Upper set monoids and length preserving morphisms[*]

Antonio Cano[1] and Jean-Éric Pin[2]

October 22, 2011

Abstract

Length preserving morphisms and inverse of substitutions are two well-studied operations on regular languages. Their connection with varieties generated by power monoids was established independently by Reutenauer and Straubing in 1979. More recently, an ordered version of this theory was proposed by Polák and by the authors. In this paper, we present an improved version of these results and obtain the following consequences. Given a variety of finite ordered monoids $\mathbf{V}$, let $\mathbf{P}^{\uparrow}\mathbf{V}$ be the variety of finite ordered monoids generated by the upper set monoids of members of $\mathbf{V}$. Then $\mathbf{P}^{\uparrow}(\mathbf{P}^{\uparrow}\mathbf{V}) = \mathbf{P}^{\uparrow}\mathbf{V}$. This contrasts with the known results for the unordered case: the operator $\mathbf{PV}$ corresponding to power monoids satisfies $\mathbf{P}^3\mathbf{V} = \mathbf{P}^4\mathbf{V}$, but the varieties $\mathbf{V}$, $\mathbf{PV}$, $\mathbf{P}^2\mathbf{V}$ and $\mathbf{P}^3\mathbf{V}$ can be distinct.

All semigroups considered in this paper are either finite, free or profinite. In particular, we use the term *variety of monoids* for *variety of finite monoids*.

Warning. This paper introduces some change of terminology and notation, compared to the existing literature. We believe that this new terminology is an improvement over the previous one, but it is fair to warn the reader of these changes.

1 Introduction

Power monoids and power varieties (varieties of the form $\mathbf{PV}$) are the topic of numerous articles [1, 2, 3, 4, 6, 8, 9, 10, 11, 12, 13, 14, 15, 20, 21] and many other references can be found in Almeida's remarkable survey [3]. Initially, the

---

[1]Departamento de Sistemas Informáticos y Computación, Universidad Politécnica de Valencia, Camino de Vera s/n, P.O. Box: 22012, E-46020 - Valencia.

[2]LIAFA, Université Paris-Diderot and CNRS, Case 7014, 75205 Paris Cedex 13, France.

study of power varieties was partly motivated by semigroup theoretic questions and partly by applications to language theory. By the way, several results on power varieties were first established by using arguments of language theory.

The key result in this direction, proved independently by Reutenauer [20] and Straubing [21] in 1979, establishes a surprising link between power varieties and two natural operations on regular languages: length preserving morphisms and inverse of substitutions. This result can be summarized as follows. Let $\mathbf{V}$ be a variety of monoids and let $\mathcal{V}$ be the corresponding variety of languages. Also let $\mathcal{PV}$ be the variety of languages corresponding to $\mathbf{PV}$. Then, for each alphabet A, $\mathcal{PV}(A^*)$ is the Boolean algebra generated by the set $\Lambda\mathcal{V}(A^*)$ of all languages of the form $\varphi(L)$, where φ is a length preserving morphism from B^* into A^* and L is a language of $\mathcal{V}(B^*)$. There is an analogous result for inverses of substitutions.

The extension of Eilenberg's variety theorem to ordered monoids [16] called for a generalization of Reutenauer's and Straubing's result to the ordered case. Such an extension was proposed by Polák [19, Theorem 4.2] and by the authors [5, 6]. First, as shown by Polák, the upper set monoid is the proper extension of the notion of power monoid to the ordered case. Let $\mathbf{P}^{\uparrow}$ be the extension of this operator to varieties of ordered monoids. Denoting by $\mathcal{P}^{\uparrow}\mathcal{V}$ the positive variety of languages corresponding to $\mathbf{P}^{\uparrow}\mathbf{V}$, it was no surprise to see that $\mathcal{P}^{\uparrow}\mathcal{V}(A^*)$ is the closure under union and intersection of $\Lambda\mathcal{V}(A^*)$.

The main result of this paper, Theorem 5.1, is an improvement of this result. It shows that $\mathcal{P}^{\uparrow}\mathcal{V}(A^*)$ is actually equal to $\Lambda\mathcal{V}(A^*)$. In particular, although $\mathcal{P}^{\uparrow}\mathcal{V}$ is closed under intersection, its description does not require this operation. As it stands, this result looks like a rather minor improvement, but this is not the case. Indeed, a major consequence of our result is that the operator $\mathbf{P}^{\uparrow}$ is idempotent. This contrasts with the corresponding result for the operator $\mathbf{P}$, which satisfies $\mathbf{P}^3 = \mathbf{P}^4$, but $\mathbf{P}^2 \neq \mathbf{P}^3$.

2   Notation and background

In this section, we briefly recall some basic facts about ordered monoids and profinite words. More details can be found in [18] for ordered monoids and in [1, 17] for profinite words.

2.1   Ordered monoids

An *ordered monoid* is a monoid M equipped with a partial order $\leqslant$ compatible with the product on M: for all $x, y, z \in M$, if $x \leqslant y$ then $zx \leqslant zy$ and $xz \leqslant yz$. Given two ordered monoids M and N, a *morphism of ordered monoids* $\varphi \colon M \to N$ is an order-preserving monoid morphism from M into N.

An ordered monoid M is a *quotient* of an ordered monoid R if there exists a surjective morphism of ordered monoids from R onto M. An *ordered submonoid* of M is a submonoid of M, equipped with the restriction of the order on M. Let M and N be ordered monoids. Then M *divides* N if M is a quotient of an ordered submonoid of N.

The *product* of a family $(M_i)_{i \in I}$ of ordered monoids, is the product monoid $\prod_{i \in I} M_i$ equipped with the product order given by

$$(s_i)_{i \in I} \leqslant (s_i')_{i \in I} \text{ if and only if, for all } i \in I, \ s_i \leqslant s_i'.$$

A *variety of monoids* is a class of monoids closed under taking submonoids, quotients and finite direct products [7]. Equivalently, a variety of monoids is a class of monoids closed under division and finite direct products. *Varieties of ordered monoids* are defined analogously [16].

2.2 Profinite words

Let A be a finite alphabet. The set of profinite words is defined as the completion of the free monoid A^* for a certain metric.

A finite monoid M *separates* two words u and v of A^* if there is a monoid morphism $\varphi : A^* \to M$ such that $\varphi(u) \neq \varphi(v)$. One can show that any pair of distinct words of A^* can be separated by a finite monoid.

Given two words $u, v \in A^*$, we set

$$r(u,v) = \min \{ |M| \mid M \text{ is a monoid that separates } u \text{ and } v \}$$

We also set $d(u,v) = 2^{-r(u,v)}$, with the usual conventions $\min \emptyset = +\infty$ and $2^{-\infty} = 0$. Then d is a metric and the completion of the metric space (A^*, d) is the set of *profinite words* on the alphabet A. Since the product of two words is a uniformly continuous function from $A^* \times A^*$ to A^*, it can be extended by continuity (in a unique way) to profinite words. The resulting topological monoid, denoted $\widehat{A^*}$, is called the *free profinite monoid* on A. It is a compact monoid. It is a well known fact that, in a compact monoid, the smallest closed subsemigroup containing a given element s has a unique idempotent, denoted s^ω.

One can show that every morphism φ from A^* onto a (discrete) finite monoid M extends uniquely to a uniformly continuous morphism from $\widehat{A^*}$ onto M. It follows that if x is a profinite word and $s = \varphi(x)$, then $\varphi(x^\omega) = s^\omega$.

For instance, the set of subsets of A is a monoid under union and the function which maps a word u onto the set of letters occurring in u is a continuous morphism, which can be extended by continuity to profinite words. The resulting map is called the *content* mapping and is denoted by c. For example, $c(abab) = \{a,b\} = c(((ab)^\omega (ba)^\omega)^\omega)$.

3 Upper set monoids

Let $(M, \leqslant)$ be an ordered monoid. A *lower set* of M is a subset E of M such that if $x \in E$ and $y \leqslant x$ then $y \in E$. An *upper set* of M is a subset F of M such that if $x \in F$ and $x \leqslant y$ then $y \in F$. Note that a subset of M is an upper set if and only if its complement is a lower set. Given an element s of M, the set

$$\uparrow s = \{ t \in M \mid s \leqslant t \}$$

is an upper set, called the *upper set generated by s*. More generally, if X is a subset of M, *the upper set generated by X* is the set

$$\uparrow X = \bigcup_{s \in X} \uparrow s$$

The *product of two upper sets* X and Y is the upper set

$$XY = \{ z \in M \mid \text{there exist } x \in X \text{ and } y \in Y \text{ such that } xy \leqslant z \}$$

This operation makes the set of upper sets of M a monoid, denoted by $\mathcal{P}^\uparrow(M)$ and called the *upper set monoid* of M. The identity element is $\uparrow 1$ and the empty set is a zero of $\mathcal{P}^\uparrow(M)$. If we omit this zero, we get the submonoid $\mathcal{P}_\star^\uparrow(M)$ of *nonempty upper sets* of M.

Let us define a relation $\leqslant$ on $\mathcal{P}^\uparrow(M)$ by setting $X \leqslant Y$ if and only if $Y \subseteq X$. In particular, one gets $X \leqslant \emptyset$ for any upper set X. We just mention for the record that the other natural way to define an order on $\mathcal{P}^\uparrow(M)$ yields exactly the same definition.

Proposition 3.1 *One has $X \leqslant Y$ if and only if, for each $y \in Y$, there exists $x \in X$ such that $x \leqslant y$.*

Proof. Suppose that $Y \subseteq X$. Then the condition of the statement is clearly satisfied by taking $x = y$. Conversely, suppose that this condition is satisfied and let $y \in Y$. Then there exists an element $x \in X$ such that $x \leqslant y$. Since X is an upper set, y is also in X and thus Y is a subset of X. $\square$

Therefore, the monoids $\mathcal{P}^\uparrow(M)$ and $\mathcal{P}_\star^\uparrow(M)$ are ordered monoids and they will be considered as such in the remainder of this paper.

Example 3.1 Let U_1 be the monoid $\{0,1\}$ under the usual multiplication of integers. We denote by U_1^+ the ordered monoid defined by the order $0 \leqslant 1$ and by U_1^- the ordered monoid defined by the order $1 \leqslant 0$. Then $\mathcal{P}_\star^\uparrow(U_1^-)$ has two elements ($\{0,1\}$ and $\{0\}$) and is isomorphic to U_1^-. Similarly, $\mathcal{P}_\star^\uparrow(U_1^+)$ has two elements ($\{0,1\}$ and $\{1\}$) and is isomorphic to U_1^+.

Example 3.2 Let 1 be the trivial monoid. Then $\mathcal{P}_\star^\uparrow(1) = 1$ and $\mathcal{P}^\uparrow(1) = U_1^-$.

The next three propositions were proved in [6].

Proposition 3.2 *Let M be an ordered monoid. Then M is a submonoid of $\mathcal{P}_\star^\uparrow(M)$, $\mathcal{P}_\star^\uparrow(M)$ is a submonoid of $\mathcal{P}^\uparrow(M)$ and $\mathcal{P}^\uparrow(M)$ is a quotient of $U_1^- \times \mathcal{P}_\star^\uparrow(M)$.*

Proposition 3.3 *Let M_1 and M_2 be two ordered monoids. Then the ordered monoid $\mathcal{P}_\star^\uparrow(M_1) \times \mathcal{P}_\star^\uparrow(M_2)$ is an ordered submonoid of $\mathcal{P}_\star^\uparrow(M_1 \times M_2)$.*

Note that the corresponding result for $\mathcal{P}^\uparrow(M)$ does not hold. Indeed, if M_1 and M_2 are the trivial monoid, $M_1 \times M_2$ is also the trivial monoid, but $\mathcal{P}^\uparrow(M_1) = \mathcal{P}^\uparrow(M_2) = \mathcal{P}^\uparrow(M_1 \times M_2) = U_1^-$ and thus $\mathcal{P}^\uparrow(M_1) \times \mathcal{P}^\uparrow(M_2)$ is not an ordered submonoid of $\mathcal{P}^\uparrow(M_1 \times M_2)$.

Proposition 3.4 *Let M be an ordered monoid and let X be a nonempty upper set of M. Then the upper subset X^ω is a semigroup and for each $x \in X^\omega$, there is an idempotent e of X^ω such that $x \leqslant_{\mathcal{J}} e$ in X^ω.*

We also need two elementary facts on lower sets.

Proposition 3.5 *Let M be an ordered monoid, let S be a lower set of M and let E be a subset of M. Then the conditions $E \cap S \neq \emptyset$ and $\uparrow E \cap S \neq \emptyset$ are equivalent.*

Proof. Since E is contained in $\uparrow E$, it suffices to prove that if S meets $\uparrow E$, then it also meets E. Let $x \in \uparrow E \cap S$. Then there exists an element $y \in E$ such that $y \leqslant x$. Since S is a lower set, one gets $y \in S$ and thus S meets E. $\square$

Proposition 3.6 *Let M be an ordered monoid and let S be a lower set of M. Then the set $\{X \in \mathcal{P}^{\uparrow}(M) \mid X \cap S \neq \emptyset\}$ is a lower set of $\mathcal{P}^{\uparrow}(M)$.*

Proof. Suppose that $X \cap S \neq \emptyset$ and $Y \leqslant X$. Then $X \subseteq Y$ and thus $Y \cap S \neq \emptyset$. $\square$

Given a variety of ordered monoids $\mathbf{V}$, we denote by $\mathbf{P}^{\uparrow}\mathbf{V}$ [$\mathbf{P}_{\star}^{\uparrow}\mathbf{V}$] the variety of ordered monoids generated by the monoids of the form $\mathcal{P}^{\uparrow}(M)$ [$\mathcal{P}_{\star}^{\uparrow}(M)$], where $M \in \mathbf{V}$. A slightly more precise description of $\mathbf{P}_{\star}^{\uparrow}\mathbf{V}$ is given in the next proposition, which is an immediate consequence of Proposition 3.3.

Proposition 3.7 *Let $\mathbf{V}$ be a variety of ordered monoids. An ordered monoid belongs to $\mathbf{P}_{\star}^{\uparrow}\mathbf{V}$ if and only if it divides a monoid $\mathcal{P}_{\star}^{\uparrow}(M)$, with $M \in \mathbf{V}$.*

Proof. Recall that an ordered monoid belongs to the variety generated by a class $\mathcal{C}$ of ordered monoids if and only if it divides a product of members of $\mathcal{C}$. The statement is therefore a consequence of Proposition 3.3. $\square$

The varieties $\mathbf{P}^{\uparrow}\mathbf{V}$ and $\mathbf{P}_{\star}^{\uparrow}\mathbf{V}$ are related as follows.

Proposition 3.8 *Let $\mathbf{V}$ be a variety of ordered monoids. Then $\mathbf{P}_{\star}^{\uparrow}\mathbf{V}$ is a subvariety of $\mathbf{P}^{\uparrow}\mathbf{V}$. Further, if $\mathbf{V}$ contains U_1^-, then $\mathbf{P}^{\uparrow}\mathbf{V} = \mathbf{P}_{\star}^{\uparrow}\mathbf{V}$.*

Proof. This is an immediate consequence of Proposition 3.2. $\square$

4 Two operations on languages

Let L be a regular language of A^*. An ordered monoid M *recognizes* L if there exists a morphism $\eta : A^* \to M$ and a lower set S of M such that $L = \eta^{-1}(S)$.

In this section, we show how two operations on languages, length preserving morphisms and inverses of substitutions, are related to upper set monoids. These results are well known in the unordered case [20, 21] and were proved in [5, 6, 19]) in a slightly different way. Let us first define our two operations.

A *length preserving morphism* is a morphism φ from A^* into B^*, such that, for each word u, the words u and $\varphi(u)$ have the same length. It is equivalent to require that, for each letter a, $\varphi(a)$ is also a letter, that is, $\varphi(A) \subseteq B$.

A *substitution* σ from A^* into B^* is a morphism from A^* into the monoid $\mathcal{P}(B^*)$ of subsets of B^*. Note that the languages $\sigma(a)$, for $a \in A$, completely determine σ. Indeed, one has $\sigma(1) = \{1\}$ and for every nonempty word $a_1 \cdots a_n$, $\sigma(a_1 \cdots a_n) = \sigma(a_1) \cdots \sigma(a_n)$.

Considered as a relation, σ has an inverse which maps a language K of A^* to the language $\sigma^{-1}(K)$ of B^* defined by

$$\sigma^{-1}(K) = \{u \in A^* \mid \sigma(u) \cap K \neq \emptyset\}$$

We shall also consider two restrictions of these operations: *surjective length preserving morphisms* and *nonempty substitutions*.

There is an obvious connection between length preserving morphisms and substitutions. If $\varphi : A^* \to B^*$ is a [surjective] length preserving morphism, then the relation $\varphi^{-1} : B^* \to A^*$ is a [nonempty] substitution σ such that $\sigma^{-1} = \varphi$. We shall see in Section 5 that there is an even tighter connection between these operations. For now, we establish a first link with upper set monoids.

Proposition 4.1 *Let L be a language of B^* recognized by an ordered monoid M and let $\sigma : A^* \to B^*$ be a substitution [nonempty substitution]. Then $\sigma^{-1}(L)$ is recognized by $\mathcal{P}^\uparrow(M)$ $[\mathcal{P}_\star^\uparrow(M)]$.*

Proof. Since M recognizes L, there is a monoid morphism $\eta : B^* \to M$ and a lower set P of M such that $L = \eta^{-1}(P)$. Define a map $\psi : A^* \to \mathcal{P}^\uparrow(M)$ by setting $\psi(u) = \uparrow\eta(\sigma(u))$. The definition of the product of two upper sets implies that ψ is a morphism. By Proposition 3.6, the set

$$\mathcal{X} = \{X \in \mathcal{P}^\uparrow(M) \mid X \cap P \neq \emptyset\}$$

is a lower set of $\mathcal{P}^\uparrow(M)$. Furthermore, one has

$$
\begin{aligned}
\psi^{-1}(\mathcal{X}) &= \{u \in A^* \mid \psi(u) \cap P \neq \emptyset\} \\
&= \{u \in A^* \mid \uparrow\eta(\sigma(u)) \cap P \neq \emptyset\} \\
&= \{u \in A^* \mid \eta(\sigma(u)) \cap P \neq \emptyset\} \quad \text{(by Proposition 3.5)} \\
&= \{u \in A^* \mid \sigma(u) \cap \eta^{-1}(P) \neq \emptyset\} \\
&= \{u \in A^* \mid \sigma(u) \cap L \neq \emptyset\} \\
&= \sigma^{-1}(L).
\end{aligned}
$$

Thus $\sigma^{-1}(L)$ is recognized by $\mathcal{P}^\uparrow(M)$.

If σ is a nonempty substitution, it suffices to replace every occurrence of $\mathcal{P}^\uparrow(M)$ by $\mathcal{P}_\star^\uparrow(M)$ to get the proof. $\square$

Since a length preserving morphism is a special case of substitution, we get as a corollary:

Corollary 4.2 *Let L be a language of A^* recognized by an ordered monoid M and let $\varphi : A^* \to B^*$ be a [surjective] length preserving morphism. Then $\varphi(L)$ is recognized by $\mathcal{P}^\uparrow(M)$ $[\mathcal{P}_\star^\uparrow(M)]$.*

5 Main result

Let us first recall a few definitions. A *class of languages* is a correspondence $\mathcal{C}$ which associates with each alphabet A a set $\mathcal{C}(A^*)$ of regular languages of A^*. A *positive variety of languages* is a class of regular languages $\mathcal{V}$ such that:

(1) for every alphabet A, $\mathcal{V}(A^*)$ is closed under union and intersection,

(2) if $\varphi\colon A^* \to B^*$ is a morphism, $L \in \mathcal{V}(B^*)$ implies $\varphi^{-1}(L) \in \mathcal{V}(A^*)$,

(3) if $L \in \mathcal{V}(A^*)$ and $u \in A^*$, then $u^{-1}L$ and Lu^{-1} are in $\mathcal{V}(A^*)$.

A *variety of languages* is a positive variety closed under complement.

There is a one to one correspondence between varieties of finite monoids (resp. varieties of finite ordered monoids) and varieties of languages (resp. positive varieties of languages) [7, 16].

Let $\mathcal{V}$ be a positive variety of languages. For each alphabet A, we denote by $\Lambda\mathcal{V}(A^*)$ [$\Lambda'\mathcal{V}(A^*)$] the set of all languages of A^* of the form $\varphi(K)$, where φ is a [surjective] length preserving morphism from B^* to A^* and K is a language of $\mathcal{V}(B^*)$.

Similarly, we denote by $\Sigma\mathcal{V}(A^*)$ [$\Sigma'\mathcal{V}(A^*)$] the set of all languages of A^* of the form $\sigma^{-1}(K)$, where σ is a [nonempty] substitution from A^* into B^* and K is a language of $\mathcal{V}(B^*)$.

Let $\mathbf{V}$ be a variety of ordered monoids. A description of the variety of languages corresponding to $\mathbf{P}^{\uparrow}\mathbf{V}$ was given by Polák [19, Theorem 4.2] and by the authors [5] and [6, Proposition 6.3]. Our main theorem gives a stronger form of these results. Indeed, contrary to the previous results, our description does not require intersection.

Theorem 5.1 *Let $\mathbf{V}$ be a variety of ordered monoids, and let $\mathcal{V}$ by the corresponding positive variety of languages. Then the positive variety of languages corresponding to $\mathbf{P}^{\uparrow}\mathbf{V}$ is equal to $\Lambda\mathcal{V}$ and to $\Sigma\mathcal{V}$.*

The proof relies on an improvement of a result of [5, 6, 19], which itself extends to positive varieties an argument of [20, 21]. However, the algebraic encoding of intersection and union makes our proof more technical than that of the weaker versions.

Proposition 5.2 *Let M be an ordered monoid of $\mathbf{V}$ and L be a language of A^* recognized by $\mathcal{P}^{\uparrow}(M)$. Then L belongs to $\Lambda\mathcal{V}(A^*)$.*

Proof. Let M be an ordered monoid of $\mathbf{V}$ and let L be a language of A^* recognized by $\mathcal{P}^{\uparrow}(M)$. Then there exists a morphism $\psi : A^* \to \mathcal{P}^{\uparrow}(M)$ and a lower set $\mathcal{S}$ of $\mathcal{P}^{\uparrow}(M)$ such that $L = \psi^{-1}(\mathcal{S})$. Since $\mathcal{S}$ is a lower set, one has $\mathcal{S} = \bigcup_{Z \in \mathcal{S}} {\downarrow}Z$ and hence

$$L = \psi^{-1}(\mathcal{S}) = \bigcup_{Z \in \mathcal{S}} \psi^{-1}({\downarrow}Z)$$

Further, one has

$$
\begin{aligned}
\psi^{-1}({\downarrow}Z) &= \{w \in A^* \mid \psi(w) \leqslant Z\} \\
&= \{w \in A^* \mid \text{ for every } z \in Z \text{ there exists } t \in \psi(w) \text{ such that } t \leqslant z\} \\
&= \{w \in A^* \mid \text{ for every } z \in Z, \psi(w) \cap {\downarrow}z \neq \emptyset\}
\end{aligned}
$$

Setting $X_z = \{w \in A^* \mid \psi(w) \cap {\downarrow}z \neq \emptyset\}$ for each $z \in Z$, we get

$$\psi^{-1}({\downarrow}Z) = \bigcap_{z \in Z} X_z$$

and finally

$$L = \bigcup_{Z \in \mathcal{S}} \bigcap_{z \in Z} X_z \tag{1}$$

Let U be the disjoint union of the sets Z, for $Z \in \mathcal{S}$ and let $N = M^U$. An element of N is a family $(m_z)_{z \in U}$ which can also be written as $\left((m_z)_{z \in Z} \right)_{Z \in \mathcal{S}}$.

Consider the lower set $\mathcal{J}$ of N defined by

$$\mathcal{J} = \{ \left((m_z)_{z \in Z} \right)_{Z \in \mathcal{S}} \in N \mid \text{ for some } Z_0 \in \mathcal{S}, \text{ for all } z \in Z_0, m_z \leqslant z \}$$

and the alphabet

$$B = \{ (a, n) \in A \times N \mid n = \left((m_z)_{z \in Z} \right)_{Z \in \mathcal{S}} \text{ where each } m_z$$
$$\text{is a minimal element of } \psi(a) \}$$

Let us define a length preserving morphism $\varphi : B^* \to A^*$ by $\varphi(a, n) = a$ and a morphism $\eta : B^* \to N$ by $\eta(a, n) = n$. Let $K = \eta^{-1}(\mathcal{J})$. Since N belongs to $\mathbf{V}$ by construction, K is a language of $\mathcal{V}(B^*)$.

We claim that $\varphi(K) = L$. First, if $a_1 \ldots a_k \in L$, there exists by (1) an element $Z_0 \in \mathcal{S}$ such that

$$a_1 \cdots a_k \in \bigcap_{z \in Z_0} X_z$$

Thus, for every $z \in Z_0$, one has $\psi(a_1 \cdots a_k) \cap \downarrow z \neq \emptyset$ and there are some elements $y_{1,z} \in \psi(a_1), \ldots, y_{k,z} \in \psi(a_k)$ such that $y_{1,z} \cdots y_{k,z} \leqslant z$. For every $z \in Z_0$, let us choose a minimal element $x_{j,z} \in \psi(a_j)$ such that $x_{j,z} \leqslant y_{j,z}$. For each $Z \neq Z_0$ in $\mathcal{S}$ and for each $z \in Z$, we also choose a minimal element $x_{j,z}$ of $\psi(a_j)$. Let $b_j = (a_j, n_j)$ be the letter of B defined by $n_j = \left((x_{j,z})_{z \in Z} \right)_{Z \in \mathcal{S}}$. Setting $v = b_1 \cdots b_k$, we get $\varphi(v) = a_1 \cdots a_k$ by construction. Furthermore, $\eta(v) = n_1 \cdots n_k$. We are only interested in the component in M^{Z_0}, whose value is $(x_{1,z})_{z \in Z_0} \cdots (x_{k,z})_{z \in Z_0}$. Now, for $z \in Z_0$, we have by definition

$$(x_{1,z})_{z \in Z_0} \cdots (x_{k,z})_{z \in Z_0} \leqslant (y_{1,z})_{z \in Z_0} \cdots (y_{k,z})_{z \in Z_0}$$

Since for each $z \in Z_0$, one has $y_{1,z} \cdots y_{k,z} \leqslant z$, we get $\eta(v) \in \mathcal{J}$, which shows that $v \in K$ and that $L \subseteq \varphi(K)$.

To prove the opposite inclusion, consider a word $v = b_1 \cdots b_k$ of K. Let us set, for $1 \leqslant j \leqslant k$, $b_j = (a_j, n_j)$, with $n_j = \left((m_{j,z})_{z \in Z} \right)_{Z \in \mathcal{S}}$. Since $\eta(v) \in \mathcal{J}$, there exists a set $Z_0 \in \mathcal{S}$, such that, for all $z \in Z_0$, $m_{1,z} \cdots m_{k,z} \leqslant z$. By definition of B, each element $m_{j,z}$ is a minimal element of $\psi(a_j)$. Therefore, one gets $\psi(a_1 \cdots a_k) \cap \downarrow z \neq \emptyset$ for every $z \in Z_0$. It follows that $\varphi(v) = a_1 \cdots a_k \in \bigcap_{z \in Z_0} X_z$. Consequently, $a_1 \cdots a_k \in L$ and thus $\varphi(K) \subseteq L$. This proves the claim and concludes the proof of the proposition. $\square$

Note that if ψ is recognised by $\mathcal{P}_\star^\uparrow(M)$, then $\psi(a)$ is never empty and the length preserving morphism $\varphi : B^* \to A^*$ is surjective. Therefore we get the following corollary.

Corollary 5.3 *Let M be an ordered monoid of $\mathbf{V}$ and L be a language of A^* recognised by $\mathcal{P}_\star^\uparrow(M)$. Then L belongs to $\Lambda' \mathcal{V}(A^*)$.*

Let us now complete the proof of our main theorem.

Proof of Theorem 5.1. Let $\mathcal{P}^\uparrow \mathcal{V}$ be the positive variety of languages corresponding to $\mathbf{P}^\uparrow \mathbf{V}$. We prove successively the inclusions $\Lambda \mathcal{V} \subseteq \Sigma \mathcal{V} \subseteq \mathcal{P}^\uparrow \mathcal{V} \subseteq \Lambda \mathcal{V}$.

The inclusion $\Lambda\mathcal{V} \subseteq \Sigma\mathcal{V}$ stems from the fact that a length preserving morphism is a special case of inverse of substitution.

Let $L \in \mathcal{V}(B^*)$ and let $\sigma : A^* \to B^*$ be a substitution. If L is recognised by M, then, by Proposition 4.1, $\sigma^{-1}(L)$ is recognised by $\mathcal{P}^{\uparrow}(M)$ and thus $\sigma^{-1}(L) \in \mathcal{P}^{\uparrow}\mathcal{V}(A^*)$. This proves the inclusion $\Sigma\mathcal{V} \subseteq \mathcal{P}^{\uparrow}\mathcal{V}$.

Let L be a language of $\mathcal{P}^{\uparrow}\mathcal{V}(A^*)$. By definition, L is recognised by a monoid of $\mathbf{P}^{\uparrow}\mathbf{V}$. It follows from Proposition 3.3 that every ordered monoid of $\mathbf{P}^{\uparrow}\mathbf{V}$ divides an ordered monoid of the form $\mathcal{P}^{\uparrow}(M)$, with $M \in \mathbf{V}$. Therefore, L is recognised by a monoid of the form $\mathcal{P}^{\uparrow}(M)$, with $M \in \mathbf{V}$. Proposition 5.2 now shows that L belongs to $\Lambda\mathcal{V}(A^*)$. $\square$

One can use Corollary 5.3 to obtain the following description of the positive variety of languages corresponding to $\mathbf{P}_{\star}^{\uparrow}\mathbf{V}$.

Theorem 5.4 *Let* $\mathbf{V}$ *be a variety of ordered monoids, and let* $\mathcal{V}$ *by the corresponding positive variety of languages. Then the positive variety of languages corresponding to* $\mathbf{P}_{\star}^{\uparrow}\mathbf{V}$ *is equal to* $\Lambda'\mathcal{V}$ *and to* $\Sigma'\mathcal{V}$.

Theorems 5.1 and 5.4 have an important consequence.

Corollary 5.5 *For every variety of ordered monoids* $\mathbf{V}$, $\mathbf{P}^{\uparrow}(\mathbf{P}^{\uparrow}\mathbf{V}) = \mathbf{P}^{\uparrow}\mathbf{V}$ *and* $\mathbf{P}_{\star}^{\uparrow}(\mathbf{P}_{\star}^{\uparrow}\mathbf{V}) = \mathbf{P}_{\star}^{\uparrow}\mathbf{V}$.

Proof. Let $\mathcal{V}$ be the positive variety of languages corresponding to $\mathbf{V}$. Since the composition of two length preserving morphisms is again length preserving, the equality $\mathcal{P}^{\uparrow}(\mathcal{P}^{\uparrow}\mathcal{V}) = \mathcal{P}^{\uparrow}\mathcal{V}$ holds for each positive variety of languages $\mathcal{V}$. It follows that $\mathbf{P}^{\uparrow}(\mathbf{P}^{\uparrow}\mathbf{V}) = \mathbf{P}^{\uparrow}\mathbf{V}$. The equality $\mathbf{P}_{\star}^{\uparrow}(\mathbf{P}_{\star}^{\uparrow}\mathbf{V}) = \mathbf{P}_{\star}^{\uparrow}\mathbf{V}$ is obtained in a similar way. $\square$

References

[1] J. Almeida, *Finite semigroups and universal algebra*, World Scientific Publishing Co. Inc., River Edge, NJ, 1994. Translated from the 1992 Portuguese original and revised by the author.

[2] J. Almeida, Power exponents of aperiodic pseudovarieties, *Semigroup Forum* **58** (1999), 18–32.

[3] J. Almeida, Power semigroups: results and problems, in *Algebraic Engineering. Proceedings of the first international conference on semigroups and algebraic engineering held in Aizu, Japan, March 24-28, 1997.*, pp. 399–415, M. Ito and C. Nehaniv (Eds.), World Scientific, Singapore, 1999.

[4] K. Auinger and B. Steinberg, On power groups and embedding theorems for relatively free profinite monoids, *Math. Proc. Cambridge Philos. Soc.* **138**,2 (2005), 211–232.

[5] A. Cano Gómez, *Semigroupes ordonnés et opérations sur les langages rationnels*, PhD thesis, Université Paris 7 and Departamento de Sistemas Informáticos y Computación, Universidad Politécnica de Valencia, 2003.

[6] A. Cano Gómez and J.-É. Pin, Shuffle on positive varieties of languages, *Theoret. Comput. Sci.* **312** (2004), 433–461.

[7] S. Eilenberg, *Automata, Languages and Machines*, vol. B, Academic Press, New York, 1976.

[8] A. P. Escada Colaço, *Contribuições para o estudo de operadores potência sobre pseudovariedades de semigrupos*, Tese para obtençao so grau de doutor em Matemática, Facultade de ciências da Universidade do Porto, 1999.

[9] S. Margolis, On M-varieties generated by power monoids, *Semigroup Forum* **22** (1981), 339–353.

[10] S. Margolis and J.-E. Pin, Minimal noncommutative varieties and power varieties, *Pacific J. Math.* **111** (1984), 125–135.

[11] S. W. Margolis and B. Steinberg, Power semigroups and polynomial closure, in *Words, languages & combinatorics, III (Kyoto, 2000)*, pp. 311–322, World Sci. Publ., River Edge, NJ, 2003.

[12] J.-F. Perrot, Variétés de langages et operations, *Theoret. Comput. Sci.* **7** (1978), 197–210.

[13] J.-E. Pin, Variétés de langages et monoïde des parties, *Semigroup Forum* **20** (1980), 11–47.

[14] J.-E. Pin, *Varieties of formal languages*, North Oxford, London and Plenum, New-York, 1986. (Translation of *Variétés de langages formels*, Masson, 1984).

[15] J.-E. Pin, *BG = PG*, a success story, in *NATO Advanced y Institute Semigroups, Formal Languages and Groups*, J. Fountain (ed.), pp. 33–47, Kluwer Academix Publisher, 1995.

[16] J.-E. Pin, A variety theorem without complementation, *Russian Mathematics (Iz. VUZ)* **39** (1995), 80–90.

[17] J.-E. Pin, Profinite methods in automata theory, in *26th International Symposium on Theoretical Aspects of Computer Science (STACS 2009)*, S. Albers and J.-Y. Marion (eds.), pp. 31–50, Internationales Begegnungs- und Forschungszentrum für Informatik (IBFI), Schloss Dagstuhl, Germany, 2009.

[18] J.-E. Pin and P. Weil, Semidirect products of ordered semigroups, *Communications in Algebra* **30** (2002), 149–169.

[19] L. Polák, Operators on classes of regular languages, in *Semigroups, Algorithms, Automata and Languages*, G. Gomes, J.-E. Pin and P. Silva (eds.), pp. 407–422, World Scientific, 2002.

[20] C. Reutenauer, Sur les variétés de langages et de monoïdes, in *Theoretical computer science (Fourth GI Conf., Aachen)*, pp. 260–265, *Lect. Notes Comp. Sci.* vol. 67, Springer, Berlin, 1979.

[21] H. Straubing, Recognizable Sets and Power Sets of Finite Semigroups, *Semigroup Forum* **18** (1979), 331–340.