}$, and the Mal'cev product, denoted $\mathbf{V} \textcircled{M} \mathbf{W}$, are binary operations. The Schützenberger product, denoted $\diamond \mathbf{V}$, is a unary operation. Giving the precise definitions of these operations would lead us too far afield, but they can be found in [2, 24, 28, 66, 64, 76, 85, 98, 104, 118] for the semidirect product, in [24, 66, 64, 72, 74, 98] for the Mal'cev product and in [100, 59, 61, 65, 66] for the Schützenberger product.

The author, generalizing an early result of Reutenauer [84], used the Schützenberger product to prove the following result [61, 66].

Theorem 5.3 *For every $n > 0$, $\mathbf{V}_{n+1} = \diamond \mathbf{V}_n$.*

A nice connection between the hierarchies \mathbf{V}_n and \mathbf{B}_n was discovered by Straubing [101] (see also Pin-Weil [76] for the half levels). A semigroup S is said to be *locally trivial* if, for every idempotent $e \in S$ and every $s \in S$, $ese = e$. Let $\mathbf{LI} = \llbracket ese = e \rrbracket$ be the variety of locally trivial semigroups. We let $\llbracket e \leq ese \rrbracket$ denote the variety of ordered semigroups, such that, for every idempotent $e \in S$ and every $s \in S$, $e \leq ese$. The dual variety $\llbracket e \geq ese \rrbracket$ is defined in the same way.

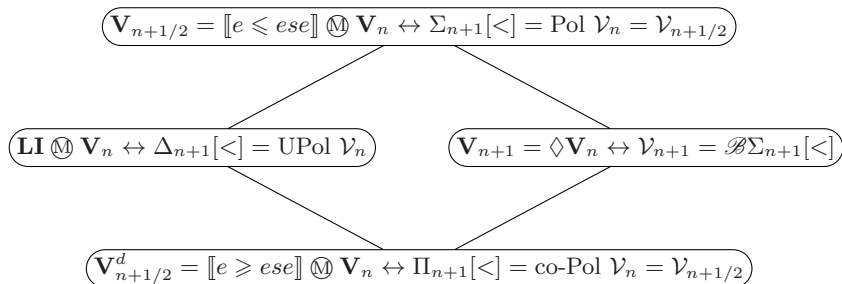


Figure 5.1:

Theorem 5.4 For every $n > 0$, $\mathbf{B}_n = \mathbf{V}_n * \mathbf{LI}$ and $\mathbf{B}_{n+1/2} = \mathbf{V}_{n+1/2} * \mathbf{LI}$.

It is very likely that this result extends to the intermediate classes Δ_n , giving $\Delta_n[\langle, S, \min, \max] = \Delta_n[\langle] * \mathbf{LI}$, but to the author's knowledge, this has only been proved [112] for $n \leq 2$.

Weil and the author [71, 74] established another useful relation.

Theorem 5.5 The variety $\mathbf{V}_{n+1/2}$ is equal to the Mal'cev product $\llbracket e \leq ese \rrbracket \otimes \mathbf{V}_n$.

A similar result holds for the varieties Δ_n , as a consequence of a more general result on the unambiguous product [58, 70].

Theorem 5.6 A language belongs to $\Delta_{n+1}[\langle]$ if and only if its syntactic monoid belongs to $\mathbf{LI} \otimes \mathbf{V}_n$.

The algebraic counterpart of the Straubing-Thérien hierarchy is summarized in Figure 5.1, in which the symbol \leftrightarrow indicates the equivalence between the algebraic characterizations and the logical descriptions. Again, one gets a similar diagram for the Brzozowski hierarchy by replacing each occurrence of \mathbf{V} by \mathbf{B} and by considering the signature $\{\langle, S, \min, \max\}$ instead of $\{\langle\}$. The algebraic approach gives algebraic characterizations of the concatenation hierarchies, but do not necessarily lead to decidability results. Let us now examine the decidability questions in more details.

6 Known decidability results

A language belongs to \mathcal{V}_0 if and only if its syntactic monoid is trivial.

6.1 Levels 1/2 and 1

The level 1/2 is also easy to study [71]. The variety $\mathcal{V}_{1/2}$ consists of the languages that are finite union of languages of the form $A^*a_1A^* \cdots a_kA^*$, where a_1, \dots, a_k are letters and the variety $\mathcal{B}_{1/2}$ consists of the languages that are finite union of languages of the form $u_0A^*u_1A^* \cdots u_{k-1}A^*u_k$, where u_0, \dots, u_k are words.

Theorem 6.1

- (1) A regular language belongs to $\mathcal{V}_{1/2}$ if and only if its ordered syntactic monoid satisfies the identity $1 \leq x$.
- (2) A language belongs to $\mathcal{B}_{1/2}$ if and only if its ordered syntactic semigroup belongs to the variety $\llbracket e \leq ese \rrbracket$.

The variety \mathcal{V}_1 consists of the languages that are Boolean combinations of languages of the form $A^*a_1A^*\cdots a_kA^*$, where a_1, \dots, a_k are letters. The decidability of \mathcal{V}_1 was obtained by Imre Simon [95] in 1975. Recall that a monoid is \mathcal{J} -trivial if two elements generating the same ideal are equal.

Theorem 6.2 *A language belongs to \mathcal{V}_1 if and only if its syntactic monoid is \mathcal{J} -trivial.*

It follows that \mathbf{V}_1 is the variety \mathbf{J} of \mathcal{J} -trivial monoids. This variety of \mathcal{J} -trivial monoids is characterized by the identities $x^{\omega+1} = x^\omega$ and $(xy)^\omega = (yx)^\omega$, or, alternatively, by the identities $y(xy)^\omega = (xy)^\omega = (xy)^\omega x$. Simon's original proof is based on a very nice argument of combinatorics on words. Simon's theorem inspired a lot of subsequent research and a number of alternative proofs have been proposed [97, 107, 1, 2, 40, 42, 43, 44]. Let me just mention two important consequences in semigroup theory. Recall that a monoid M divides a monoid N if M is a quotient of a submonoid of N . The first result is due to Straubing [99] and the second one to Straubing and Thérien [107].

Theorem 6.3 *A monoid is \mathcal{J} -trivial if and only if it divides a monoid of upper unitriangular Boolean matrices.*

Theorem 6.4 *A monoid is \mathcal{J} -trivial if and only if it is a quotient of an ordered monoid satisfying the identity $1 \leq x$.*

The languages of dot-depth one are the Boolean combinations of languages of the form $u_0A^*u_1A^*\cdots u_{k-1}A^*u_k$, where $k \geq 0$ and $u_0, u_1, \dots, u_k \in A^+$. The decidability of \mathcal{B}_1 was obtained by Knast [45, 46] and the proof was improved by Thérien [111]. This result also had a strong influence on subsequent developments, notably in finite semigroup theory.

Theorem 6.5 *A regular language belongs to \mathcal{B}_1 if and only if its syntactic semigroup satisfies Knast identity:*

$$(x^\omega py^\omega qx^\omega)^\omega py^\omega s(x^\omega ry^\omega sx^\omega)^\omega = (x^\omega py^\omega qx^\omega)^\omega (x^\omega ry^\omega sx^\omega)^\omega.$$

6.2 The classes Δ_2

The variety UPol \mathcal{V}_1 is equal to $\Delta_2[<]$. According to a result of Schützenberger [88], it consists of the finite disjoint unions of the unambiguous products of the form $A_0^*a_1A_1^*a_2\cdots a_kA_k^*$, where $a_1, \dots, a_k \in A$ and A_0, A_1, \dots, A_k are subsets of A . It corresponds to the variety \mathbf{DA} of all monoids in which each regular \mathcal{D} -class is an idempotent subsemigroup [88]. This variety can be defined by the profinite identity $(xy)^\omega y(xy)^\omega = (xy)^\omega$. Therefore we have

Theorem 6.6 *A language belongs to $\Delta_2[<]$ if and only if its syntactic monoid belongs to \mathbf{DA} .*

The variety **DA** has numerous applications, nicely summarized by Tesson and Thérien in their survey *Diamonds are forever: the variety DA* [109].

The first application relates **DA** to another fragment of first-order logic. Let $\mathbf{FO}^k[\prec]$ be the class of languages that can be defined by a first-order sentence using at most k variables and let $\mathbf{FO}[\prec] = \bigcup_{k \geq 0} \mathbf{FO}^k[\prec]$. We have already seen that $\mathbf{FO}[\prec]$ is the variety of star-free languages. One can show that $\mathbf{FO}[\prec] = \mathbf{FO}^3[\prec]$ and it is not difficult to see that a language is in $\mathbf{FO}^1[\prec]$ if and only if its syntactic monoid is idempotent and commutative. The following result is due to Thérien and Wilke [112].

Theorem 6.7 *A language belongs to $\mathbf{FO}^2[\prec]$ if and only if its syntactic monoid belongs to **DA**.*

Etesami, Vardi and Wilke proved in [29] that $\mathbf{FO}^2[\prec]$ is also the class of languages captured by a fragment of temporal logic called unary temporal logic. Finally, Schwentick, Thérien and Vollmer [89] proved that a language is accepted by a partially ordered two-way automaton if and only if its syntactic monoid belongs to **DA**. See also the article of Diekert, Gastin and Kufleitner [27] for alternative proofs of these results.

Let us now consider the signature $\{\prec, S\}$. We already mentioned that the variety corresponding to $\Delta_2[\prec, S]$ is $\mathbf{DA} * \mathbf{LI}$. Moreover, Almeida [3] proved that $\mathbf{DA} * \mathbf{LI} = \mathbf{LDA}$, the variety of all finite semigroups S such that, for all $e \in S$, $eSe \in \mathbf{DA}$. It follows that $\Delta_2[\prec, S]$ is also decidable.

6.3 Level 3/2

Two general decidability results are consequences of the results of Section 5. The first one is due to Straubing [101] (see also [76] for the half levels) and is a consequence of Theorem 5.4, except for the case $n = 1$, which follows from Theorems 6.2 and 6.5.

Theorem 6.8 *For each $n \geq 1$, the variety \mathbf{B}_n is decidable if and only if the variety \mathbf{V}_n is decidable. Similarly, the variety $\mathbf{B}_{n+1/2}$ is decidable if and only if the variety $\mathbf{V}_{n+1/2}$ is decidable.*

Given a set of profinite identities defining a variety of finite monoids \mathbf{V} , Weil and the author [72] gave a set of identities defining the varieties $\llbracket e \leq ese \rrbracket \textcircled{\mathbf{V}}$ and $\mathbf{LI} \textcircled{\mathbf{V}}$. This leads in particular to a set of profinite identities for $\mathbf{V}_{3/2}$ [72].

Theorem 6.9 *A language belongs to $\mathcal{V}_{3/2}$ if and only if its ordered syntactic monoid satisfies the profinite identities $u^\omega \leq u^\omega v u^\omega$, where u and v are idempotent profinite words on the same alphabet. This condition is decidable.*

The decidability of $\mathcal{V}_{3/2}$ was also proved by Arfi [8, 9] as a consequence of Hashiguchi's results [38]. See also the model theoretic approach of Selivanov [91] for alternative proofs.

The decidability of $\mathcal{B}_{3/2}$ now follows from Theorem 6.8. A direct characterization of $\mathcal{V}_{3/2}$ and $\mathcal{B}_{3/2}$ using forbidden patterns was given Glaßer and Schmitz [32, 34]. It leads to an **NL**-algorithm for the membership problem for $\mathcal{B}_{3/2}$.

Very recently, Almeida, Bartonova, Klíma and Kunc [5] improved the result of Weil and the author [72] to get the following decidability result.

Theorem 6.10 *If $\Sigma_n[<]$ is decidable, then $\Delta_{n+1}[<]$ is decidable.*

This result can be translated in two ways. In terms of varieties of languages:

if $\text{Pol } \mathcal{V}_{n-1}$ is decidable, then $\text{UPol } \mathcal{V}_n$ is decidable,

or in terms of varieties of monoids:

if $\mathbf{V}_{n-1/2}$ is decidable, then $\mathbf{LI} \otimes \mathbf{V}_n$ is decidable.

6.4 Level 2 and beyond

Let us return to the level 2 of the Straubing-Thérien hierarchy. A simple description of the languages of \mathcal{V}_2 was obtained by Straubing and the author [68] in 1981:

Theorem 6.11 *A language belongs to $\mathcal{V}_2(A)$ if and only if it is a Boolean combination of languages of the form $A_0^* a_1 A_1^* a_2 \cdots a_k A_k^*$, where $a_1, \dots, a_k \in A$ and A_0, A_1, \dots, A_k are subsets of A .*

In the same article, Straubing and the author gave an algebraic characterisation of \mathbf{V}_2 similar to Theorem 6.3.

Theorem 6.12 *A monoid belongs to \mathbf{V}_2 if and only if it divides a monoid of upper triangular Boolean matrices.*

However, it is not clear whether Theorem 6.12 leads to an effective characterization and despite numerous partial results [6, 7, 26, 75, 102, 103, 108, 117, 118], the decidability of \mathcal{V}_2 remained a major open problem for 20 years. It was finally settled by Place and Zeitoun in 2014 [78].

Theorem 6.13 *The variety of languages $\mathcal{V}_2 = \mathcal{B}\Sigma_2[<]$ is decidable.*

In the same paper [78], Place and Zeitoun also obtained three other decidability results.

Theorem 6.14 *The positive varieties of languages $\Sigma_3[<]$, $\Pi_3[<]$ and $\Delta_3[<]$ are decidable.*

On a two-letter alphabet, this result was first established in [33]. The algebraic translation of Theorem 6.14 states that the varieties of ordered monoids $\mathcal{V}_{5/2}$ and $\mathcal{V}_{5/2}^d$ are decidable. In view of Theorem 6.10, this also gives the decidability of $\Delta_4[<]$.

To obtain these results, Place and Zeitoun considered a more general question than membership, the separation problem. Let us say that a language S separates two languages K and L if $K \subseteq S$ and $L \cap S = \emptyset$. The separation problem can be formulated for any class \mathcal{C} of languages.

Problem 2 *Is the following problem decidable: given two disjoint regular languages K and L , is there a language $S \in \mathcal{C}$ separating K and L .*

Note that if the separation problem is decidable for \mathcal{C} , then \mathcal{C} is decidable. Indeed, since L is the unique language separating L and L^c , L belongs to \mathcal{C} if and only if L and L^c are separable.

As shown by Almeida [4], the separation problem is related to a problem on finite semigroups (finding the 2-pointlike sets relative to a variety of semigroups). The separation problem for star-free languages was first solved by Henckell [39] in its semigroup form. Successive improvements can be found in [41, 79, 82].

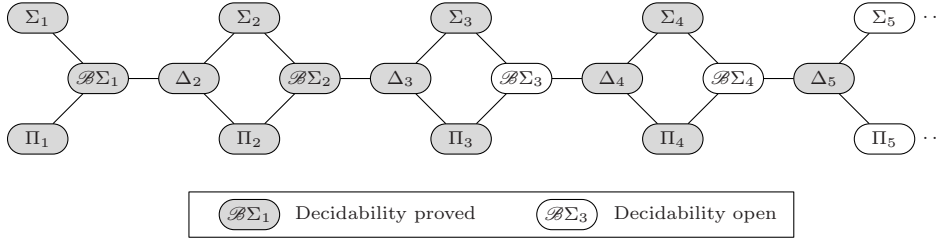
A major result of Place and Zeitoun [78] is the following much stronger result.

Theorem 6.15 *If the separation problem for $\Sigma_n[<]$ is decidable, then $\Sigma_{n+1}[<]$ is decidable.*

The latest result, due to Place [77] states that the separation problem is decidable for $\Sigma_3[<]$ and $\Pi_3[<]$. New decidability results follow, as a corollary of Theorem 6.15 and 6.10.

Theorem 6.16 *The positive varieties of languages $\Sigma_4[<]$, $\Pi_4[<]$ and the varieties of languages $\Delta_4[<]$ and $\Delta_5[<]$ are decidable.*

The decidability of the other levels is still open and the following diagram summarizes the known results on the quantifier alternation hierarchy. Due to the lack of space, the signature is omitted. Thus Σ_n stands for $\Sigma_n[<]$.



For the signature $\{<, S, \min, \max\}$, the decidability of Σ_n and Π_n , for $n \leq 4$ and that of $B\Sigma_n$, for $n \leq 2$, follows from Theorem 6.8. The decidability of Δ_n , for $n \leq 4$, follows from the decidability of Σ_n and Π_n . Finally Δ_5 seems to be the only fragment known to be decidable in the signature $\{<\}$, but still pending for the signature $\{<, S\}$.

We have seen the importance of the operation $\mathbf{V} \rightarrow \mathbf{V} * \mathbf{LI}$, where \mathbf{V} is a variety of monoids. However, Auinger proved that decidability is not always preserved by this operation [10]. In other words, there exists a decidable variety \mathbf{V} such that $\mathbf{V} * \mathbf{LI}$ is not decidable. Surprisingly, as shown by Steinberg [96], the same operation preserves the decidability of pointlikes. This implies the following result, which was recently reproved by Place and Zeitoun [80] in a simpler way.

Theorem 6.17 *Let \mathbf{V} be a variety of finite monoids. If separability is decidable in the variety of languages corresponding to \mathbf{V} , then it is also decidable in the variety corresponding to $\mathbf{V} * \mathbf{LI}$.*

In the same paper, Place and Zeitoun [80] proved the following result.

Theorem 6.18 *Let \mathcal{F} be one of the fragments Σ_n , Π_n or $\mathcal{B}\Sigma_n$. If separation is decidable in $\mathcal{F}[<]$, then it is decidable in $\mathcal{F}[<, S, \min, \max]$.*

It follows that separation is decidable for $\Delta_2[<, S]$ and for $\Sigma_n[<, S, \min, \max]$ and $\Pi_n[<, S, \min, \max]$ for $n \leq 3$.

To the knowledge of the author, the decidability of the other levels is still open. We recommend the recent survey of Place and Zeitoun [81] for a presentation of the new ideas and new results on the expressiveness of fragments of first-order logic.

7 Other developments

In this section, we list several research topics related to concatenation hierarchies. We apologize for not giving much details, but presenting any of these topics would require an independent article. However, we tried to give some relevant bibliography for the interested reader.

7.1 Other hierarchies

Several subhierarchies of star-free languages were presented in Brzozowski's survey [20]. An interesting subhierarchy of \mathcal{V}_1 is obtained by limiting the number of marked products [94]. In particular, if only one product is allowed, one gets the variety of languages \mathcal{J}_1 consisting of the Boolean closure of the languages of the form A^*aA^* . As was already mentioned, this variety is equal to $\mathbf{FO}^1[<]$ and the corresponding variety of monoids is the variety \mathbf{J}_1 of idempotent and commutative monoids.

A subhierarchy of \mathcal{B}_1 can be defined in a similar way. The first level of this subhierarchy is the *li*-variety of *locally testable languages*, which consists of the Boolean closure of the languages of the form uA^* , A^*v and A^*wA^* , where u , v and w are words of A^* . An algebraic characterization of this class was obtained independently by McNaughton [54] and by Brzozowski and Simon [22]. Let us say that a semigroup S is locally idempotent and commutative if, for each idempotent $e \in S$, the semigroup eSe is idempotent and commutative. We let \mathbf{LJ}_1 denote the variety of all locally idempotent and commutative semigroups.

Theorem 7.1 *A language is locally testable if its syntactic semigroup is locally idempotent and commutative.*

In fact, it is relatively easy to prove that a language is locally testable if its syntactic semigroup belongs to the variety $\mathbf{J}_1 * \mathbf{LI}$, but the really difficult part of the proof is the equality $\mathbf{LJ}_1 = \mathbf{J}_1 * \mathbf{LI}$. Historically, this result was the first decidability for a variety of the form $\mathbf{V} * \mathbf{LI}$ and it became very influential for this reason. Locally testable languages give another parameter to play with: one can assume in the definition that $|u|, |v| < k$ and $|w| \leq k$, which leads to the notion of *k-testable language*.

By limiting iteratively the number of marked products, one can also define tree-like hierarchies [12, 13, 14, 59, 61], which also admit an algebraic counterpart in terms of Schützenberger product.

Another interesting way to obtain subhierarchies is to limit the number of Boolean operations. Such Boolean hierarchies were extensively studied by Selivanov and his coauthors [35, 47, 91, 90, 93].

Finally, several subhierarchies of $\Delta_2 = \mathbf{FO}^2$ were considered in the recent years [30, 48, 49, 50, 51, 53, 52, 106, 119].

7.2 Connection with complexity classes

Bearing comparison with Brzozowski's original motivation, a result of Barrington and Thérien [11] gives evidence that the dot-depth provides a useful measure of complexity for Boolean circuits. More precisely, these authors found a remarkable correspondence between languages of dot-depth n and Boolean \mathbf{AC}^0 -circuits of depth n .

Another surprising connection between language hierarchies and the structure of complexity classes is offered by the theory of leaf languages [16, 17, 18, 19, 23, 36, 37, 92, 93, 116].

8 Conclusion

Several surveys related to concatenation hierarchies can be found in the literature [20, 31, 60, 64, 67, 81, 110]. Moreover, the study of concatenation hierarchies is not limited to words and similar hierarchies were considered for infinite words, for traces, for data words [15] and even for tree languages.

Since its introduction in 1971, the dot-depth hierarchy has been the topic of numerous investigations. The reason for this success is to be found in the variety of approaches successively proposed to solve the difficult problems raised by this hierarchy. Automata theory, combinatorics on words, semigroup theory, finite model theory, all these areas joined forces to produce increasingly sophisticated tools, leading to substantial progress, notably on decidability questions. Let us hope that the next 45 years will see even more progress and that the decidability of the dot-depth hierarchy will finally be established.

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