A tutorial on sequential functions

Jean-Éric Pin

LIAFA, CNRS and University Paris 7

30 January 2006, CWI, Amsterdam

Outline

(1) Sequential functions
(2) A characterization of sequential transducers
(3) Minimal sequential transducers
(4) Minimization of sequential transducers
(5) Composition of sequential transducers
(6) An algebraic approach
(7) The wreath product principle

Part I

Sequential functions

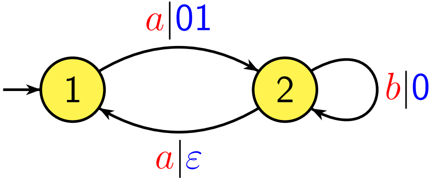
Informal definitions

A transducer (or state machine) is an automaton equipped with an output function. A transducer computes a relation on $A^* \times B^*$.

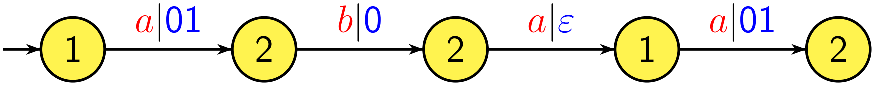
A sequential transducer is a transducer whose underlying automaton is deterministic (but not necessarily complete). A sequential transducer computes a partial function from A^* into B^*.

A pure sequential transducer computes a partial function φ preserving prefixes: if u is a prefix of v, then $\varphi(u)$ is a prefix of $\varphi(v)$.

An example of a pure sequential transducer



On the input $abaa$, the output is 01001.

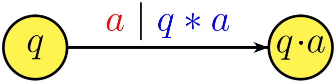


Pure sequential transducers

A pure sequential transducer is a 6-tuple

$$\mathcal{A} = (Q, A, B, i, \cdot, *)$$

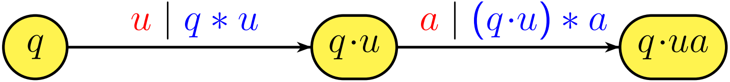
where the input function $(q, a) \to q{\cdot}a \in Q$ and the output function $(q, a) \to q * a \in B^*$ are defined on the same domain $D \subseteq Q \times A$.



Extensions of the transition and output functions

The transition function is extended to $Q \times A^* \to Q$.
Set $q{\cdot}\varepsilon = q$ and, if $q{\cdot}u$ and $(q{\cdot}u){\cdot}a$ are defined,
$q{\cdot}(ua) = (q{\cdot}u){\cdot}a$.

The output function is extended to $Q \times A^* \to B^*$.
Set $q * \varepsilon = \varepsilon$ and, if $q * u$ and $(q{\cdot}u) * a$ are defined,
$q * (ua) = (q * u)((q{\cdot}u) * a)$.



Pure sequential functions

The function $\varphi\colon A^* \to B^*$ defined by

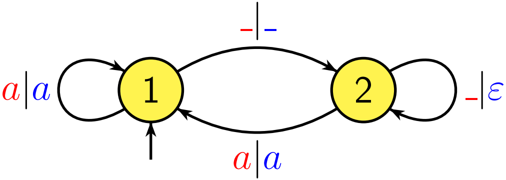
$$\varphi(u) = i * u$$

is called the function realized by $\mathcal{A}$.

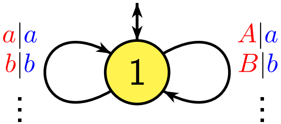
A function is pure sequential if it can be realized by some pure sequential transducer.

Examples of pure sequential functions

Replacing consecutive white spaces by a single one:



Converting upper case to lower case letters:

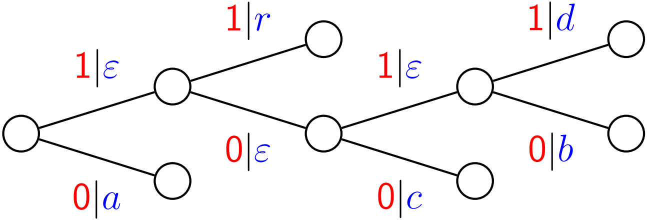


Coding and decoding

Consider the coding

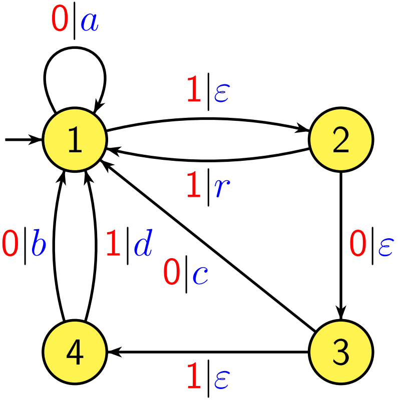
$a \to 0 \quad b \to 1010 \quad c \to 100 \quad d \to 1011 \quad r \to 11$

Decoding function



Decoding

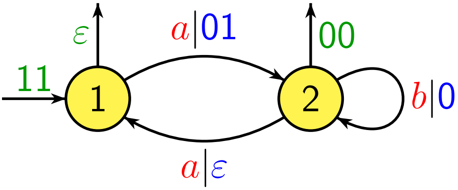
$a \to 0 \quad b \to 1010 \quad c \to 100 \quad d \to 1011 \quad r \to 11$



$0101011010001011010101110 \to abracadabra$

Sequential transducers: informal definition

A sequential transducer is a transducer whose underlying automaton is deterministic (but not necessarily complete). There is an initial prefix and a terminal function.



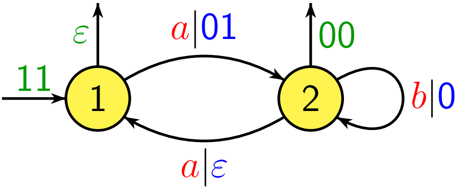
On the input $abaa$, the output is 110100100.

Sequential transducers

A sequential transducer is a 8-tuple

$$\mathcal{A} = (Q, A, B, i, \cdot, *, m, \rho)$$

where $(Q, A, B, i, \cdot, *)$ is a pure sequential transducer, $m \in B^*$ is the initial prefix and $\rho\colon Q \to B^*$ is a partial function, called the terminal function.



Sequential functions

The function $\varphi\colon A^* \to B^*$ defined by

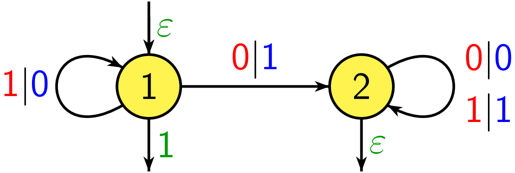
$$\varphi(u) = m(i * u)\rho(i{\cdot}u)$$

is called the function realized by $\mathcal{A}$.

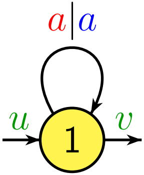
A function is sequential if it can be realized by some sequential transducer.

Some examples of sequential functions

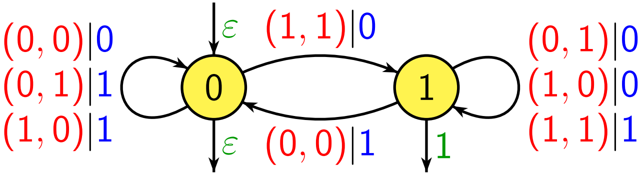
The function $x \to x + 1$ (in reverse binary)



The map $\varphi : A^* \to A^*$ defined by $\varphi(x) = uxv$.



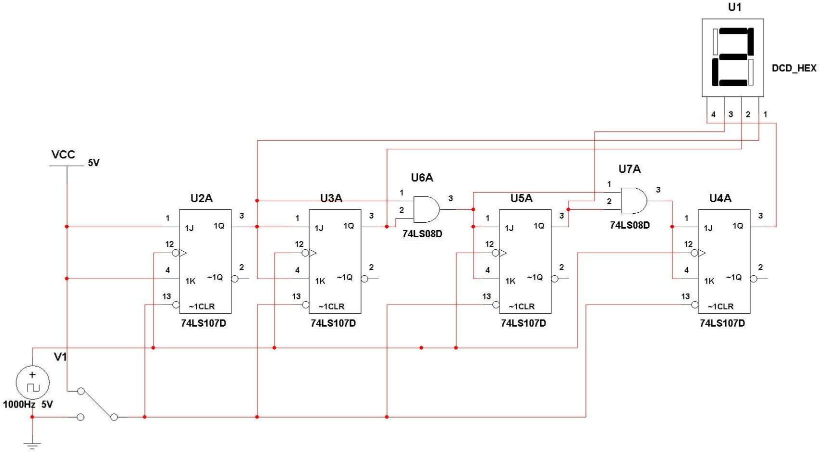
Addition (in reverse binary)



In inverse binary notation, $22 = 2 + 4 + 16 \to 01101$ and $13 = 1 + 4 + 8 \to 10110$. Taking as input $(0,1)(1,0)(1,1)(0,1)(1,0)$, the output is 110001, the inverse binary representation of $35 = 1 + 2 + 32$.

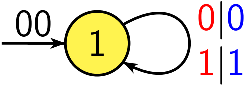
Hardware applications (Wikipedia)

The circuit diagram for a 4 bit TTL counter

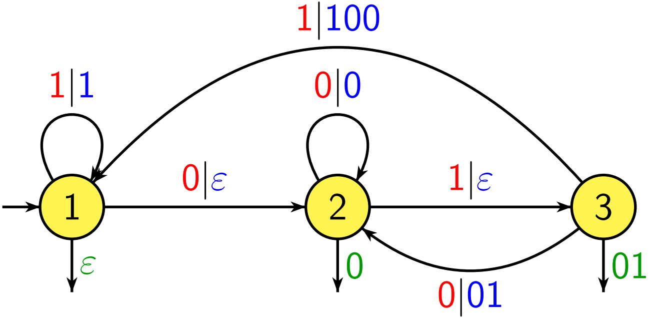


Other examples

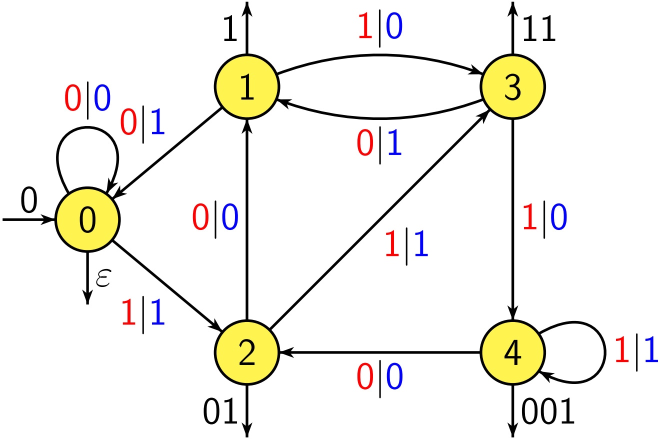
Multiplication by 4



Replacing each occurrence of 011 by 100.



Multiplication by 10

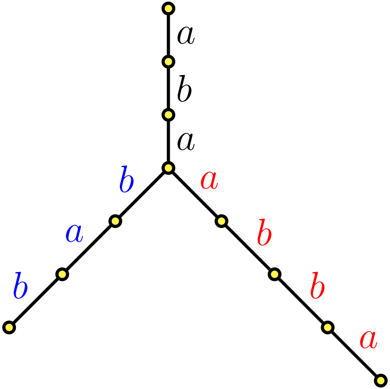


Part II

A characterization

The geodesic metric

The distance between $ababab$ and $abaabba$ is 7.



The geodesic metric (2)

Denote by $u \wedge v$ the longest common prefix of the words u and v. Then

$$d(u,v) = |u| + |v| - 2|u \wedge v|$$

Example: $d(ababab, abaabba) = 6 + 7 - 2 \times 3 = 7$.
One can show that d is a metric.

(1) $d(u,v) = 0$ iff $u = v$,

(2) $d(u,v) = d(v,u)$,

(3) $d(u,v) \leqslant d(u,w) + d(w,v)$.

A characterization of sequential functions

A function $\varphi\colon : A^* \to B^*$ is Lipschitz if there exists some $K > 0$ such that, for all $u, v \in A^*$,

$$d(\varphi(u), \varphi(v)) \leqslant K d(u, v)$$

Theorem (Choffrut 1979)

Let $\varphi : A^* \to B^*$ be a function whose domain is closed under taking prefixes. TFCAE:

(1) φ *is sequential,*

(2) φ *is Lipschitz, and* φ^{-1} *preserves regular sets.*

A characterization of pure sequential functions

Theorem (Ginsburg-Rose 1966)

Let $\varphi : A^* \to B^*$ *be a function whose domain is closed under taking prefixes. TFCAE:*

(1) φ *is a pure sequential function,*

(2) φ *is Lipschitz and preserves prefixes, and* φ^{-1} *preserves regular sets.*

Part III

Minimal sequential transducers

Residuals of a language

Let L be a language over A^*. Let $u \in A^*$. The (left) residual of L by u is the set

$$u^{-1}L = \{x \in A^* \mid ux \in L\}.$$

It is easy to see that $v^{-1}(u^{-1}L) = (uv)^{-1}L$.

Let $A = \{a, b\}$ and $L = A^*abaA^*$. Then

$$1^{-1}L = L \qquad a^{-1}L = A^*abaA^* \cup baA^*$$
$$b^{-1}L = L \qquad (ab)^{-1}L = A^*abaA^* \cup aA^*, \text{ etc.}$$

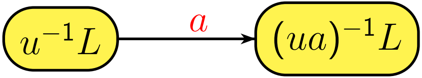
Minimal automaton of a language

The minimal automaton of a language L is equal to

$$\mathcal{A}(L) = (Q, A, \cdot, i, F)$$

where $Q = \{u^{-1}L \neq \emptyset \mid u \in A^*\}$, $i = L$ and $F = \{u^{-1}L \mid u \in L\})$. The transition function is given by

$$(u^{-1}L)\cdot a = a^{-1}(u^{-1}L) = (ua)^{-1}L.$$



Example of a minimal automaton

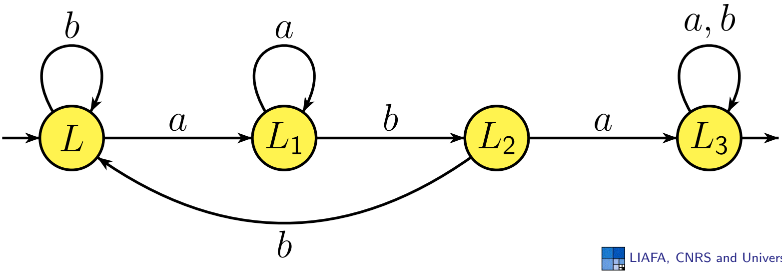
Let $A = \{a, b\}$ and $L = A^*abaA^*$. Then

$$a^{-1}L = A^*abaA^* \cup baA^* = L_1 \qquad b^{-1}L = L$$

$$b^{-1}L_1 = A^*abaA^* \cup aA^* = L_2 \qquad a^{-1}L_1 = L_1$$

$$a^{-1}L_2 = A^* = L_3 \qquad b^{-1}L_2 = L$$

$$a^{-1}L_3 = b^{-1}L_3 = L_3$$



Residuals of a sequential function

Let $\varphi : A^* \to B^*$ be a function and let $u \in A^*$. The **residual** of φ by u is the function $u^{-1}\varphi : A^* \to B^*$ defined by

$$(u^{-1}\varphi)(x) = (\varphi * u)^{-1}\varphi(ux)$$

where $(\varphi * u)$ is the longest common prefix of the words $\varphi(ux)$, for $ux \in \mathrm{Dom}(\varphi)$.

In other words, $u^{-1}\varphi$ can be obtained from the function $x \to \varphi(ux)$ by deleting the prefix $\varphi * u$ of $\varphi(ux)$.

The function $n \to 6n$

n	x	$\varphi(x)$
0	ε	0
1	1	011
2	01	0011
3	11	01001
4	001	00011
5	101	01111
6	011	001001
7	111	010101
8	0001	000011

n	x	$\varphi(x)$
9	1001	011011
10	0101	001111
11	1101	0100001
12	0011	0001001
13	1011	0111001
14	0111	0010101
15	1111	0101101
16	00001	0000011
17	10001	0110011

Let $\varepsilon^{-1}\varphi = \varphi_0$. Then φ_0 represents $n \to 3n$

n	x	$\varphi(x)$
0	ε	0
1	1	011
2	01	0011
3	11	01001
4	001	00011
5	101	01111
6	011	001001
7	111	010101
8	0001	000011

n	x	$\varphi(x)$
9	1001	011011
10	0101	001111
11	1101	0100001
12	0011	0001001
13	1011	0111001
14	0111	0010101
15	1111	0101101
16	00001	0000011
17	10001	0110011

The function φ_0, representing $n \to 3n$

n	x	$\varphi_0(x)$
0	ε	ε
1	1	11
2	01	011
3	11	1001
4	001	0011
5	101	1111
6	011	01001
7	111	10101
8	0001	00011

n	x	$\varphi_0(x)$
9	1001	11011
10	0101	01111
11	1101	100001
12	0011	001001
13	1011	111001
14	0111	010101
15	1111	101101
16	00001	000011
17	10001	110011

Residuals of φ_0

Let φ_0, φ_1 and φ_2 be the functions representing $n \to 3n$, $n \to 3n+1$ and $n \to 3n+2$, respectively.

$$\varphi_0 * 0 = 0$$
$$(0^{-1}\varphi_0)(x) = 0^{-1}\varphi_0(0x) = \varphi_0(x)$$
$$\varphi_0 * 1 = 1$$
$$(1^{-1}\varphi_0)(x) = 1^{-1}\varphi_0(1x) = \varphi_1(x)$$

Indeed, if x represents n, $1x$ represents $2n+1$, $\varphi_0(1x)$ represents $3(2n+1) = 6n+3$ and $1^{-1}\varphi_0(1x)$ represents $((6n+3)-1)/2 = 3n+1$.

The function φ_1, representing $n \rightarrow 3n+1$

n	x	$\varphi_1(x)$
0	ε	ε
1	1	001
2	01	111
3	11	0101
4	001	1011
5	101	00001
6	011	11001
7	111	01101
8	0001	10011

n	x	$\varphi_1(x)$
9	1001	00111
10	0101	100001
11	1101	010001
12	0011	101001
13	1011	000101
14	0111	110101
15	1111	011101
16	00001	100011
17	10001	001011

Residuals of φ_1

$$\varphi_1 * 0 = 1$$

$$(0^{-1}\varphi_1)(x) = 1^{-1}\varphi_1(0x) = \varphi_0(x)$$

Indeed, if x represents n, $0x$ represents $2n$, $\varphi_1(0x)$ represents $3(2n) + 1 = 6n + 1$ and $1^{-1}\varphi_1(0x)$ represents $((6n + 1) - 1)/2 = 3n$.

$$\varphi_1 * 1 = 0$$

$$(1^{-1}\varphi_1)(x) = 0^{-1}\varphi_1(1x) = \varphi_2(x)$$

Indeed, if x represents n, $1x$ represents $2n + 1$, $\varphi_1(1x)$ represents $3(2n + 1) + 1 = 6n + 4$ and $0^{-1}\varphi_1(1x)$ represents $(6n + 4)/2 = 3n + 2$.

The function φ_2, representing $n \to 3n + 2$

n	x	$\varphi_2(x)$
0	ε	ε
1	1	101
2	01	0001
3	11	1101
4	001	0111
5	101	10001
6	011	00101
7	111	11101
8	0001	01011

n	x	$\varphi_2(x)$
9	1001	10111
10	0101	000001
11	1101	110001
12	0011	011001
13	1011	100101
14	0111	001101
15	1111	111101
16	00001	010011
17	10001	101011

Residuals of φ_2

$$\varphi_2 * 0 = 0$$
$$(0^{-1}\varphi_2)(x) = 0^{-1}\varphi_2(0x) = \varphi_1(x)$$

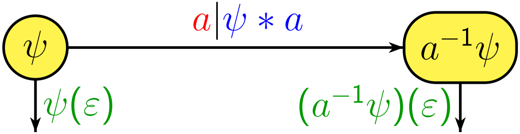
Indeed, if x represents n, $0x$ represents $2n$, $\varphi_2(0x)$ represents $3(2n) + 2 = 6n + 2$ and $0^{-1}\varphi_2(0x)$ represents $(6n + 2)/2 = 3n + 1$.

$$\varphi_2 * 1 = 1$$
$$(1^{-1}\varphi_2)(x) = 1^{-1}\varphi_2(1x) = \varphi_2(x)$$

Indeed, if x represents n, $1x$ represents $2n + 1$, $\varphi_2(1x)$ represents $3(2n + 1) + 2 = 6n + 5$ and $1^{-1}\varphi_2(1x)$ represents $((6n + 5) - 1)/2 = 3n + 2$.

Minimal sequential transducer of a function φ

It is the sequential transducer whose states are the residuals of φ and transitions are of the form



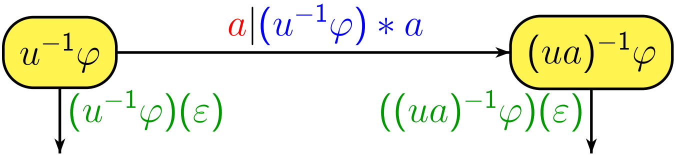
Recall that $\psi * a$ is the longest common prefix of the words $\psi(ax)$, for $ax \in \mathrm{Dom}(\varphi)$.
The initial state is $\varepsilon^{-1}\varphi$ and the initial prefix is $\varphi * \varepsilon$.

More formally...

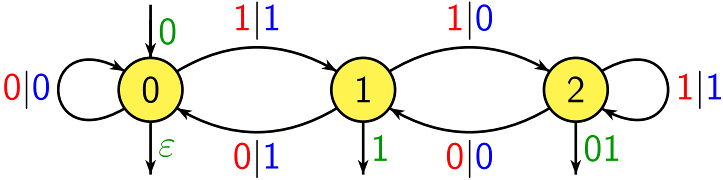
It is the sequential transducer
$\mathcal{A}_\varphi = (Q, A, B, i, \cdot, *, m, \rho)$ defined by

$$Q = \{u^{-1}\varphi \mid u \in A^* \text{ and } \mathrm{Dom}(\varphi\cdot u) \neq \emptyset\}$$
$$i = \varepsilon^{-1}\varphi,\ m = \varphi * \varepsilon \text{ and, for } q \in Q,\ \rho(q) = q(\varepsilon)$$

A typical transition of $\mathcal{A}_\varphi$:



The minimal sequential function of $n \to 6n$



$185 = 1 + 8 + 16 + 32 + 128$ and
$6 \times 185 = 1110 = 2 + 4 + 16 + 64 + 1024$.
Thus $\varphi(10011101) = 0110101001$

Part IV

Minimizing sequential transducers

The three steps of the algorithm

How to minimize a sequential transducer?

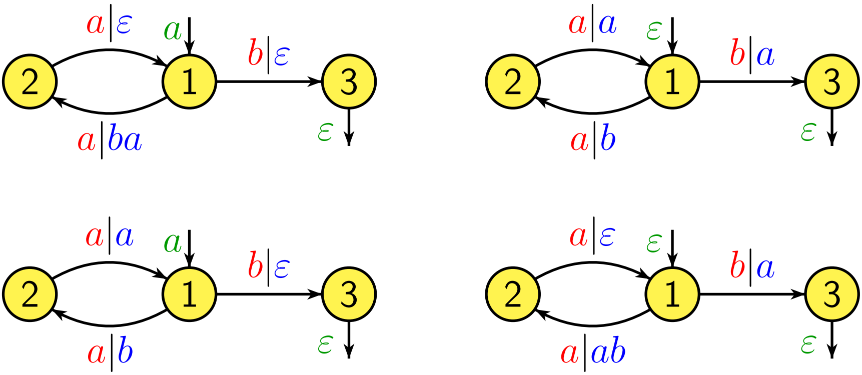
(1) Obtain a trim transducer (easy)
(2) Normalise the transducer (tricky)
(3) Merge equivalent states (standard)

Obtaining a trim transducer

Let $\mathcal{A} = (Q, A, B, i, \cdot, *, m, \rho)$ be a sequential transducer and let $F = \mathrm{Dom}(\rho)$. The transducer $\mathcal{A}$ is trim if the automaton $(Q, A, \cdot, q_0, F)$ is trim: all states are accessible from the initial state and one can reach a final state from any state.

Algorithm: it suffices to remove the useless states.

Equivalent transducers



These four sequential transducers realize exactly the same function $\varphi : \{a, b\}^* \to \{a, b\}^*$, with domain $(aa)^*b$, defined, for all $n \geqslant 0$, by $\varphi(a^{2n}b) = (ab)^n a$.

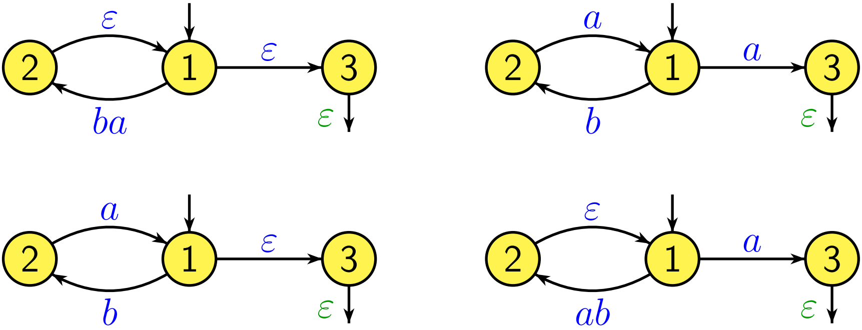
Normalized transducer

Let $\mathcal{A} = (Q, A, B, i, \cdot, *, m, \rho)$ be a sequential transducer. For each state q, denote by m_q the greatest common prefix of the words $(q * u)\rho(q \cdot u)$, where u ranges over the domain of the sequential function.

Equivalently, $m_q = \varphi_q * \varepsilon$, where φ_q is the sequential function realized by the transducer derrived from $\mathcal{A}$ by taking q as initial state and the empty word as initial prefix.

A sequential transducer is normalized if, for all states q, m_q is the empty word.

$m_q = (q * u)\rho(q \cdot u)$



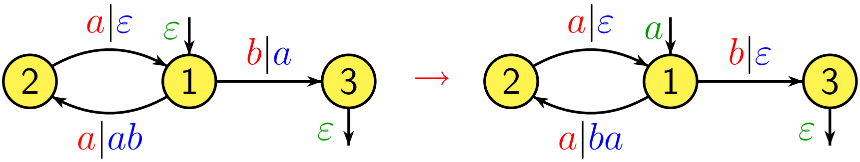
(1)	$m_1 = \varepsilon$	$m_2 = \varepsilon$	$m_3 = \varepsilon$
(2)	$m_1 = \varepsilon$	$m_2 = a$	$m_3 = \varepsilon$
(3)	$m_1 = \varepsilon$	$m_2 = a$	$m_3 = \varepsilon$
(4)	$m_1 = a$	$m_2 = a$	$m_3 = \varepsilon$

Normalising a transducer

Let $\mathcal{A} = (Q, A, B, i, \cdot, *, m, \rho)$ be a trim sequential transducer. One obtains a normalised transducer by changing the initial prefix, the output function and the terminal function as follows:

$$q *' a = m_q^{-1}(q * a)m_{q \cdot a}$$
$$m' = mm_i$$
$$\rho'(q) = m_q^{-1}\rho(q)$$

Normalisation on an example



One has $m_1 = a$, $m_2 = a$, $m_3 = \varepsilon$. Thus

$$m' = mm_1 = \varepsilon a = a$$

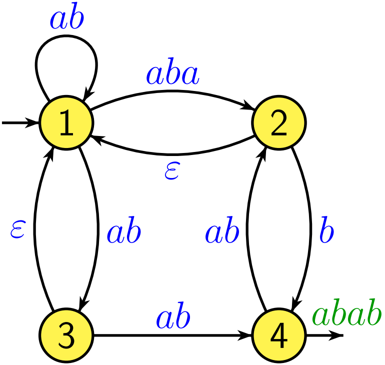
$$1 *' a = m_1^{-1}(1 * a)m_2 = a^{-1}(ab)a = ba$$

$$2 *' a = m_2^{-1}(1 * a)m_1 = a^{-1}(\varepsilon)a = \varepsilon$$

$$1 *' b = m_1^{-1}(1 * b)m_3 = a^{-1}(a)\varepsilon = \varepsilon$$

$$\rho'(3) = m_3^{-1}\rho(3) = \varepsilon^{-1}\varepsilon = \varepsilon$$

Computing the m_q is not so easy...



$$X_1 = abX_1 + abaX_2 + abX_3$$
$$X_2 = X_1 + bX_4$$
$$X_3 = X_1 + abX_4$$
$$X_4 = abX_2 + abab$$

Solving the system

$$X_1 = abX_1 + abaX_2 + abX_3$$
$$X_2 = X_1 + bX_4$$
$$X_3 = X_1 + abX_4$$
$$X_4 = abX_2 + abab$$

We work on $k = A^* \cup \{0\}$. Addition is the least common prefix operator ($u + 0 = 0 + u = u$ by convention). Observe that $u + u = u$ and $u(v_1 + v_2) = uv_1 + uv_2$ (but $(v_1 + v_2)u = v_1u + v_2u$ does not hold in general). Thus k is a left semiring.

Choffrut's algorithm (2003)

The prefix order is a partial order $\leqslant$ on k (with $u \leqslant 0$ by convention). One can extend this order to k^n componentwise.

Proposition

For all $u, v \in k^n$, the function $f(x) = ux + v$ is increasing. The sequence $f^n(0)$ is decreasing and converges to the greatest fixpoint of f.

The greatest solution of our system is exactly (m_1, m_2, m_3, m_4).

Example of Choffrut's algorithm

$$X_1 = abX_1 + abaX_2 + abX_3$$
$$X_2 = X_1 + bX_4$$
$$X_3 = X_1 + abX_4$$
$$X_4 = abX_2 + abab$$

The sequence $f^n(0)$ is $(0, 0, 0, 0)$, $(0, 0, 0, abab)$, $(0, babab, ababab, abab)$, $(abababab, babab, ababab, ab)$, $(abab, \varepsilon, ababab, ab)$, $(aba, \varepsilon, abab, ab)$, $(aba, \varepsilon, aba, ab)$, $(aba, \varepsilon, aba, ab)$.
Thus $m_1 = aba$, $m_2 = \varepsilon$, $m_3 = aba$, $m_4 = ab$.

Merging states

Two states are equivalent if they are equivalent in the input automaton $(Q, A, i, F, \cdot)$, have the same output functions and the same terminal functions:

$$p \sim q \iff \begin{cases} p \cdot a \sim q \cdot a \\ p * a = q * a \\ \rho(p) = \rho(q) \end{cases}$$

Part V

Composition of sequential functions

Composition of two pure sequential transducers

Theorem

Pure sequential functions are closed under composition.

Let σ and τ be two pure sequential functions realized by the transducers

$$\mathcal{A} = (Q, A, B, q_0, \cdot, *) \text{ and } \mathcal{B} = (P, B, C, p_0, \cdot, *)$$

The wreath product of $\mathcal{B}$ by $\mathcal{A}$ is obtained by taking as input for $\mathcal{B}$ the output of $\mathcal{A}$. It realizes $\tau \circ \sigma$.

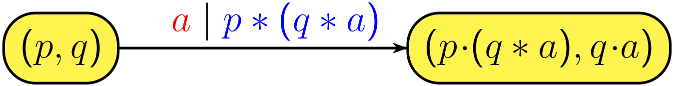
Wreath product of two pure sequential transducers

The wreath product is defined by

$$\mathcal{B} \circ \mathcal{A} = (P \times Q, A, C, (p_0, q_0), \cdot, *)$$

$$(p, q)\cdot a = (p\cdot(q * a), q\cdot a)$$

$$(p, q) * a = p * (q * a)$$



Composition of two sequential transducers

Theorem

Sequential functions are closed under composition.

Let φ and ψ be two sequential functions realized by the transducers $\mathcal{A}$ (equipped with the initial word n and the terminal function ρ) and $\mathcal{B}$ (equipped with the initial word m and the terminal function σ).

The wreath product of $\mathcal{B}$ by $\mathcal{A}$ is obtained by taking $m(p_0 * n)$ as initial word and, as terminal function, $\omega(p, q) = (p * \rho(q))\sigma(p \cdot \rho(q))$.

Iterating sequential functions...

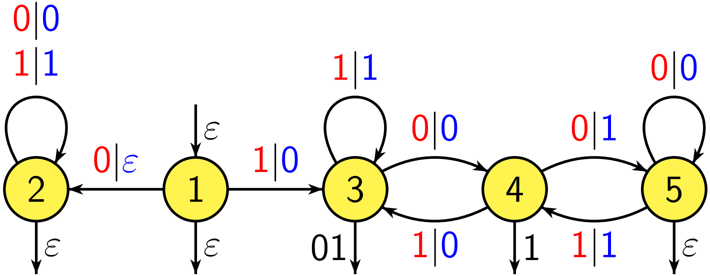
Iterating sequential functions can lead to difficult problems...

Let $f(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$

It is conjectured that for each positive integer n, there exists k such that $f^k(n) = 1$. The problem is still open.

Minimal transducer of the $3n+1$ function

Let $f(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$



Iterating the $3n + 1$ function...

31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1.

A useful result

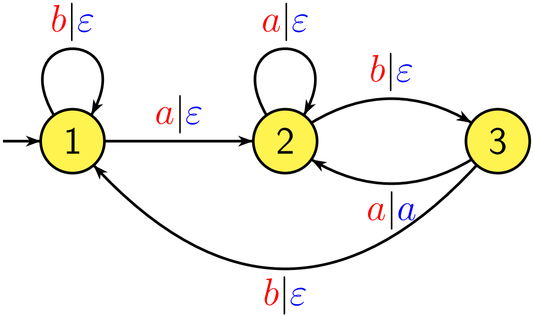
Let $\varphi : A^* \to B^*$ be a pure sequential function realized by $\mathcal{A} = (Q, A, B, q_0, \cdot, *)$. Let L be the regular language over B^* recognized by the DFA $\mathcal{B} = (P, B, \cdot, p_0, F)$. The wreath product of $\mathcal{B}$ by $\mathcal{A}$ is the DFA $\mathcal{B} \circ \mathcal{A} = (P \times Q, A, (p_0, q_0), \cdot)$ defined by $(p, q){\cdot}a = (p{\cdot}(q * a), q{\cdot}a)$.

Theorem

The language $\varphi^{-1}(L)$ is recognized by $\mathcal{B} \circ \mathcal{A}$.

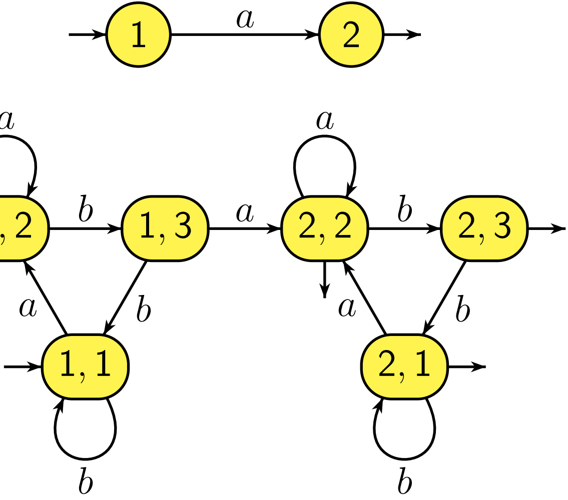
Example 1

Let $\varphi(u) = a^n$, where n is the number of occurrences of aba in u. This function is pure sequential:



Then $\varphi^{-1}(a)$ is the set of words containing exactly one occurrence of aba.

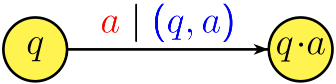
Wreath product of the two automata



The operation $L \to LaA^*$

Let $\mathcal{A} = (Q, A, B, q_0, F, \cdot)$ be a DFA. Let $B = Q \times A$ and let $\sigma \colon A^* \to B^*$ be the pure sequential function defined by

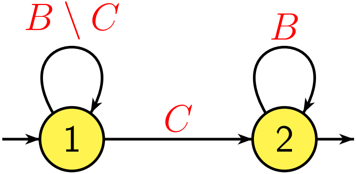
$$\sigma(a_1 \cdots a_n) = (q_0, a_1)(q_0{\cdot}a_1, a_2) \cdots (q_0 a_1 \cdots a_{n-1}, a_n)$$



Let $a \in A$ and let $C = F \times \{a\} \subseteq B$. Then $\sigma^{-1}(B^* C B^*) = LaA^*$.[Proof on blackboard!]

Example 2

Therefore $\mathcal{B} \circ \mathcal{A}$ recognizes LaA^*, where $\mathcal{B}$ is the minimal automaton of B^*CB^*.



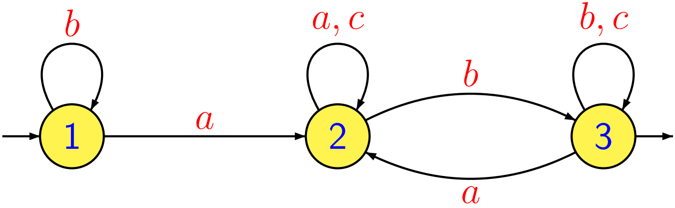
Note that if φ is a formula of linear temporal logic, then $L(F(p_a \wedge X\varphi)) = A^*aL(\varphi)$

Part VI

The algebraic approach

Idea: replace automata by monoids.

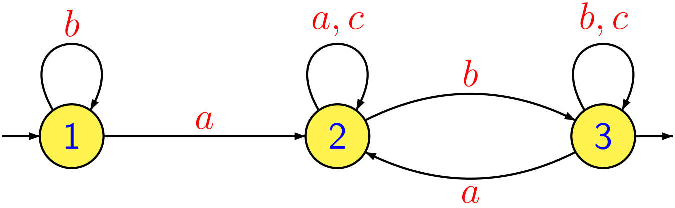
Transformation monoid of an automaton



1	1	2	3
a	2	2	2
b	1	3	3
c	-	2	3

Relations:

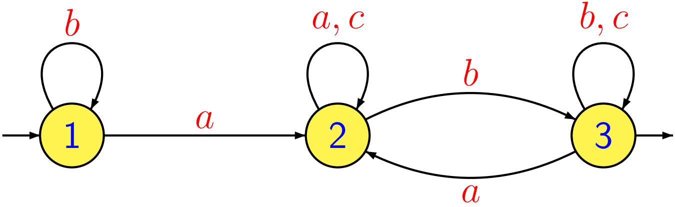
Transformation monoid of an automaton



1	1	2	3
a	2	2	2
b	1	3	3
c	-	2	3

Relations:
$aa = a$

Transformation monoid of an automaton



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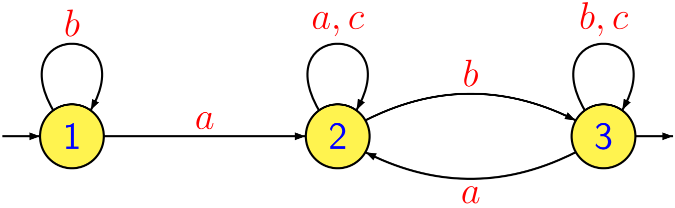

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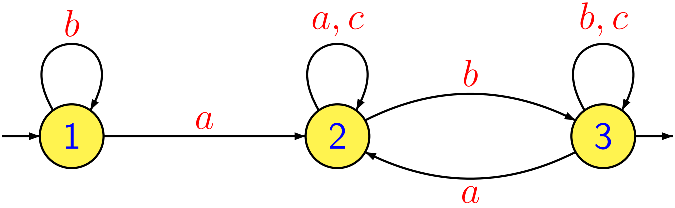
Transformation monoid of an automaton



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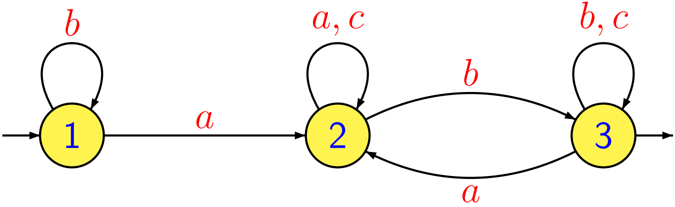
$ba = a$

Transformation monoid of an automaton



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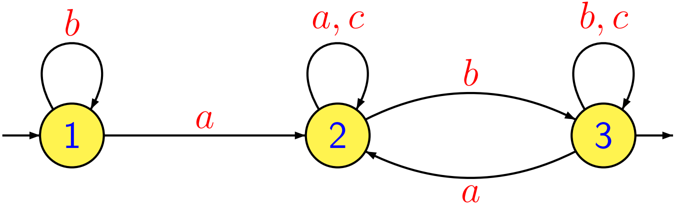
ab	3	3	3

Relations:
$aa = a$
$ac = a$
$ba = a$
$bb = b$

Transformation monoid of an automaton



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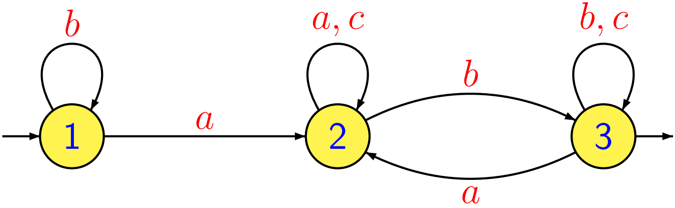
Transformation monoid of an automaton



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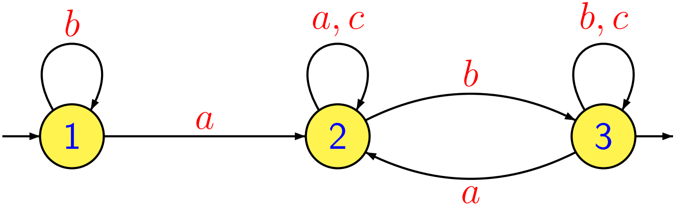

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Relations:
$aa = a$
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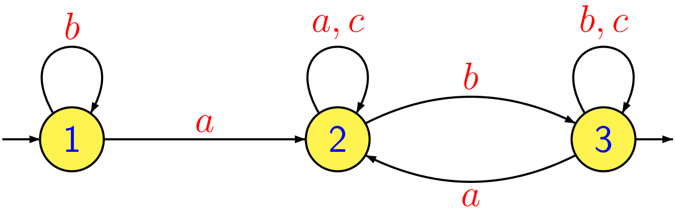
$ba = a$
$bb = b$
$cb = bc$

Transformation monoid of an automaton



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Relations:
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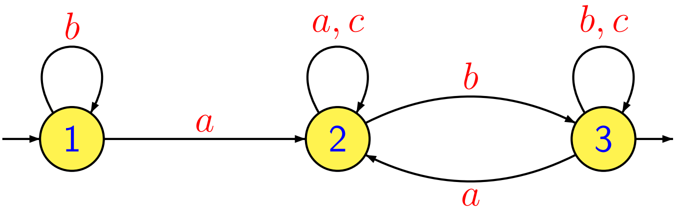
$cb = bc$
$cc = c$

Transformation monoid of an automaton



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$ac = a$
$ba = a$
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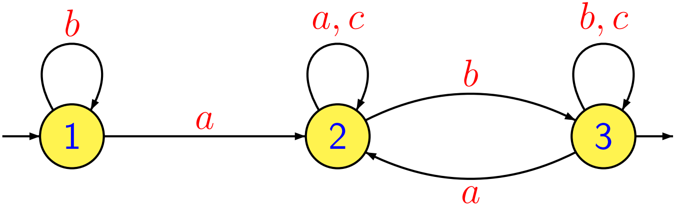
$cc = c$

$abc = ab$

Transformation monoid of an automaton



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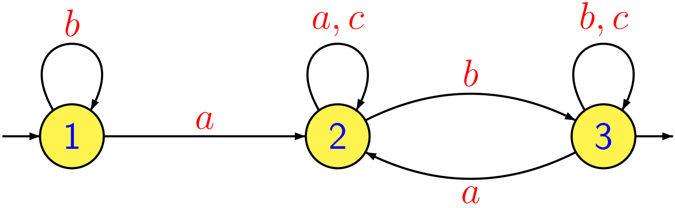
$abc = ab$
$bca = ca$

Transformation monoid of an automaton



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ca	-	2	2

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$ac = a$
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$abc = ab$
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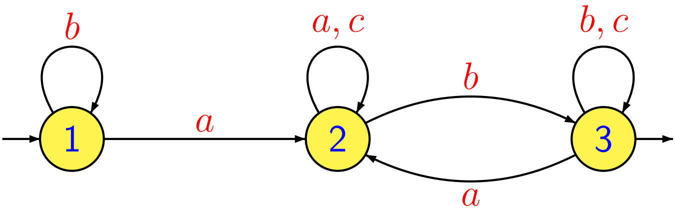
Transformation monoid of an automaton



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ca	-	2	2

Relations:
$aa = a$
$ac = a$
$ba = a$
$bb = b$
$cb = bc$
$cc = c$

$abc = ab$
$bca = ca$
$cab = bc$
The end!

Recognizing by a morphism

Definition

Let M be a monoid and let L be a language of A^*. Then M recognizes L if there exists a monoid morphism $\varphi : A^* \to M$ and a subset P of M such that $L = \varphi^{-1}(P)$.

Proposition

A language is recognized by a finite monoid iff it is recognized by a finite deterministic automaton.

Syntactic monoid

Definition (algorithmic)

The syntactic monoid of a language is the transition monoid of its minimal automaton.

Definition (algebraic)

The syntactic monoid of a language $L \subset A^*$ is the quotient monoid of A^* by the syntactic congruence of L: $u \sim_L v$ iff, for each $x, y \in A^*$, $xvy \in L \Leftrightarrow xuy \in L$

Part VII

The wreath product principle

The wreath product $N \circ K$ of two monoids N and K is defined on the set $N^K \times K$ by the following product:

$$(f_1, k_1)(f_2, k_2) = (f, k_1k_2) \text{ with } f(k) = f_1(k)f_2(kk_1)$$

Straubing's wreath product principle provides a description of the languages recognized by the wreath product of two automata (or monoids).

The wreath product principle

Proposition

Let M and N be monoids. Every language of A^* recognized by $M \circ N$ is a finite union of languages of the form $U \cap \sigma_\varphi^{-1}(V)$, where $\varphi : A^* \to N$ is a monoid morphism, U is a language of A^* recognized by φ and V is a language of B_N^* recognized by M.

The wreath product principle 2

Theorem

Let $L \subseteq A^*$ be a language recognized by an wreath product of the form $(P, Q \times A) \circ (Q, A)$. Then L is a finite union of languages of the form $W \cap \sigma^{-1}(V)$, where $W \subseteq A^*$ is recognized by (Q, A), σ is a $\mathcal{C}$-sequential function associated with the action (Q, A) and $V \subseteq (Q \times A)^*$ is recognized by $(P, Q \times A)$.

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C. Choffrut, A generalization of Ginsburg and Rose's characterization of gsm mappings, in *Automata, Languages and Programming*, H. A. Maurer (éd.), pp. 88–103, *Lecture Notes in Comput. Sci.* vol. 71, Springer, 1979.

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- S. Ginsburg and G. F. Rose, A characterization of machine mappings, *Canad. J. Math.* **18** (1966), 381–388.