The kernel of a monoid morphism

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Outline

(1) Kernels and extensions
(2) The synthesis theorem
(3) The finite case
(4) Group radical and effective characterization
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Basic definitions

An element $e$ of a semigroup is idempotent if $e^2 = e$. The set of idempotents of a semigroup $S$ is denoted by $E(S)$.

A semigroup is idempotent if each of its elements is idempotent (that is, if $E(S) = S$). A semilattice is a commutative and idempotent monoid.

A variety of finite monoids is a class of finite monoids closed under taking submonoids, quotient monoids and finite direct products.

The kernel of a group morphism

Let $\pi : H \rightarrow G$ be a surjective group morphism. The kernel of $\pi$ is the group $T = \text{Ker}(\pi) = \pi^{-1}(1)$ and $H$ is an extension of $G$ by $T$.

The synthesis problem in finite group theory consists in constructing $H$ given $G$ and $T$.

◬ Is there a similar theory for semigroups?
### A specific example

A monoid $M$ is an extension of a group by a semilattice if there is a surjective morphism $\pi$ from $M$ onto a group $G$ such that $\pi^{-1}(1)$ is a semilattice.

- How to characterize the extensions of a group by a semilattice?
- Is there a synthesis theorem in this case?
- In the finite case, what is the variety generated by the extensions of a group by a semilattice?

### The kernel category of a morphism

Let $G$ be a group and let $\pi : M \rightarrow G$ be a surjective morphism. The kernel category $Ker(\pi)$ of $\pi$ has $G$ as its object set and for all $g, h \in G$, $Mor(u, v) = \{(u, m, v) \in G \times M \times G \mid u \pi(m) = v\}$

Note that $Mor(u, u)$ is a monoid equal to $\pi^{-1}(1)$ and that $G$ acts naturally (on the left) on $Ker(\pi)$:

```
  u → m → v
  ↓      ↓
gu → gm → gv
```

### A first necessary condition

**Proposition**

Let $\pi : H \rightarrow G$ be a surjective group morphism and let $K = \pi^{-1}(1)$. Then $\pi(h_1) = \pi(h_2)$ iff $h_1 h_2^{-1} \in K$.

**Proof.** As $\pi^{-1}(1)$ is a semilattice, $\pi^{-1}(1) \subseteq E(M)$.

### A second necessary condition

**Proposition**

Let $\pi$ be a surjective morphism from a monoid $M$ onto a group $G$ such that $\pi^{-1}(1)$ is a semilattice. Then $\pi^{-1}(1) = E(M)$ and the idempotents of $M$ commute.

**Proof.** As $\pi^{-1}(1)$ is a semilattice, $\pi^{-1}(1) \subseteq E(M)$.

### A second necessary condition (2)

**Proposition**

Let $\pi$ be a surjective morphism from a monoid $M$ onto a group $G$ such that $\pi^{-1}(1) = E(M)$. Then $M$ is $E$-unitary dense.

**Proof.** If $ex = f$ then $\pi(e)\pi(x) = \pi(f)$, that is $\pi(x) = 1$. Thus $x \in E(M)$ and $M$ is $E$-unitary.

Let $x \in M$ and let $g = \pi(x)$. Let $\bar{x}$ be such that $\pi(\bar{x}) = g^{-1}$. Then $\pi(x\bar{x}) = 1 = \pi(x\bar{x})$. Therefore $\bar{x}$ and $x\bar{x}$ are idempotent. Thus $M$ is $E$-dense.
The fundamental group $\pi_1(M)$

Let $F(M)$ be the free group with basis $M$. Then there is a natural injection $m \mapsto (m)$ from $M$ into $F(M)$. The fundamental group $\pi_1(M)$ of $M$ is the group with presentation

$$\langle M \mid (m)(n) = (mn) \text{ for all } m, n \in M \rangle$$

**Fact.** If $M$ is an $E$-dense monoid with commuting idempotents, then $\pi_1(M)$ is the quotient of $M$ by the congruence $\sim$ defined by $u \sim v$ if there exists an idempotent $e$ such that $eu = ev$.

Characterization of extensions of groups

**Theorem (Margolis-Pin, J. Algebra 1987)**

Let $M$ be a monoid whose idempotents form a subsemigroup. TFCAE:

1. there is a surjective morphism $\pi : M \to G$ onto a group $G$ such that $\pi^{-1}(1) = E(M)$,
2. the surjective morphism $\pi : M \to \Pi_1(M)$ satisfies $\pi^{-1}(1) = E(M)$,
3. $M$ is $E$-unitary dense.

Categories

Notation: $u$ and $v$ are objects, $x, y, p, q, p + x, p + x + y$ are morphisms, $p, q, x + y, y + x$ are loops.

For each object $u$, there is a loop $0_u$ based on $u$ such that, for every morphism $x$ from $u$ to $v$, $0_u + x = x$ and $x + 0_v = x$.

The local monoid at $u$ is the monoid formed by the loops based on $u$.

Groups acting on a category (1)

An action of a group $G$ on a category $C$ is given by a group morphism from $G$ into the automorphism group of $C$. We write $gx$ for the result of the action of $g \in G$ on an object or morphism $x$. Note that for all $g \in G$ and $p, q \in C$:

- $g(p + q) = gp + gq$,
- $g0_u = 0_{gu}$.

The group $G$ acts freely on $C$ if $gx = x$ implies $g = 1$. It acts transitively if the orbit of any object of $C$ under $G$ is $\text{Obj}(C)$.

The monoid $C_u$

Let $G$ be a group acting freely and transitively on a category $C$. Let $u$ be an object of $C$ and let

$$C_u = \{(p, g) \mid g \in G, p \in \text{Mor}(u, gu)\}$$

Then $C_u$ is a monoid under the multiplication defined by $(p, g)(q, h) = (p + gq, gh)$.
A property of the monoid $C_u$

**Proposition**

Let $G$ be a group acting freely and transitively on a category. Then for each object $u$, the monoid $C_u$ is isomorphic to $C/G$.

The synthesis theorem

**Theorem (Margolis-Pin, J. Algebra 1987)**

Let $M$ be a monoid. The following conditions are equivalent:

1. $M$ is an extension of a group by a semilattice,
2. $M$ is $E$-unitary dense with commuting idempotents,
3. $M$ is isomorphic to $C/G$, where $G$ is a group acting freely and transitively on a connected, idempotent and commutative category.

The covering theorem

Let $M$ and $N$ be monoids with commuting idempotents. A cover is a surjective morphism $\gamma: M \to N$ which induces an isomorphism from $E(M)$ to $E(N)$.

**Theorem (Fountain, 1990)**

Every $E$-dense monoid with commuting idempotents has an $E$-unitary dense cover with commuting idempotents.

Part III

The finite case

Closure properties

**Proposition**

The class of extensions of groups by semilattices is closed under taking submonoids and direct product.

**Proof.** Let $\pi$ be a surjective morphism from a monoid $M$ onto a group $G$ such that $\pi^{-1}(1)$ is a semilattice. If $N$ be a submonoid of $M$, then $\pi(N)$ is a submonoid of $G$ and hence is group $H$. Thus $N$ is an extension of $H$ and $\pi^{-1}(1) \cap N$ is a semilattice.

Direct products: easy.

Variety generated by finite extensions

Let $V$ be the variety generated by extensions of groups by semilattices.

A monoid belongs to $V$ iff it is a quotient of an extension of a group by a semilattice.

\[ \gamma: M \rightarrow N \]
\[ \pi: M \rightarrow G \]
\[ \pi^{-1}(1) = E(M) \]

The monoid $N$ belongs to $V$. 

The monoid $N$ belongs to $V$. 

Variety generated by finite extensions

Let \( \mathbf{V} \) be the variety generated by extensions of groups by semilattices.

A monoid belongs to \( \mathbf{V} \) iff it is a quotient of an extension of a group by a semilattice.

\[
\begin{array}{ccc}
\gamma & \longrightarrow & M \\
\downarrow & & \downarrow \pi \\
N & \longrightarrow & G
\end{array}
\]

\[\pi^{-1}(1) = E(M)\]

The monoid \( N \) belongs to \( \mathbf{V} \).

This diagram is typical of a relational morphism.

Relational morphisms

Let \( M \) and \( N \) be monoids. A relational morphism from \( M \) to \( N \) is a map \( \tau : M \rightarrow \mathcal{P}(N) \) such that:

1. \( \tau(s) \) is nonempty for all \( s \in M \),
2. \( \tau(s) \cap \tau(t) \subseteq \tau(st) \) for all \( s, t \in M \),
3. \( 1 \in \tau(1) \).

Examples of relational morphisms include:

- Morphisms
- Inverses of surjective morphisms
- The composition of two relational morphisms

Graph of a relational morphism

The graph \( R \) of \( \tau \) is a submonoid of \( M \times N \). Let \( \alpha : R \rightarrow M \) and \( \beta : R \rightarrow N \) be the projections.

Then \( \alpha \) is surjective and \( \tau = \beta \circ \alpha^{-1} \).

\[
\begin{array}{ccc}
\alpha & \longrightarrow & M \\
\uparrow & & \uparrow \tau \\
\beta & \longrightarrow & N
\end{array}
\]

\[\alpha(m, n) = m \quad \tau(m) = \beta(\alpha^{-1}(m))\]

\[\beta(m, n) = n \quad \tau^{-1}(n) = \alpha(\beta^{-1}(n))\]

An example of relational morphism

Let \( Q \) be a finite set. Let \( S(Q) \) the symmetric group on \( Q \) and let \( I(Q) \) be the monoid of all injective partial functions from \( Q \) to \( Q \) under composition.

Let \( \tau : I(Q) \rightarrow S(Q) \) be the relational morphism defined by \( \tau(f) = \{ \text{Bijections extending } f \} \)

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\hline
f & 3 & 1 & 2 \\
\hline
h_1 & 3 & 1 & 2 \\
\hline
h_2 & 3 & 4 & 2 \\
\hline
\end{array}
\]

\[\tau(f) = \{ h_1, h_2 \}\]

Relational morphisms

Proposition

Let \( \tau : M \rightarrow N \) be a relational morphism. If \( T \) is a subsemigroup of \( N \), then

\[\tau^{-1}(T) = \{ x \in M \mid \tau(x) \cap T \neq \emptyset \}\]

is a subsemigroup of \( M \).

In our example, \( \tau^{-1}(1) \) is a semilattice since

\[\tau^{-1}(1) = \{ f \in I(Q) \mid \text{the identity extends } f \}\]

\[= \{ \text{subidentities on } Q \} \equiv (\mathcal{P}(Q), \cap)\]

Finite extensions and relational morphisms

A monoid belongs to \( \mathbf{V} \) iff it is a quotient of an extension of a group by a semilattice.

\[
\begin{array}{ccc}
\gamma & \longrightarrow & M \\
\downarrow & & \downarrow \pi \\
N & \longrightarrow & G
\end{array}
\]

\[\pi^{-1}(1) \text{ semilattice} \quad \gamma(\pi^{-1}(1)) \text{ semilattice}\]

Proposition

A monoid \( N \) belongs to \( \mathbf{V} \) iff there is a relational morphism \( \tau \) from \( N \) onto a group \( G \) such that

\( \tau^{-1}(1) \) is a semilattice.
Finite extensions and relational morphisms (2)

Consider the canonical factorization of $\tau$:

$$
\begin{array}{c}
N \\
\alpha \\
\beta \downarrow \tau \\
\downarrow \\
G
\end{array}
$$

Then $\alpha$ induces an isomorphism from $\beta^{-1}(1)$ onto $\tau^{-1}(1)$ since

$$
\beta^{-1}(1) = \{(n, 1) \in R \mid 1 \in \tau(n)\}
$$

and

$$
\tau^{-1}(1) = \{n \in N \mid 1 \in \tau(n)\}.
$$

A non effective characterization

**Theorem (Margolis-Pin, J. Algebra 1987)**

Let $N$ be a finite monoid. TFCAE

1. $N$ belongs to $V$,
2. $N$ is a quotient of an extension of a group by a semilattice,
3. $N$ is covered by an extension of a group by a semilattice,
4. there is a relational morphism $\tau$ from $N$ onto a group $G$ such that $\tau^{-1}(1)$ is a semilattice.

The finite covering theorem

**Theorem (Ash, 1987)**

Every finite monoid with commuting idempotents has a finite $E$-unitary cover with commuting idempotents.

**Corollary**

The variety $V$ is the variety of finite monoids with commuting idempotents.

Group radical of a monoid

Let $M$ be a finite monoid. The group radical of $M$ is the set

$$
K(M) = \bigcap_{\tau : M \to G} \tau^{-1}(1)
$$

where the intersection runs over the set of all relational morphisms from $M$ into a finite group.

Universal relational morphisms

**Proposition**

For each finite monoid $M$, there exists a finite group $G$ and a relational morphism $\tau : M \to G$ such that $K(M) = \tau^{-1}(1)$.

**Proof.** There are only finitely many subsets of $M$. Therefore $K(M) = \tau_1^{-1}(1) \cap \cdots \cap \tau_n^{-1}(1)$ where $\tau_1 : M \to G_1, \ldots, \tau_n : M \to G_n$. Let $\tau : M \to G_1 \times \cdots \times G_n$ be the relational morphism defined by $\tau(m) = \tau_1(m) \times \cdots \times \tau_n(m)$. Then $\tau^{-1}(1) = K(M)$. $\square$
### Another characterization of $V$

**Theorem**

Let $M$ be a finite monoid. TFCAE:

1. $M$ belongs to $V$,
2. $K(M)$ is a semilattice,
3. The idempotents of $M$ commute and $K(M) = E(M)$.

- Is there an algorithm to compute $K(M)$?

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### Ash’s small theorem

**Theorem (Ash 1987)**

If $M$ is a finite monoid with commuting idempotents, then $K(M) = E(M)$.

**Corollary**

The variety $V$ is the variety of finite monoids with commuting idempotents.

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### Ash’s big theorem

Denote by $D(M)$ the least submonoid $T$ of $M$ closed under weak conjugation: if $t \in T$ and $a \bar{a}a = a$, then $ata \in T$ and $\bar{a}ata \in T$.

**Theorem (Ash 1991)**

For each finite monoid $M$, one has $K(M) = D(M)$.

**Corollary**

One can effectively compute $K(M)$.

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### Part V

**The topological approach**

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### The pro-group topology

The pro-group topology on $A^*$ [on $FG(A)$] is the least topology such that every morphism from $A^*$ on a finite (discrete) group is continuous.

**Proposition**

Let $L$ be a subset of $A^*$ and $u \in A^*$. Then $u \in T$ iff, for every morphism $\beta$ from $A^*$ onto a finite group $G$, $\beta(u) \in \beta(L)$.

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### A topological characterization of $K(M)$

**Theorem (Pin, J. Algebra 1991)**

Let $\alpha : A^* \to M$ be surjective morphism. Then $m \in K(M)$ iff $1 \in \alpha^{-1}(m)$.

1. $1 \in \alpha^{-1}(m) \iff$ for all $\beta : A^* \to G, 1 \in \beta(\alpha^{-1}(m))$  
   $\iff$ for all $\tau : M \to G, 1 \in \tau(m)$  
   $\iff$ for all $\tau : M \to G, m \in \tau^{-1}(1)$  
   $\iff m \in K(M)$
Finitely generated subgroups of the free group

Theorem (M. Hall 1950)
Every finitely generated subgroup of the free group is closed.

Theorem (Ribes-Zalesskii 1993)
Let $H_1, \ldots, H_n$ be finitely generated subgroups of the free group. Then $H_1H_2\cdots H_n$ is closed.

Computation of the closure of a set

Theorem (Pin-Reutenauer, 1991 (mod R.Z.))
There is a simple algorithm to compute the closure of a given rational subset of the free group.

Theorem (Pin, J. Algebra 1991 (mod P.R.))
There is a simple algorithm to compute the closure of a given rational language of the free monoid.

Another proof of Ash’s big theorem

Given the simple algorithm to compute the closure of a rational language, one has $K(M) = D(M)$.

Therefore, Ribes-Zalesskii’s theorem gives another proof of Ash’s big theorem.

Theorem
Given a decidable variety $\mathcal{V}$, the variety generated by $\mathcal{V}$-extensions of groups is decidable.