MPRI, Fondations mathématiques de la théorie des automates

Olivier Carton, Jean-Éric Pin

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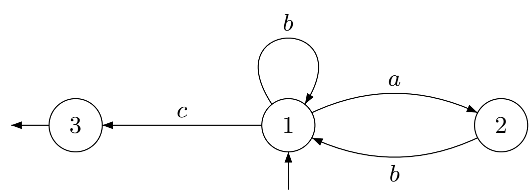
Avertissement : On attachera une grande importance à la clarté, à la précision et à la concision de la rédaction.

Partie 1. Étude d'un langage.

Soit $A = \{a, b, c\}$. On considère l'automate $\mathcal{A}$ représenté ci-dessous



et soit L le langage reconnu par $\mathcal{A}$.

Question 1. Donner une expression rationnelle pour L.

Question 2. Calculer le monoïde syntactique M de L. On donnera la liste des éléments et des relations permettant de définir M (vous devriez trouver 8 éléments, en comptant l'élément neutre).

Question 3. Donner la liste des idempotents de M.

Question 4. Déterminer la structure en $\mathcal{D}$-classes de M et dessiner les diagrammes boîtes à œufs.

Question 5. Montrer que le langage L est sans-étoile.

Question 6. Donner une expression sans étoile pour le langage L.

Question 7. Donner une formule du premier ordre définissant L (on pourra introduire des sous-formules si nécessaire).

Partie 2. Étude d'un monoïde.

Soit $\mathcal{A} = (Q, A, \cdot\,)$ un automate fini. Si u est un mot de A^*, on pose $Q \cdot u = \{q \cdot u \mid q \in Q\}$ et le *rang* de u est l'entier $r(u) = |Q \cdot u|$. On note M le monoïde de transitions de $\mathcal{A}$.

Question 8. Montrer que pour tout $u, v \in A^*$, $r(uv) \leqslant \min\{r(u), r(v)\}$.

Question 9. Montrer que si dans M, $u \mathcal{J} v$, alors $r(u) = r(v)$.

On s'intéresse désormais à l'exemple suivant. On prend $A = \{a, b\}$, $Q = \{0, 1, 2, 3, 4\}$ et si les transitions sont données par le tableau suivant:

	1	2	3	4	0
a	3	3	0	2	0
b	4	4	1	4	0

On a donc $r(a) = r(b) = 3$ puisque $Q \cdot a = \{0, 2, 3\}$ et $Q \cdot b = \{0, 1, 4\}$.

Question 10. Montrer que si $r(u) = 1$, alors u est un zéro de M.

Question 11. Montrer que si $r(u) \leqslant 2$ et si u est élément d'un groupe, alors ce groupe est trivial.

Question 12. Montrer que dans M, $aba = a$ et $bab = b$ et que $r(a^2) = r(b^2) = 2$.

Question 13. En déduire (sans calculer tout le monoïde M!) que tous les groupes de M sont triviaux.

Question 14. En déduire que M est apériodique.

MPRI, Mathematical foundations of automata theory

Olivier Carton, Jean-Éric Pin

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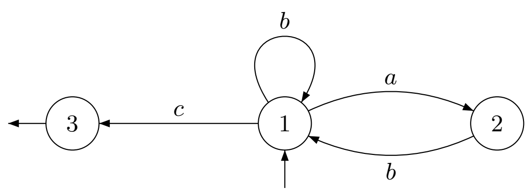
Warning : Clearness, accuracy and concision of the writing will be rewarded.

Part 1. Study of a language

Let $A = \{a, b, c\}$. Let us consider the automaton $\mathcal{A}$ represented below:



Let L be the language recognised by $\mathcal{A}$.

Question 1. Give a rational expression for L.

Question 2. Compute the syntactic monoid M of L. Give the list of elements and the defining relations of M. (Hint: you should find 8 elements, including the identity).

Question 3. Give the list of all idempotents of M.

Question 4. Give the $\mathcal{D}$-class structure of M and draw the corresponding egg-box pictures.

Question 5. Show that L is star-free.

Question 6. Give a star-free expression for L.

Question 7. Give a first order sentence defining L (you can introduce subformulas if needed).

Part 2. Study of a monoid.

Let $\mathcal{A} = (Q, A, \cdot)$ be a finite automaton. If u is a word of A^*, let $Q \cdot u = \{q \cdot u \mid q \in Q\}$. The *rank* of u is the integer $r(u) = |Q \cdot u|$. We denote M the transition monoid of $\mathcal{A}$.

Question 8. Show that for all $u, v \in A^*$, $r(uv) \leqslant \min\{r(u), r(v)\}$.

Question 9. Show that if in M, $u \mathrel{\mathcal{J}} v$, then $r(u) = r(v)$.

One now consider the following example. Let $A = \{a, b\}$, $Q = \{0, 1, 2, 3, 4\}$ and the transitions are given by the following table:

	1	2	3	4	0
a	3	3	0	2	0
b	4	4	1	4	0

Then $r(a) = r(b) = 3$ since $Q \cdot a = \{0, 2, 3\}$ and $Q \cdot b = \{0, 1, 4\}$.

Question 10. Show that if $r(u) = 1$, then u is a zero of M.

Question 11. Show that if $r(u) \leqslant 2$ and if u is element of a group, then this group is trivial.

Question 12. Show that in M, $aba = a$ and $bab = b$. Show also that $r(a^2) = r(b^2) = 2$.

Question 13. Conclude (without computing the whole monoid M!) that all groups in M are trivial.

Question 14. Conclude that M is aperiodic.

Solution

Question 1. $L = (ab \cup b)^*c$.

Question 2. The syntactic monoid of L is

	1	2	3
$* \ 1$	1	2	3
a	2	0	0
$* \ b$	1	1	0
c	3	0	0
$* \ a^2$	0	0	0
$* \ ab$	1	0	0
$* \ ba$	2	2	0
bc	3	3	0

Relations:

$$ac = 0 \qquad b^2 = b \qquad a^2 = ca = cb = c^2 = 0 \qquad aba = a \qquad abc = c \qquad bab = b$$

Question 3. Idempotents:

$$E(S) = \{1, b, a^2, ab, ba\}$$

Question 4. $\mathcal{D}$-classes:

^{*1}

a	$^*ab$
$^*ba$	$^*b$

c
bc

$^*a^2$

Question 5. Since the syntactic monoid of L is aperiodic, L is star-free.

Question 6. One has $L = \big(\{1\} \cup (A^*b - A^*aaA^*)\big)c$. Thus a star-free expression for L is

$$\big(\{1\} \cup (\emptyset^c b - \emptyset^c aa\emptyset^c)\big)c.$$

Question 7. Let us introduce the following macros

$$
\begin{aligned}
x \leqslant y &: (x < y) \vee (x = y) \\
\max &: \exists x \ \forall y \ (y \leqslant x) \\
(y = Sx) &: \exists y \ (x < y) \wedge (\forall z \ (x < z) \rightarrow (y \leqslant z)) \\
\text{No\_}\mathbf{aa} &: \forall x \ \neg(\mathbf{a}x \wedge \mathbf{a}Sx)
\end{aligned}
$$

Then L can be defined by the first order formula

$$\mathbf{c}\max \wedge \text{No\_}\mathbf{aa} \wedge (\forall x \ ((Sx = \max) \rightarrow \mathbf{b}x))$$

Part 2. Study of a monoid.

Question 8. One has $Q\cdot uv = (Q\cdot u)\cdot v \subseteq Q\cdot v$. Thus $r(uv) \leqslant r(v)$. Moreover, $|(Q\cdot u)\cdot v| \leqslant |(Q\cdot u)|$ and thus $r(uv) \leqslant r(u$.

Question 9. Since $0 \in Q\cdot u$ for all u, the condition $r(u) = 1$ is equivalent to $Q\cdot u = \{0\}$ and thus u is a zero.

Question 10. If $u \leqslant_{\mathcal{J}} v$, there exist $x, y \in M$ such that $u = xvy$. It follows by Question 8 that $r(u) \leqslant r(v)$. Thus if $u \mathrel{\mathcal{J}} v$, then $r(u) = r(v)$.

Question 11. Let u be a group element of rank $\leqslant 2$. If $r(u) = 1$, then u is a zero and its $\mathcal{H}$-class is trivial. Otherwise $Q\cdot u = \{q, 0\}$ for some $q \in Q - 0$. Note that $Q\cdot u^2 \subseteq Q\cdot u$. Thus either $Q\cdot u^2 = \{0\}$ and the $\mathcal{H}$-class of u is not a group, or $Q\cdot u^2 = \{q, 0\}$ and then $q\cdot u = q$ and $u = u^2$. Thus u is idempotent.

Question 12. A direct computation that the relations $aba = a$ and $bab = b$ hold in M. Moreover $Q\cdot a^2 = \{0, 3\}$ and $Q\cdot b^2 = \{0, 4\}$ and thus $r(a^2) = r(b^2) = 2$.

Question 13. First, all words having a^2 or b^2 as a factor have rank $\leqslant 2$. The only words of rank > 2 are thus 1, a, b, ab and ba (since $aba = a$ and $bab = b$). The $\mathcal{D}$-class structure of these elements is

*1

*ba	b
a	*ab

and it contains no nontrivial group. By Question 11, the words of rank $\leqslant 2$ generate no nontrivial group. Thus M contains no nontrivial group.

Question 14. It follows that M is aperiodic.