On uniformly continuous functions for some profinite topologies

Dedicated to Antonio Restivo for his 70th birthday

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Abstract

Given a variety of finite monoids $V$, a subset of a monoid is a $V$-subset if its syntactic monoid belongs to $V$. A function between two monoids is $V$-preserving if it preserves $V$-subsets under preimages and it is hereditary $V$-preserving if it is $W$-preserving for every subvariety $W$ of $V$. The aim of this paper is to study hereditary $V$-preserving functions when $V$ is one of the following varieties of finite monoids: groups, $p$-groups, aperiodic monoids, commutative monoids and all monoids.

1 Introduction

This article is a follow-up of [12], where the authors started the study of $V$-preserving functions. Let us first remind the definition. Let $M$ be a monoid and let $V$ be a variety of finite monoids. A recognizable subset $S$ of $M$ is said to be a $V$-subset if its syntactic monoid belongs to $V$. A function $f : M \to N$ is called $V$-preserving if, for each $V$-subset of $N$, $f^{-1}(L)$ is a $V$-subset of $M$. A function is hereditary $V$-preserving if it is $W$-preserving for every subvariety $W$ of $V$.

Let us first consider the case where $f$ is a function from $A^*$ to $B^*$, where $A$ and $B$ are finite alphabets. If $V$ is the variety $M$ of all finite monoids, a $V$-preserving function is also called regularity-preserving, according to the terminology used in [5, 16, 18]. The characterization of regularity-preserving functions is a long-term objective, but in spite of intensive research (see [10] for a detailed bibliography), it is still out of reach. For the variety $G_p$ of finite $p$-groups, the situation is more advanced. Indeed, the authors gave in [13] a characterization of $G_p$-preserving functions when $B$ is a one-letter alphabet.
and a preliminary step towards a general solution can be found in [10]. For the variety $G$ of finite groups and for the variety $A$ of finite aperiodic monoids, the only known contribution to the study of $V$-preserving functions seems to be the article of Reutenauer and Schützenberger on rational functions [14].

This paper focuses on hereditary $V$-preserving functions when $V$ is one of the varieties $M$, $G$, $G_p$ and $A$. We consider functions from a free monoid or a free commutative monoid to $\mathbb{N}$ and, in the case of the varieties $G$ and $G_p$, we also study functions from $A^*$ to $\mathbb{Z}$ or from $\mathbb{Z}^k$ to $\mathbb{Z}$. The case of a one-letter alphabet was also discussed in [3]. Our results are summarized in the table below.

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Characterization of hereditary $V$-preserving functions.

## 2 Preliminaries

In this section, we review the basic notions used in this paper.

### 2.1 Varieties

A *variety of finite monoids* is a class of finite monoids closed under taking submonoids, quotients and finite direct products. In the sequel, we shall use freely the term *variety* instead of *variety of finite monoids*.

We denote by $M$ (respectively $\text{Com}$, $G$, $\text{Ab}$, $A$) the variety of all finite monoids (respectively finite commutative monoids, finite groups, finite abelian groups, finite aperiodic monoids). Given a prime number $p$, we denote by $G_p$ the variety of all finite $p$-groups and by $\text{Ab}_p$ the variety of all finite abelian $p$-groups. Each finite monoid $M$ generates a variety, denoted by $(M)$. The join of a family of varieties $(V_i)_{i \in I}$ is the least variety containing all the varieties $V_i$, for $i \in I$.

For $n > 0$, $C_n$ denotes the cyclic group of order $n$. Throughout the paper, we shall use the well-known structure theorem for finite abelian groups [15], which shows that $\text{Ab}$ is the variety generated by the finite cyclic groups.

**Proposition 2.1** Every finite abelian group is isomorphic to a direct product of finite cyclic groups.

### 2.2 Ultrametrics and pseudo-ultrametrics

A *pseudo-ultrametric* on a set $X$ is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties, for all $x, y, z \in X$:

$$(P_1)\quad d(x, y) \geq 0,$$
\( (P_2) \) \( d(x, x) = 0, \)
\( (P_3) \) \( d(x, y) = d(y, x), \)
\( (P_4) \) \( d(x, z) \leq \max\{d(x, y), d(y, z)\}. \)

An ultrametric satisfies a stronger version of \((P_2):\)
\( (P_5) \) \( d(x, y) = 0 \) if and only if \( x = y. \)

2.3 Uniformly continuous functions

Given two pseudometric spaces \((X_1, d_1)\) and \((X_2, d_2)\), a function \( f : X_1 \to X_2 \) is uniformly continuous if, for every positive real number \( \epsilon \) there exists a positive real number \( \delta > 0 \) such that for all \( (x, y) \in X^2, \)
\[
d_1(x, y) < \delta \text{ implies } d_2(f(x), f(y)) < \epsilon. \tag{2.1}
\]

It follows in particular that if \( d_1(x, y) = 0, \) then \( d_2(f(x), f(y)) = 0. \) Moreover this condition is sufficient if 0 is an isolated point in the range of \( d_1 \) and \( d_2. \) We shall only need a weaker version of this result.

**Proposition 2.2** If \( d_1 \) and \( d_2 \) have finite range, a function \( f : (X_1, d_1) \to (X_2, d_2) \) is uniformly continuous if and only if
\[
d_1(x, y) = 0 \text{ implies } d_2(f(x), f(y)) = 0. \tag{2.2}
\]

**Proof.** Since \( d_2 \) has finite range, there exists a positive real number \( \epsilon \) such that \( d_2(u, v) < \epsilon \) implies \( d_2(u, v) = 0. \) If \( f \) is uniformly continuous, there exists \( \delta \) such that \( d_1(x, y) < \delta \) implies \( d_2(f(x), f(y)) < \epsilon. \) By the choice of \( \epsilon, \) this actually implies \( d_2(f(x), f(y)) = 0 \) and thus \((2.2)\) holds.

Since \( d_1 \) has finite range, there exists a positive real number \( \delta \) such that \( d_1(u, v) < \delta \) implies \( d_1(u, v) = 0. \) Suppose that \( (2.2) \) holds and let \( \epsilon \) be a positive integer. If \( d_1(u, v) < \delta \) then \( d_1(u, v) = 0 \) and by \( (2.2), \) \( d_2(f(x), f(y)) = 0. \) It follows in particular that \( d_2(f(x), f(y)) < \epsilon \) and thus \( f \) is uniformly continuous. \( \Box \)

Nonexpansive functions form an interesting subclass of the class of uniformly continuous functions. A function \( f : (X_1, d_1) \to (X_2, d_2) \) is nonexpansive if, for all \( (x, y) \in X_1 \times X_1, \)
\[
d_2(f(x), f(y)) \leq d_1(x, y)
\]

We shall use nonexpansive functions in Section 3.

2.4 Pro-V metrics

For the remainder of this section, let \( \mathbf{V} \) denote a variety of finite monoids. Let \( M \) be a monoid and let \( u, v \in M. \) We say that a monoid \( N \) separates \( u \) and \( v \) if there exists a monoid morphism \( \varphi : M \to N \) such that \( \varphi(u) \neq \varphi(v). \) A monoid \( M \) is residually \( \mathbf{V} \) if any two distinct elements of \( M \) can be separated by a monoid in \( \mathbf{V}. \)

We shall use the conventions \( \min \emptyset = \infty \) and \( 2^{-\infty} = 0. \) For all \( u, v \in M, \) let
\[
r_{\mathbf{V}}(u, v) = \min \left\{ |N| \mid N \text{ is in } \mathbf{V} \text{ and separates } u \text{ and } v \right\}
\]
and $d_{V}(u, v) = 2^{-r_{V}(u, v)}$. Then $d_{V}$ is a pseudo-ultrametric, called the pro-$V$ metric on $M$ (see [12]). If the monoid is residually $V$, then $d_{V}$ is an ultrametric.

In this paper, we consider free monoids, free commutative monoids and free abelian groups of finite rank: they are all finitely generated and residually $V$ for the main varieties considered in this paper: monoids, (abelian) groups, abelian $p$-groups, (commutative) aperiodic monoids.

2.5 $V$-uniform continuity and $V$-hereditary continuity

Let $M$ and $N$ be monoids. A function $f : M \to N$ is said to be $V$-uniformly continuous if it is uniformly continuous for the pro-$V$ pseudometric on $M$ and $N$. The following result was proved in [12, Theorem 4.1].

Proposition 2.3 A function $f : M \to N$ is $V$-preserving if and only if it is $V$-uniformly continuous.

We say that $f$ is $V$-hereditarily continuous if it is $W$-uniformly continuous for each subvariety $W$ of $V$. Closure properties of this notion under various operators are analysed in [12, Subsection 4.3].

A monoid $N$ is called $V$-projective if the following property holds: if $\alpha : N \to R$ is a morphism and if $\beta : T \to R$ is a surjective morphism, where $T$ (and hence $R$) is a monoid of $V$, then there exists a morphism $\gamma : N \to T$ such that $\alpha = \beta \circ \gamma$.

For example, any free monoid (in particular $\mathbb{N}$) is $V$-projective for every variety of finite monoids. Similarly, any free group (in particular $\mathbb{Z}$) is $V$-projective for every variety of finite groups. Note that a $V$-projective monoid is $W$-projective for every subvariety $W$ of $V$.

The following results were proved in [12]:

Proposition 2.4 [12, Proposition 5.7] Let $V$ be the join of a family $(V_{i})_{i \in I}$ of varieties of finite commutative monoids. A function from a monoid to a $V$-projective monoid is $V$-hereditarily continuous if and only if it is $V_{i}$-hereditarily continuous for all $i \in I$.

Proposition 2.5 [12, Proposition 5.4] A function from a monoid to a commutative monoid is $V$-hereditarily continuous if and only if it is $(V \cap \text{Com})$-hereditarily continuous.

In contrast, note that a $V$-uniformly continuous function from a monoid to a commutative monoid is not necessarily $(V \cap \text{Com})$-hereditarily continuous. For instance, the function $f$ from $\{a, b\}^{*}$ to $\mathbb{N}$ defined by $f(ab) = 1$ and $f(u) = 0$ if $u \neq ab$ is $M$-uniformly continuous but is not $\text{Com}$-uniformly continuous.
2.6 \textit{p}-adic valuations

Let \( p \) be a prime number. If \( n \) is a non-zero integer, the \textit{p-adic valuation} of \( n \) is the integer
\[
v_p(n) = \max \{ k \in \mathbb{N} \mid p^k \text{ divides } n \}\]
By convention, \( v_p(0) = +\infty \). Note that the equality \( v_p(nm) = v_p(n) + v_p(m) \) holds for all integers \( n, m \).

The \textit{p-adic norm} of \( n \) is the real number
\[
|n|_p = p^{-v_p(n)}.
\]
The \( p \)-adic norm satisfies the following properties, for all \( n, m \in \mathbb{Z} \):
1. \((N_1)\) \( |n|_p \geq 0 \),
2. \((N_2)\) \( |n|_p = 0 \) if and only if \( n = 0 \),
3. \((N_3)\) \( |mn|_p = |m|_p|n|_p \),
4. \((N_4)\) \( |m + n|_p \leq \max\{ |m|_p, |n|_p \} \).

The \( p \)-adic valuation and the \( p \)-adic norm can be extended to \( \mathbb{Z}^k \) as follows.
Given \( n = (n_1, \ldots, n_k) \in \mathbb{Z}^k \), we set
\[
v_p(n) = \min_{1 \leq j \leq k} \{ v_p(n_j) \} \quad \text{and} \quad |n|_p = p^{-v_p(n)} = \max_{1 \leq j \leq k} \{ |n_j|_p \}.
\]
The \( p \)-adic norm on \( \mathbb{Z}^k \) still satisfies \((N_1), (N_2)\) and \((N_4)\), as well as the following weaker version of \((N_3)\):
\((N_5)\) for all \( n, m \in \mathbb{Z}^k \), \( |mn|_p \leq |m|_p|n|_p \).

The \( p \)-adic norm on \( \mathbb{Z}^k \) induces the \textit{\( p \)-adic ultrametric} \( d_p \) on \( \mathbb{Z}^k \), defined by
\[
d_p(u, v) = |u - v|_p.
\]
Note that the pro-\( \text{Ab}_p \) metric \( d_{\text{Ab}_p} \) and \( d_p \) are strongly equivalent metrics.

2.7 Binomial coefficients

Let \( A \) be a finite alphabet. We denote by \( A^* \) the free monoid on \( A \). Note that if \( |A| = 1 \), then \( A^* \) is isomorphic to the additive monoid \( \mathbb{N} \).

Let \( u \) and \( v \) be two words of \( A^* \). Let \( u = a_1 \cdots a_n \), with \( a_1, \ldots, a_n \in A \). Then \( u \) is a subword of \( v \) if there exist \( v_0, \ldots, v_n \in A^* \) such that \( v = v_0a_1v_1 \cdots a_nv_n \).

Set
\[
\binom{v}{u} = |\{(v_0, \ldots, v_n) \mid v = v_0a_1v_1 \cdots a_nv_n\}|.
\]

Note that if \( A = \{a\} \), \( u = a^n \) and \( v = b^m \), then \( \binom{a}{a} = \binom{b}{b} \) and hence these numbers constitute a generalization of the classical binomial coefficients. See [7, Chapter 6] for more details. Sometimes, it will be useful to use the convention \( \binom{m}{n} = 0 \) for \( m \geq 0 \) and \( n \in \mathbb{Z} \setminus \{0, \ldots, m\} \), which is compatible with the usual properties of binomial coefficients.

2.8 Mahler expansions

For a fixed \( v \in A^* \), we can view the generalized binomial coefficient \( \binom{v}{u} \) as a function from \( A^* \) to \( \mathbb{N} \). The functions \( \{ \binom{v}{u} \mid u \in A^* \} \) constitute a \textit{locally finite} family of functions in the sense that, for each \( u \in A^* \), the image of \( u \) is 0 for all but finitely many elements of the family.
It is clear that the sum of a locally finite family of functions is well defined. In particular, if \((g_v)_{v \in A^*}\) is a family of elements of an abelian group \(G\), then there is a well-defined function \(f\) from \(A^*\) into \(G\) defined by the formula (in additive notation)

\[ f(u) = \sum_{v \in A^*} g_v(u) \]

The generalized binomial coefficients provide a unique decomposition of the functions from \(A^*\) into \(G\), which will be referred as Mahler expansion:

**Proposition 2.6 (Lothaire [7])** Let \(G\) be an abelian group and let \(f : A^* \to G\) be an arbitrary function. Then there exists a unique family \(\langle f, v \rangle_{v \in A^*}\) of elements of \(G\) such that, for all \(u \in A^*\), \(f(u) = \sum_{v \in A^*} \langle f, v \rangle (uv)\). This family is given by the inversion formula

\[ \langle f, v \rangle = \sum_{w \in A^*} (-1)^{|v|+|w|} \binom{v}{w} f(w) \]  

(2.3)

A similar result holds for functions from \(\mathbb{N}^k\) to an abelian group \(G\). If \(r\) is an element of \(\mathbb{N}^k\) (or more generally of \(\mathbb{Z}^k\)), we denote by \(r_i\) its \(i\)-th component, so that \(r = (r_1, \ldots, r_k)\). First observe that the family

\[ \{(r_1)^{i_1} \cdots (r_k)^{i_k} \mid r \in \mathbb{N}^k\} \]

is a locally finite family of functions from \(\mathbb{N}^k\) into \(\mathbb{N}\). Thus, given a family \((g_r)_{r \in \mathbb{N}^k}\), the formula

\[ f(n) = \sum_{r \in \mathbb{N}^k} g_r(n_1^{i_1} \cdots n_k^{i_k}) \]

defines a function \(f : \mathbb{N}^k \to G\). Conversely, each function from \(\mathbb{N}^k\) to \(G\) admits a unique Mahler expansion, a result proved in a more general setting in [2, 1].

**Proposition 2.7** Let \(G\) be an abelian group and let \(f : \mathbb{N}^k \to G\) be an arbitrary function. Then there exists a unique family \(\langle f, r \rangle_{r \in \mathbb{N}^k}\) of elements of \(G\) such that, for all \(n \in \mathbb{N}^k\),

\[ f(n) = \sum_{r \in \mathbb{N}^k} \langle f, r \rangle (n_1^{i_1} \cdots n_k^{i_k}). \]

The coefficients \(\langle f, r \rangle\) are given by

\[ \langle f, r \rangle = \sum_{i_1=0}^{r_1} \cdots \sum_{i_k=0}^{r_k} (-1)^{r_1+\cdots+r_k+i_1+\cdots+i_k} \binom{r_1}{i_1} \cdots \binom{r_k}{i_k} f(i). \]

3 \(G_p\)-hereditary continuity

Let \(p\) be a prime number. We proved in [11, 13] that \(G_p\)-uniformly continuous functions from \(A^*\) to \(\mathbb{Z}\) can be characterized by properties of their Mahler expansions. The case where \(A\) is a one-letter alphabet corresponds to the classical Mahler’s Theorem from \(p\)-adic number theory [8, 9].

**Theorem 3.1** Let \(f : A^* \to \mathbb{Z}\) be a function and let \(f(u) = \sum_{v \in A^*} \langle f, v \rangle (u)\) be its Mahler expansion. Then the following conditions are equivalent:
(1) $f$ is $G_p$-uniformly continuous;
(2) $\lim_{|v| \to \infty} |(f,v)|_p = 0$.

A similar result (Amice, [2]) holds when $A^*$ is replaced by $\mathbb{Z}^k$ (see also [13, Corollary 6.3] for an alternative proof). In this section, we obtain analogous results for $G_{p^n}$-hereditary continuity. A first step is to reduce $G_{p^n}$-hereditary continuity to a simpler property.

**Lemma 3.2** A function from a monoid to a $G_{p^n}$-projective commutative monoid is $G_{p^n}$-hereditarily continuous if and only if it is $(C_{p^n})$-uniformly continuous for all $n > 0$.

**Proof.** By Proposition 2.5, $f$ is $G_{p^n}$-hereditarily continuous if and only if it is $(G_p \cap \text{Com})$-hereditarily continuous. Since

$$G_p \cap \text{Com} = G_p \cap \text{Ab} = \bigvee_{n>0} (C_{p^n})$$

by Proposition 2.1, Proposition 2.4 implies that $f$ is $G_{p^n}$-hereditarily continuous if and only if $f$ is $(C_{p^n})$-hereditarily continuous for every $n \in \mathbb{N}$. Since the only subvarieties of $(C_{p^n})$ are those of the form $(C_{p^i})$ with $i \leq n$, the lemma follows. □

Let $V$ be a variety of groups. Since any morphism from $\mathbb{N}^k$ to a finite group extends uniquely to a morphism from $\mathbb{Z}^k$ to that same group, the pro-$V$ pseudo-metric on $\mathbb{N}^k$ is the restriction of the pro-$V$ pseudo-metric on $\mathbb{Z}^k$. Therefore the forthcoming results hold for $\mathbb{N}^k$ even though they are stated and proved for $\mathbb{Z}^k$.

We denote by $e_1, \ldots, e_k$ the canonical generators of both $\mathbb{N}^k$ and $\mathbb{Z}^k$. Thus $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 occurs in position $j$.

**Lemma 3.3** Let $n \in \mathbb{N}$ and let $d$ be the pro-$(C_{p^n})$ pseudo-metric on $\mathbb{Z}^k$. For $r, s \in \mathbb{Z}^k$, one has $d(r, s) = 2^{-r_m}$ where

$$m = \min \{ i \leq n \mid \text{there exists } j \in \{1, \ldots, k\} \text{ such that } r_j \neq s_j \mod p^i \}.$$

**Proof.** Suppose that $r_j \neq s_j \mod p^i$ for some $i \leq n$ and $j \in \{1, \ldots, k\}$. Let $f : \mathbb{Z}^k \to C_{p^n}$ be defined by $f(n) = n_j$. Clearly, $C_{p^n} \in (C_{p^n})$ and $f$ separates $r$ and $s$, hence $d(r, s) \geq 2^{-r_m}$ and so $d(r, s) \geq 2^{-r_m}$. Note that this last inequality holds trivially if $m = \infty$.

If $d(r, s) = 0$, equality follows. Otherwise, we may assume that $f : \mathbb{Z}^k \to G \in (C_{p^n})$ is a morphism that separates $r$ and $s$ with $|G|$ minimum. By Proposition 2.1, $G$ is a direct product of cyclic groups. Since their order must divide $|G|$ which is a power of $p$, each one of these factor groups is of the form $C_{p^i}$. Since any group in $(C_{p^n})$ must satisfy the identity $x^{p^n} = 1$, we conclude that $i \leq n$ in each case. If $G$ were a nontrivial direct product, we could decompose $f$ into its components and contradict the minimality of $G$, thus $G = C_{p^i}$ with $i \leq n$.

Suppose that $r_j \equiv s_j \mod p^i$ for every $j \in \{1, \ldots, k\}$. Then $r_j = s_j$ in $C_{p^i}$ for every $j$ and so

$$f(r) = \sum_{j=1}^{k} r_j f(e_j) = \sum_{j=1}^{k} s_j f(e_j) = f(s),$$
The next corollary shows how the pro-\((C_p^n)\) pseudo-metric relates to the \(p\)-adic norm:

**Corollary 3.4** Let \(n \in \mathbb{N}\) and let \(d\) denote the pro-\((C_p^n)\) pseudo-metric on \(\mathbb{Z}^k\).

For all \(r, s \in \mathbb{Z}^k\), we have

\[
d(r, s) = \begin{cases} 2 - \frac{1}{p^n} & \text{if } |r - s|_p > p^{-n} \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** Let

\[
m = \min \left\{ i \leq n \mid \text{exists } j \in \{1, \ldots, k\} \text{ such that } r_j \not\equiv s_j \pmod{p^i} \right\}.
\]

It is easy to check that

\[
m = \begin{cases} v_p(r - s) + 1 & \text{if } v_p(r - s) < n \\ \infty & \text{otherwise.} \end{cases}
\]

Clearly, \(v_p(r - s) < n\) if and only if \(|r - s|_p > p^{-n}\). In this case,

\[
p^m = p^{v_p(r-s)+1} = \frac{p}{|r-s|_p}
\]

and the claim follows from Lemma 3.3. \(\square\)

We arrive to our characterization of \(G_p\)-hereditarily continuous functions.

**Theorem 3.5** A function from \(\mathbb{Z}^k\) to \(\mathbb{Z}\) is \(G_p\)-hereditarily continuous if and only if it is nonexpansive for the \(p\)-adic norm.

**Proof.** Let \(d_n\) denote the pro-\((C_p^n)\) pseudo-metric. By Lemma 3.2, \(f\) is hereditarily \(G_p\)-uniformly continuous if and only if, for all \(n > 0\), it is uniformly continuous for \(d_n\). By Proposition 2.2, this holds if and only if, for all \(r, s \in \mathbb{Z}^k\),

\[
d_n(r, s) = 0 \text{ implies } d_n(f(r), f(s)) = 0.
\]

By Corollary 3.4, \(d_n(r, s) = 0\) if and only if \(|r-s|_p \leq p^{-n}\), thus (3.4) is equivalent to stating that for all \(r, s \in \mathbb{Z}^k\),

\[
|r-s|_p \leq p^{-n} \text{ implies } |f(r) - f(s)|_p \leq p^{-n}.
\]

(3.5)

Clearly, (3.5) holds for every \(n\) if and only if \(|f(r) - f(s)|_p \leq |r-s|_p\), which proves the result. \(\square\)

It follows easily from Theorem 3.5 that all polynomial functions from \(\mathbb{Z}^k\) to \(\mathbb{Z}\) are \(G_p\)-hereditarily continuous. We shall use the Mahler expansion of functions given by Proposition 2.7 to characterize all the \(G_p\)-hereditarily continuous functions from \(\mathbb{N}^k\) to \(\mathbb{Z}\). Polynomial functions will appear then as the finitely generated case. We shall need a few lemmas:
Lemma 3.6 The sum of a locally finite family of $G_p$-hereditarily continuous functions from $\mathbb{N}^k$ to $\mathbb{Z}$ is $G_p$-hereditarily continuous.

Proof. Let $\{f_i : \mathbb{N}^k \to \mathbb{Z} \mid i \in I\}$ be a locally finite family of $G_p$-hereditarily continuous functions and let $f = \sum_{i \in I} f_i$. By Theorem 3.5, each $f_i$ is nonexpansive for the $p$-adic norm, and since the $p$-adic norm satisfies $(N_4)$, $f$ is also nonexpansive. $\Box$

The following result is due to Kummer [6]. See also [17, 4].

Proposition 3.7 Let $n, r \in \mathbb{N}$ with $0 \leq r \leq n$. Then $v_p \left( \binom{n}{r} \right)$ is equal to the number of carries it takes to add $r$ and $n-r$ in base $p$.

Taking $n = p^s$ yields the following corollary

Lemma 3.8 Let $r, s \in \mathbb{N}$ with $0 < r \leq p^s$. Then $v_p \left( \binom{p^s}{r} \right) = s - v_p(r)$.

We also need a result stated in [3, Lemma 2.8], for which we give a shorter proof.

Lemma 3.9 Let $n, r, s \in \mathbb{N}$. Then

$$p^s \text{ divides } \left( \frac{\text{lcm}_{1 \leq j \leq r} j \right) \left( \binom{n+p^s}{r} - \binom{n}{r} \right)$$

or equivalently,

$$s \leq \max_{1 \leq j \leq r} v_p(j) + v_p \left( \binom{n+p^s}{r} - \binom{n}{r} \right).$$

Proof. Since $\binom{n+p^s}{r} - \binom{n}{r} = \sum_{j=1}^{r} \binom{p^s}{j} \binom{n}{r-j}$, one gets by Lemma 3.8 the relation

$$\left| \binom{n+p^s}{r} - \binom{n}{r} \right|_p \leq \max_{1 \leq j \leq r} \left| \binom{p^s}{j} \right|_p \left| \binom{n}{r-j} \right|_p \leq \max_{1 \leq j \leq r} \left| \binom{p^s}{j} \right|_p \left| \binom{n}{r-j} \right|_p = \max_{1 \leq j \leq r} v_p(j) - s$$

or equivalently,

$$v_p \left( \binom{n+p^s}{r} - \binom{n}{r} \right) \geq \min_{1 \leq j \leq r} (s - v_p(j)) = s - \max_{1 \leq j \leq r} v_p(j)$$

which gives (3.7). $\Box$

We shall need two elementary results on nonexpansive functions.

Lemma 3.10 Let $f : \mathbb{N} \to \mathbb{Z}$ be a nonexpansive function for the $p$-adic norm and let $s \in \mathbb{N}$. Then for $0 \leq i \leq p^s$, $p^s$ divides $\binom{p^s}{i}$ $(f(i) - f(0))$, or equivalently, $s \leq v_p \left( \binom{p^s}{i} \right) + v_p(f(i) - f(0))$.

Proof. Since $f$ is nonexpansive, one has $|f(i) - f(0)|_p \leq |i - 0|_p$ and thus $v_p(f(i) - f(0)) \geq v_p(i)$. Since $v_p \left( \binom{p^s}{i} \right) = s - v_p(i)$ by Lemma 3.8, the relation $s \leq v_p \left( \binom{p^s}{i} \right) + v_p(f(i) - f(0))$ follows immediately. $\Box$
Corollary 3.11 Let $f : \mathbb{N} \to \mathbb{Z}$ be a nonexpansive function for the $p$-adic norm and let $s \in \mathbb{N}$. Then $p^s$ divides $\sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i} f(i)$.

Proof. Newton’s binomial formula yields

$$0 = (1 - 1)^{p^s} = \sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i},$$

hence

$$\sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i} f(i) = \sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i} (f(i) - f(0)).$$

The result now follows from Lemma 3.10. \qed

Theorem 3.12 Let $f(n) = \sum_{r \in \mathbb{N}^k} \langle f, r \rangle \langle r_1 \rangle \cdots \langle r_k \rangle$ be the Mahler expansion of a function $f : \mathbb{N}^k \to \mathbb{Z}$. Then the following conditions are equivalent:

1. $f$ is $G_p$-hereditarily continuous,
2. $v_p(j) \leq v_p((f, r))$ holds for all $j, r$ such that $1 \leq j \leq \max\{r_1, \ldots, r_k\}$.

Proof. (1) $\Rightarrow$ (2). For all $r, t \in \mathbb{N}^k$, let us set

$$m_r(t) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_k=0}^{r_k} (-1)^{i_1 + \cdots + r_k + i_1 + \cdots + i_k} \binom{r_1}{i_1} \binom{r_2}{i_2} \cdots \binom{r_k}{i_k} f(i + t).$$

By Proposition 2.7, we have $m_r(0, \ldots, 0) = \langle f, r \rangle$. We next show that

$$\min_{t \in \mathbb{N}^k} \{v_p(m_r(t))\} \leq \min_{t \in \mathbb{N}^k} \{v_p(m_{r+s}(t))\} \tag{3.8}$$

for all $r, s \in \mathbb{N}^k$. By transitivity, we may assume that $s_1 + \cdots + s_k = 1$. By symmetry, we may assume that $s = (1, 0, \ldots, 0)$. Let $\ell = \min_{t \in \mathbb{N}^k} \{v_p(m_r(t))\}$. For all $t \in \mathbb{N}^k$, we have

$$m_{r+s}(t) = \sum_{i_1=0}^{r_1+1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{i_1 + \cdots + r_k + i_1 + \cdots + i_k} \binom{r_1+1}{i_1} \binom{r_2}{i_2} \cdots \binom{r_k}{i_k} f(i + t)$$

$$= \sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{i_1 + \cdots + r_k + i_1 + \cdots + i_k} \binom{r_1}{i_1} \binom{r_2}{i_2} \cdots \binom{r_k}{i_k} f(i + t)$$

$$+ \sum_{i_1=0}^{r_1+1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{i_1 + \cdots + r_k + i_1 + \cdots + i_k} \binom{r_1+1}{i_1} \binom{r_2}{i_2} \cdots \binom{r_k}{i_k} f(i + t)$$

$$= - \sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{i_1 + \cdots + r_k + i_1 + \cdots + i_k} \binom{r_1}{i_1} \binom{r_2}{i_2} \cdots \binom{r_k}{i_k} f(i + t)$$

$$+ \sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{i_1 + \cdots + r_k + i_1 + \cdots + i_k} \binom{r_1}{i_1} \binom{r_2}{i_2} \cdots \binom{r_k}{i_k} f(i + t + s)$$

$$= -m_r(t) + m_r(t + s).$$

Since $p^\ell \mid m_r(t)$ and $p^\ell \mid m_r(t + s)$, it follows that $p^\ell \mid m_{r+s}(t)$ and so (3.8) holds.
Now we show that
\[ s \leq v_p(m_{p^r e_j}(t)) \]  
for all \( s \in \mathbb{N}, t \in \mathbb{N}^k \) and \( j = 1, \ldots, k \).

By symmetry, we may assume that \( j = 1 \), so that (3.9) becomes
\[ p^s \mid \sum_{i=0}^{p^r} (-1)^{p^r+i} \binom{p^r}{i} f(i + t_1, t_2, \ldots, t_k). \]  
(3.10)

Fix \( t \in \mathbb{N}^k \) and let \( g : \mathbb{N} \rightarrow \mathbb{Z} \) be the function defined by
\[ g(n) = f(n + t_1, t_2, \ldots, t_k). \]

By Theorem 3.5, \( g \) is \( G_p \)-hereditarily continuous and thus (3.10) follows from Corollary 3.11. Therefore (3.10) holds and so does (3.9).

We now show that
\[ 1 \leq j \leq \max\{r_1, \ldots, r_k\} \Rightarrow v_p(j) \leq v_p(m_e(t)) \]  
(3.11)
holds for all \( j \in \mathbb{N} \) and \( r, t \in \mathbb{N}^k \).

We use induction on \( q = r_1 + \ldots + r_k \). The claim holds trivially for \( q = 0 \), hence we assume that \( q > 0 \) and (3.11) holds for smaller values of \( q \). By symmetry, we may assume that \( r_1 > 0 \).

Assume first that \( 1 \leq j \leq \max\{r_1 - 1, \ldots, r_k\} \). By the induction hypothesis on \( q \), we have \( v_p(j) \leq v_p(m_{r_1-1, r_2, \ldots, r_k}(t)) \) for all \( t \in \mathbb{N}^k \). Thus \( v_p(j) \leq v_p(m_e(t)) \) by (3.8).

The remaining case corresponds to \( j = r_1 > \max\{r_1 - 1, \ldots, r_k\} \). If \( j \) is not a power of \( p \), then we may write \( j = j_1 j_2 \) with \( j_1 < j \) and \( v_p(j_1) = v_p(j) \), falling into the previous case. Thus we may assume that \( j = p^i \) for some \( i \in \mathbb{N} \). By (3.9), we have \( i \leq v_p(m_{p^i, 0, \ldots, 0}(t)) \) for all \( t \in \mathbb{N}^k \). Since \( r_1 = j = p^i \), it follows from (3.8) that \( v_p(j) = i \leq v_p(m_e(t)) \) and (3.11) holds.

Considering now the particular case \( t = 0 \), we obtain Condition (2).
(2) \( \Rightarrow \) (1). By Lemma 3.6, it is enough to show that the function
\[ g(n) = (f, r)^{r_1} \cdots (r_k) \]
is \( G_p \)-hereditarily continuous for a fixed \( r \in \mathbb{N}^k \). Write \( m = (f, r) \). Let \( x, y \in \mathbb{N}^k \) and assume that \( p^s \mid x - y \). By Theorem 3.5, it suffices to show that
\[ p^s \mid m \left( \binom{x_1}{r_1} \cdots \binom{x_k}{r_k} - \binom{y_1}{r_1} \cdots \binom{y_k}{r_k} \right). \]  
(3.12)

We have \( p^s \mid x - y \) if and only if \( y = x + p^\ell z \) for some \( z \in \mathbb{Z}^k \). Clearly, we can obtain \( y \) from \( x \) by successively adding or subtracting \( p^\ell e_i \) (i = 1, \ldots, k). Since \( p^s \mid \ell \) and \( p^s \mid \ell' \) together imply \( p^s \mid \ell - \ell' \), we may assume without loss of generality that \( x = y + p^\ell e_i \). By symmetry, we may also assume that \( i = 1 \).

Therefore (3.12) will follow from
\[ p^s \mid m \left( \binom{y_1+p^\ell}{r_1} - \binom{y_1}{r_1} \right). \]  
(3.13)

By condition (2), we have \( v_p(j) \leq v_p(m) \) if \( 1 \leq j \leq r_1 \), hence Lemma 3.9 yields
\[ s \leq \max_{1 \leq j \leq r_1} v_p(j) + v_p \left( \binom{y_1+p^\ell}{r_1} - \binom{y_1}{r_1} \right) \leq v_p(m) + v_p \left( \binom{y_1+p^\ell}{r_1} - \binom{y_1}{r_1} \right) \]
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and (3.13) holds as required. □

It followed from Theorem 3.5 that all polynomial functions \( f : \mathbb{N}^k \to \mathbb{Z} \) with integer coefficients are \( G_p \)-hereditarily continuous. There are of course only countably many such functions. Theorem 3.12 implies the existence of uncountably many \( G_p \)-hereditarily continuous functions:

**Corollary 3.13** There are uncountably many \( G_p \)-hereditarily continuous functions \( f : \mathbb{N}^k \to \mathbb{Z} \).

**Proof.** For every \( r \in \mathbb{N}^k \), let

\[
\ell_r = \max\{v_p(j) \mid 1 \leq j \leq \max\{r_1, \ldots, r_k\}\}.
\]

By Theorem 3.12 and Proposition 2.7, the map

\[
(n_r)_{r \in \mathbb{N}^k} \mapsto \sum_{r \in \mathbb{N}^k} p^\ell_r n_r(z_1) \cdots (z_k)
\]

is a bijection between \( \mathbb{Z}^{(\mathbb{N}^k)} \) and the set of all \( G_p \)-hereditarily continuous functions from \( \mathbb{N}^k \) to \( \mathbb{Z} \). □

We now consider functions from a free monoid \( A^* \) to \( \mathbb{Z} \). Let \( h : A^* \to \mathbb{N}^A \) be the canonical morphism defined by \( h(u) = ([u]_a)_{a \in A} \), where \([u]_a\) denotes as usual the number of occurrences of the letter \( a \) in \( u \). Let \( \sim \) be the commutative equivalence, formally defined by \( u \sim v \) if and only if \( h(u) = h(v) \).

**Lemma 3.14** Let \( f : A^* \to \mathbb{Z} \) be a \( G_p \)-hereditarily continuous function and let \( u, v \in A^* \) be commutatively equivalent. Then \( f(u) = f(v) \).

**Proof.** Let us choose \( s \) such that \( p^s > |f(u) - f(v)| \) and let \( d \) (respectively \( d' \)) be the pro-\((C_p)\)-pseudo-metric on \( A^* \) (respectively \( \mathbb{Z} \)). Since \( f \) is hereditarily \( G_p \)-uniformly continuous, it is in particular \((C_p)\)-uniformly continuous. Now, if \( u \) and \( v \) are commutatively equivalent, then \( d(x, y) = 0 \) and hence \( d'(f(x), f(y)) = 0 \), which means that \( f(x) \equiv f(y) \mod p^s \). Since \( p^s > |f(u) - f(v)| \), this finally implies that \( f(u) = f(v) \). □

**Lemma 3.15** Let \( u \in A^* \) and \( r = (r_a)_{a \in A} \in \mathbb{N}^A \). Then \( \sum_{v \in h^{-1}(r)} \binom{u}{v} = \prod_{a \in A} \binom{([u]_a)}{r_a} \).

**Proof.** Let \( \mathbb{Z}[A] \) be the ring of polynomials in noncommutative variables in \( A \) with integer coefficients. The monoid morphism \( \mu \) from \( A^* \) to the multiplicative monoid \( \mathbb{Z}[A] \) defined, for each letter \( a \in A \), by \( \mu(a) = 1 + a \), is called the Magnus transformation. By [7, Proposition 6.3.6], the following formula holds for all \( u \in A^* \):

\[
\mu(u) = \sum_{v \in A^*} \binom{u}{v} v
\]

(3.14)

Let \( \mathbb{Z}[A] \) be the ring of polynomials in commutative variables in \( A \) with integer coefficients. The commutative version of the Magnus transformation is the monoid morphism \( \bar{\mu} \) from \( A^* \) to the multiplicative monoid \( \mathbb{Z}[A] \) defined, for
each letter \( a \in A \), by \( \mu(a) = 1 + a \). Thus by definition, one has, for each word \( v \in A^* \),

\[
\mu(u) = \prod_{a \in A} (1 + a)^{|u|_a} = \prod_{a \in A} \left( \sum_{0 \leq r_a \leq |u|_a} \binom{|u|_a}{r_a} a^{r_a} \right) = \sum_{0 \leq r_a \leq |u|_a} \left( \prod_{a \in A} \binom{|u|_a}{r_a} \right) \prod_{a \in A} a^{r_a}
\]

and on the other hand, \((3.14)\) shows that

\[
\mu(u) = \sum_{v \in A^*} \binom{u}{v} \prod_{a \in A} a^{|v|_a} = \sum_{r \in \mathbb{N}^A} \left( \sum_{v \in h^{-1}(r)} \binom{u}{v} \right) \prod_{a \in A} a^{r_a}
\]

Comparing \((3.15)\) and \((3.16)\) now gives the formula \( \sum_{v \in h^{-1}(r)} \binom{u}{v} = \prod_{a \in A} (\binom{|u|_a}{r_a}) \). \( \square \)

**Lemma 3.16** Let \( f : A^* \to G \) be a function from \( A^* \) to some abelian group with Mahler expansion \( f(.) = \sum_{w \in A^*} f(w) \binom{.}{w} \). Then the following conditions are equivalent:

1. for any two commutatively equivalent words \( u \) and \( v \), \( (f, u) = (f, v) \),
2. for any two commutatively equivalent words \( u \) and \( v \), \( f(u) = f(v) \).

**Proof.** (1) implies (2). Suppose that (2) holds. For each \( r \in \mathbb{N}^k \), let \( (k, r) \) be the common value of \( (f, v) \) for all \( v \in h^{-1}(r) \). With the help of Lemma 3.15, we now obtain

\[
f(u) = \sum_{v \in A^*} (f, v) \binom{u}{v} = \sum_{r \in \mathbb{N}^A} \sum_{v \in h^{-1}(r)} (k, r) \binom{u}{v} = \sum_{r \in \mathbb{N}^k} \sum_{v \in h^{-1}(r)} (k, r) \binom{u}{v} = \sum_{r \in \mathbb{N}^k} (k, r) \prod_{a \in A} \binom{|u|_a}{r_a}
\]

It follows immediately that if \( u \) and \( v \) are commutatively equivalent, then \( f(u) = f(v) \).

(2) implies (1). Let \( g : A^* \to G \) be the function defined by \( g(u) = (-1)^{|u|} (f, u) \). It follows from the inversion formula \((2.3)\) that \( (g, v) = (-1)^{|v|} f(v) \). Thus if (2) holds, then for any two commutatively equivalent words \( u \) and \( v \), \( (g, u) = (g, v) \). By the first part of the proof applied to \( g \), it follows that \( g(u) = g(v) \) and thus \( (f, u) = (f, v) \). \( \square \)

**Lemma 3.17** Let \( g : \mathbb{N}^k \to \mathbb{Z} \) be a function and let \( V \) be a variety of finite groups. Then \( g \) is \( V \)-hereditarily continuous if and only if \( g \circ h \) is \( V \)-hereditarily continuous.

**Proof.** By Proposition 2.5, \( g \) or \( g \circ h \) are \( V \)-hereditarily continuous if and only if they are \((V \cap \text{Ab})\)-hereditarily continuous. Let \( W \) be a subvariety of \( V \cap \text{Ab} \) and let \( d \) denote the pro-\( W \) pseudo-metric. Since \( h \) is surjective, every element of \( \mathbb{N}^k \) can be written in the form \( h(u) \) for some \( u \in A^* \). Therefore \( g \) is \( W \)-uniformly continuous if and only if for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that, for all \( u, v \in A^* \),

\[
d(h(u), h(v)) < \delta \text{ implies } d(g \circ h(u), g \circ h(v)) < \varepsilon \quad (3.17)
\]
Since any morphism from $A^*$ to an abelian group factors through $\mathbb{N}^k$, one has $d(u, v) = d(h(u), h(v))$ for all $u, v \in A^*$. Therefore (3.18) can be rewritten as
\[
d(u, v) < \delta \text{ implies } d(g \circ h(u), g \circ h(v)) < \varepsilon
\]
and thus $g$ is $\mathcal{W}$-uniformly continuous if and only if $g \circ h$ is $\mathcal{W}$-uniformly continuous. □

**Lemma 3.18** Let $g : \mathbb{N}^k \to \mathbb{Z}$ be a function and let
\[
g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}
\]
and $g \circ h(u) = \sum_{v \in A^*} \langle g \circ h, v \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$ be the Mahler expansions of $g$ and $g \circ h$. Then $\langle g, r \rangle = \langle g \circ h, a_1^{r_1} \cdots a_k^{r_k} \rangle$ for every $r \in \mathbb{N}^k$.

**Proof.** We have
\[
g(n) = g \circ h(a_1^{r_1} \cdots a_k^{r_k}) = \sum_{v \in A^*} \langle g \circ h, v \rangle \binom{a_1^{r_1} \cdots a_k^{r_k}}{v}
\]
\[
= \sum_{v \in a_1^{r_1} \cdots a_k^{r_k}} \langle g \circ h, v \rangle \binom{a_1^{r_1} \cdots a_k^{r_k}}{v} = \sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} \sum_{a_1^{r_1} \cdots a_k^{r_k} \in A} \langle g \circ h, a_1^{r_1} \cdots a_k^{r_k} \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}.
\]
By the uniqueness of the Mahler expansion in Proposition 2.7, we conclude that $\langle g, r \rangle = \langle g \circ h, a_1^{r_1} \cdots a_k^{r_k} \rangle$ for every $r \in \mathbb{N}^k$. □

**Theorem 3.19** Let $f : A^* \to \mathbb{Z}$ be a function and let $f(u) = \sum_{v \in A^*} \langle f, v \rangle \binom{u}{v}$ be its Mahler expansion. Then $f$ is $\mathcal{G}_p$-hereditarily continuous if and only if it satisfies the following conditions:

1. for any two commutatively equivalent words $u$ and $v$, $\langle f, u \rangle = \langle f, v \rangle$,
2. $v_p(j) \leq v_p((f, v))$ holds for all $v \in A^*$ and $1 \leq j \leq \max_{a \in A} |v|_a$.

**Proof.** Assume that $f$ is $\mathcal{G}_p$-hereditarily continuous. By Lemmas 3.14 and 3.16, condition (1) holds. Moreover, by Lemma 3.14, we may write $f = g \circ h$, where $h : A^* \to \mathbb{N}^k$ is the canonical morphism and $g : \mathbb{N}^k \to \mathbb{Z}$ is defined by
\[
g(n) = f(a_1^{n_1} \cdots a_k^{n_k}).
\]
By Lemma 3.18, the Mahler expansion
\[
g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}
\]
of $g$ is defined by $\langle g, r \rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle$.

Assume that $v \in A^*$ and $j \in \mathbb{N}$ are such that $1 \leq j \leq |v|_{a_i}$ for every $i \in \{1, \ldots, k\}$. Let $r = (|v|_{a_1}, \ldots, |v|_{a_k})$. By Lemma 3.17, $g$ is $\mathcal{G}_p$-hereditarily continuous and so we get $v_p(j) \leq v_p((g, r)) = v_p((f, a_1^{r_1} \cdots a_k^{r_k}))$ by Theorem 3.12. Since $v \sim a_1^{r_1} \cdots a_k^{r_k}$, we get $\langle f, v \rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle$ by Lemma 3.16 and so $v_p(j) \leq v_p((f, v))$. Thus condition (2) holds.
Conversely, assume that conditions (1) and (2) hold. By Lemma 3.16, if \( f(u) = f(v) \) whenever \( u \sim v \) and so there exists a function \( g : \mathbb{N}^k \to \mathbb{Z} \) such that \( f = g \circ h \). By Lemma 3.17, it suffices to show that \( g \) is \( G_p \)-hereditarily continuous. Let
\[
g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}
\]
be the Mahler expansion of \( g \) and suppose that \( 1 \leq j \leq \max \{r_1, \ldots, r_k\} \). By Theorem 3.12, we only need to show that
\[
v_p(j) \leq v_p(\langle g, r \rangle).
\] (3.19)
By Lemma 3.18, we have \( \langle g, r \rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle \). Since
\[
1 \leq j \leq \max \{r_1, \ldots, r_k\} = \max \{|a_1^{r_1} \cdots a_k^{r_k}|_{a_1}, \ldots, |a_1^{r_1} \cdots a_k^{r_k}|_{a_k}\},
\]
it follows from condition (2) that \( v_p(j) \leq v_p(\langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle) \) and so (3.19) holds as required. \( \square \)

4 G-hereditary continuity
Let \( \mathbb{P} \) denote the set of all positive primes.

Theorem 4.1 A function from \( \mathbb{Z}^k \) to \( \mathbb{Z} \) is \( G \)-hereditarily continuous if and only if, for each prime \( p \), it is nonexpansive for the \( p \)-adic norm.

Proof. Since \( G \cap \text{Com} = \bigvee_{p \in \mathbb{P}} (G_p \cap \text{Com}) \), it follows from Propositions 2.4 and 2.5 that a function from \( \mathbb{Z}^k \) to \( \mathbb{Z} \) is \( G \)-hereditarily continuous if and only if it is \( G_p \)-hereditarily continuous for every \( p \in \mathbb{P} \). It now remains to apply Theorem 3.5 to conclude. \( \square \)

Theorem 3.12 yields:

Theorem 4.2 Let \( f(u) = \sum_{r \in \mathbb{N}^k} (f, r) \binom{n_1}{r_1} \cdots \binom{n_k}{r_k} \) be the Mahler expansion of a function \( f : \mathbb{N}^k \to \mathbb{Z} \). Then the following conditions are equivalent:

1. \( f \) is \( G \)-hereditarily continuous;
2. \( j \) divides \( (f, r) \) for all \( r \in \mathbb{N}^k \) such that \( 1 \leq j \leq \max \{r_1, \ldots, r_k\} \).

We present now the analogue of Theorem 3.19 through an adaptation of its proof. We keep the notation introduced in Section 3.

Theorem 4.3 Let \( f : A^* \to \mathbb{Z} \) be a function and let \( f(u) = \sum_{v \in A^*} (f, v) \binom{u}{v} \) be its Mahler expansion. Then \( f \) is \( G \)-hereditarily continuous if and only if it satisfies the following conditions:

1. if \( u \) and \( v \) are commutatively equivalent, then \( (f, u) = (f, v) \);
2. \( j \) divides \( (f, v) \) for all \( v \in A^* \) and \( 1 \leq j \leq \max_{a \in A} |v|_a \).
Proof. Assume that \( f \) is \( G \)-hereditarily continuous. Since \( G \)-hereditarily continuous implies \( G_p \)-hereditarily continuous, Lemma 3.14 remains valid for \( G \). Together with Lemma 3.16, this yields condition (1). Moreover, by Lemma 3.14, we may write \( f = gh \), where \( h : A^* \to \mathbb{N}^k \) is the canonical morphism and \( g : \mathbb{N}^k \to \mathbb{Z} \) is defined by 
\[
g(n) = f(a_1^{n_1} \cdots a_k^{n_k})
\]
By Lemma 3.18, the Mahler expansion 
\[
g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}
\]
of \( g \) is defined by \( \langle g, r \rangle = (f, a_1^{r_1} \cdots a_k^{r_k}) \).

Assume that \( 1 \leq j \leq |v|_{a_i} \) for some \( v \in A^* \) and \( i \in \{1, \ldots, k\} \). Let \( r = ([v]_{a_1}, \ldots, [v]_{a_k}) \). By Lemma 3.17, \( g \) is \( G \)-hereditarily continuous and so we get 
\[
\langle g, r \rangle = (f, a_1^{r_1} \cdots a_k^{r_k})
\]
by Theorem 4.2. Since \( v \sim a_1^{r_1} \cdots a_k^{r_k} \), we get 
\[
\langle f, v \rangle = (f, a_1^{r_1} \cdots a_k^{r_k})
\]
by Lemma 3.16 and so \( j \mid \langle f, v \rangle \). Thus condition (2) holds.

Conversely, assume that conditions (1) and (2) hold. By Lemma 3.16, \( f(u) = f(v) \) whenever \( u \sim v \) and so there exists a function \( g : \mathbb{N}^k \to \mathbb{Z} \) such that \( f = gh \).

By Lemma 3.17, it suffices to show that \( g \) is \( G \)-hereditarily continuous. Let 
\[
g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}
\]
be the Mahler expansion of \( g \) and suppose that \( 1 \leq j \leq \max\{r_1, \ldots, r_k\} \). By Theorem 4.2, we only need to show that 
\[
j \mid \langle g, r \rangle.
\]
By Lemma 3.18, we have \( \langle g, r \rangle = (f, a_1^{r_1} \cdots a_k^{r_k}) \). Since 
\[
1 \leq j \leq \max\{r_1, \ldots, r_k\} = \max\{|a_1^{r_1} \cdots a_k^{r_k}|_{a_1}, \ldots, |a_1^{r_1} \cdots a_k^{r_k}|_{a_k}\},
\]
it follows from condition (2) that \( j \mid \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle \) and so (4.20) holds as required. \( \square \)

5 A-uniform continuity

Given a variety \( V \), let \( CV = \mathbf{Com} \cap V \). In particular \( CA \) is the variety of commutative and aperiodic monoids. For each \( t \in \mathbb{N} \), let \( A_t = [x^{t+1} = x^t] \) and \( CA_t = \mathbf{Com} \cap A_t \) be the variety of commutative aperiodic monoids of exponent \( t \).

Let also \( N_t \) denote the monogenic monoid presented by \( \langle x \mid x^t = x^{t+1} \rangle \). We usually view \( N_t \) as a quotient of \( \mathbb{N} \) in order to represent its elements by natural numbers. The following results are folklore.

Proposition 5.1 Every variety of commutative monoids is generated by its monogenic monoids. In particular \( CA_t = (N_t) \) for every \( t \in \mathbb{N} \). Moreover, if \( V \subseteq CA \), then \( V = CA_t \) for some \( t \in \mathbb{N} \).
Given \( m, n \in \mathbb{N} \), let us set
\[
(m \land n) = \begin{cases} 
\min\{m, n\} & \text{if } m \neq n \\
\infty & \text{if } m = n
\end{cases}
\]

More generally, for \( u, v \in \mathbb{N}^k \), we set write

\[
(u \land v) = \min\{u_1 \land v_1, \ldots, u_k \land v_k\}.
\]

**Lemma 5.2** Let \( u, v \in \mathbb{N}^k \) and \( t \in \mathbb{N} \). Then:

1. \( r_A(u, v) = r_{CA}(u, v) = (u \land v) + 2 \);
2. \( r_{A_t}(u, v) = r_{CA_t}(u, v) = \begin{cases} 
(u \land v) + 2 & \text{if } (u \land v) < t \\
\infty & \text{otherwise}
\end{cases} \)

**Proof.** We may assume that \( u \neq v \). Let \( V \subseteq A \). Since \( CV \subseteq V \) and every quotient of \( \mathbb{N}^k \) in \( V \) is necessarily in \( CV \), we have \( r_V(u, v) = r_{CV}(u, v) \). We show next that

\[
r_{CV}(u, v) = \min\{|N_t| \mid N_t \in CV \text{ and separates } u \text{ and } v}\]. \hspace{1cm} (5.21)

Indeed, if \( M \in CV \) separates \( u \) and \( v \) through \( \psi: \mathbb{N}^k \to M \), it follows from the proof of Proposition 5.1 that there exists an onto homomorphism \( \varphi: N_{t_1} \times \cdots \times N_{t_n} \to M \), where each \( N_{t_i} \) may be assumed to be a submonoid of \( M \). Since \( \mathbb{N}^k \) is a free commutative monoid, we may factor \( \psi \) through \( \theta: \mathbb{N}^k \to N_{t_1} \times \cdots \times N_{t_n} \).

Since \( \psi(u) \neq \psi(v) \), one of the component morphisms \( \theta_i: \mathbb{N}^k \to N_{t_i} \) must separate \( u \) and \( v \). Therefore the smallest \( M \in CV \) separating \( u \) and \( v \) must be of the form \( N_t \) and so (5.21) holds.

(1) By (5.21), we have

\[
r_{CA}(u, v) = \min\{|N_t| \mid N_t \text{ separates } u \text{ and } v\}. \hspace{1cm} (5.22)
\]

If \( u \land v = u_i \land v_i \), it is immediate that the projection on the \( i \)-th component induces a morphism from \( \mathbb{N}^k \) to \( N_{(u \land v) + 1} \) separating \( u \) and \( v \).

Suppose now that \( \eta: \mathbb{N}^k \to N_t \) separates \( u \) and \( v \) with \( t \leq (u \land v) \). Since
\[
\sum_{i=1}^k \eta(u_i e_i) = \eta(u) \neq \eta(v) = \sum_{i=1}^k \eta(v_i e_i),
\]
we have \( \eta(u_i e_i) \neq \eta(v_i e_i) \) for some \( i \in \{1, \ldots, k\} \). Hence \( \eta(e_i) \geq 1 \). Since \( u_i, v_i \geq t \), it follows that \( \eta(u_i e_i) = t = \eta(v_i e_i) \), a contradiction.

Thus \( N_{(u \land v) + 1} \) is the smallest \( N_t \) separating \( u \) and \( v \). In view of (5.21), it follows that

\[
r_{CA}(u, v) = |N_{(u \land v) + 1}| = (u \land v) + 2.
\]
(2) By $(5.21)$ and Proposition 5.1, we have
\[ r_{CA_t}(u,v) = \min\{|N_s| \mid s \leq t \text{ and } N_s \text{ separates } u \text{ and } v \}. \]

In view of $(5.22)$, it follows that
\[ r_{CA_t}(u,v) = \begin{cases} r_{CA_t}(u,v) & \text{if } r_{CA_t}(u,v) \leq t + 1 = |N_t| \\ \infty & \text{otherwise} \end{cases} \]

By $(1)$, $r_{CA_t}(u,v) \leq t+1$ is equivalent to $(u \land v) < t$ and the claim follows. □

**Theorem 5.3** Let $f : \mathbb{N} \to \mathbb{N}$ be a mapping. Then the following conditions are equivalent:

1. $f$ is $A$-uniformly continuous,
2. for all $n \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that, for all $u,v \in \mathbb{N}$, $u \land v \geq s$ implies $f(u) \land f(v) \geq n$,
3. for every $n \in \mathbb{N}$, $f^{-1}(n)$ is either finite or cofinite.

**Proof.** $(1) \iff (2)$. It follows from the definition that $f$ is $A$-uniformly continuous if and only if for all $n \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that, for all $u,v \in \mathbb{N}$,
\[ r_A(u,v) \geq s \text{ implies } r_A(f(u),f(v)) \geq n, \]
that is equivalent to $(2)$ by Lemma 5.2.

$(2) \implies (3)$. Suppose that $f^{-1}(m)$ is neither finite nor cofinite. Let $s \in \mathbb{N}$ be arbitrary. Take $u_s \in f^{-1}(m)$ and $v_s \in \mathbb{N} \setminus f^{-1}(m)$ such that $u_s,v_s \geq s$. Thus the relation
\[ u_s \land v_s \geq s \text{ and } f(u_s) \land f(v_s) \leq m \]
holds for all $s \in \mathbb{N}$, and so $(2)$ fails.

$(3) \implies (2)$. Let $n \in \mathbb{N}$. Suppose first that $f^{-1}(m)$ is cofinite for some $m \in \mathbb{N}$. Let $s = \max(\mathbb{N} \setminus f^{-1}(m))$. If $u \land v \geq s + 1$, then $u \neq v$ implies $u,v \geq s + 1$ and so $f(u) = m = f(v)$, hence we have $f(u) = f(v)$ in any case and $f(u) \land f(v) > n$ trivially.

Assume now that $f^{-1}(i)$ is finite for every $i \in \mathbb{N}$. Let $s = \max(\bigcup_{i=0}^{m-1} f^{-1}(i))$. If $u \land v \geq s + 1$ and $u \neq v$, then $u,v \geq s + 1$ and so $u,v \notin \bigcup_{i=0}^{m-1} f^{-1}(i)$. Hence $f(u),f(v) \geq n$ and so $f(u) \land f(v) \geq n$. Therefore $(2)$ holds. □

Similarly, we get

**Theorem 5.4** Let $f : \mathbb{N}^k \to \mathbb{N}$ be a mapping. Then the following conditions are equivalent:

1. $f$ is $A$-uniformly continuous;
2. for all $n \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that for all $u,v \in \mathbb{N}^k$, $u \land v \geq s$ implies $f(u) \land f(v) \geq n$.

However, there is no analogue of condition $(3)$ of Theorem 5.3 in this case: if we define $f : \mathbb{N}^2 \to \mathbb{N}$ by $f(m,n) = m$, it is immediate that $f$ is $A$-uniformly continuous and $f^{-1}(m)$ is infinite for every $m \in \mathbb{N}$.
Theorem 5.5 Let \( f : \mathbb{N}^k \to \mathbb{N} \) be a mapping and \( t \in \mathbb{N} \). Then the following conditions are equivalent:

1. \( f \) is \( A_t \)-uniformly continuous,
2. for all \( u, v \in \mathbb{N}^k \), if \( u \wedge v \geq t \) implies \( f(u) \wedge f(v) \geq t \).

Proof. Since \( \mathbb{N}^k \) and \( \mathbb{N} \) are commutative, the pseudo-metrics \( d_{A_t} \) and \( d_{CA_t} \) coincide in both monoids. Hence \( f \) is \( A_t \)-uniformly continuous if and only if it is \( CA_t \)-uniformly continuous.

Since \( \text{Im} r_{CA_t} = \{2, \ldots, t+1, \infty\} \) by Lemma 5.2 (2), it follows from Proposition 2.2 that \( f \) is \( CA_t \)-uniformly continuous if and only if for all \( u, v \in \mathbb{N}^k \), \( r_{CA_t}(u, v) = \infty \) implies \( r_{CA_t}(f(u), f(v)) = \infty \). Now the claim follows from the same Lemma 5.2 (2). □

6 A-hereditary continuity

Lemma 6.1 A function from a monoid \( M \) to \( \mathbb{N} \) is \( A \)-hereditarily continuous if and only if it is \( CA \)-uniformly continuous for every \( t \in \mathbb{N} \).

Proof. By Proposition 2.5, a function is \( A \)-hereditarily continuous if and only if it is \( CA \)-hereditarily continuous. The lemma now follows from [12, Proposition 5.9]. □

Theorem 6.2 Let \( f : \mathbb{N}^k \to \mathbb{N} \) be a mapping. Then the following conditions are equivalent:

1. \( f \) is \( A \)-hereditarily continuous,
2. for all \( u, v \in \mathbb{N}^k \), if \( u \wedge v \leq f(u) \wedge f(v) \),
3. \( f \) is \( A \)-nonexpansive.

Proof. (1) is equivalent to (2). By Lemma 6.1, \( f \) is \( A \)-hereditarily continuous if and only it is \( CA_t \)-uniformly continuous for every \( t \in \mathbb{N} \). In view of Lemma 5.2, this amounts to stating that, for all \( t \in \mathbb{N} \) and for all \( u, v \in \mathbb{N}^k \), if \( u \wedge v \geq t \) implies \( f(u) \wedge f(v) \geq t \).

(2) is equivalent to (3). By Lemma (5.2) (1), an equivalent formulation of (2) is that, for all \( u, v \in \mathbb{N}^k \), \( r_A(u, v) \leq r_A(f(u), f(v)) \), which is equivalent to (3). □

We now look for a more explicit characterization of \( A \)-hereditary continuity. Given a function \( f : \mathbb{N}^k \to \mathbb{N} \), we say that \( g : \mathbb{N} \to \mathbb{N} \) is a slice function of \( f \) if there exists some \( j \in \{1, \ldots, k\} \) and \( a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k \in \mathbb{N} \) such that \( g(x) = f(a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_k) \) for every \( x \in \mathbb{N} \).

A function \( f : \mathbb{N} \to \mathbb{N} \) is said to be extensive if \( x \leq f(x) \) for every \( x \in \mathbb{N} \) and truncated if there exists some \( m \in \mathbb{N} \) such that \( x \leq f(x) \) for \( x \leq m \) and \( f(x) = m \) for \( x > m \). Functions that are either extensive or truncated can be described by the following single property:

(C) if \( b = \min\{x \in \mathbb{N} \mid f(x) < x\} \), then \( f(x) = b - 1 \) for every \( x \geq b \).

Indeed, the case \( b = \infty \) corresponds to extensive functions and the case \( b \) finite corresponds to truncated functions.
Lemma 6.3 Let \( f : \mathbb{N}^k \to \mathbb{N} \) be a mapping satisfying condition (C) and assume that \( f(a_1, \ldots, a_k) < \min\{a_1, \ldots, a_k\} \). Then:

1. \( f(x_1, \ldots, x_k) = f(a_1, \ldots, a_k) \) for all \( x_1 \geq a_1, \ldots, x_k \geq a_k \);
2. there exists some \( c \leq \min\{a_1, \ldots, a_k\} \) such that \( f(a_1, \ldots, a_k) = f(c, \ldots, c) = c - 1 \).

Proof. (1) We use induction on \( k \). For \( k = 1 \), assume that \( f(a) < a \) and \( x \geq a \). Then there exists \( b = \min\{y \in \mathbb{N} \mid f(y) < y\} \) and so, by condition (C), \( x \geq a \) implies \( f(x) = f(a) = b - 1 \).

Assume now that \( k > 1 \) and (1) holds for smaller values of \( k \). Let \( x_1 \geq a_1, \ldots, x_k \geq a_k \). By condition (C), we have \( f(a_1, \ldots, a_{k-1}, x_k) = f(a_1, \ldots, a_k) \): indeed, if we take \( b = \min\{x \in \mathbb{N} \mid f(a_1, \ldots, a_{k-1}, x) < x\} \), then \( b \leq a_k \leq x_k \) and so \( f(a_1, \ldots, a_{k-1}, x) = b - 1 = f(a_1, \ldots, a_k) \).

Define now \( g : \mathbb{N}^{k-1} \to \mathbb{N} \) by \( g(y_1, \ldots, y_{k-1}) = f(y_1, \ldots, y_{k-1}, x_k) \). Since \( f \) satisfies (C), so does \( g \). Moreover,

\[
g(a_1, \ldots, a_{k-1}) = f(a_1, \ldots, a_{k-1}, x) = b - 1 = f(a_1, \ldots, a_k) < \min\{a_1, \ldots, a_{k-1}\}.
\]

By the induction hypothesis, we get \( g(x_1, \ldots, x_{k-1}) = g(a_1, \ldots, a_{k-1}) \) since \( x_1 \geq a_1, \ldots, x_{k-1} \geq a_{k-1} \). Thus

\[
f(x_1, \ldots, x_k) = g(x_1, \ldots, x_{k-1}) = g(a_1, \ldots, a_{k-1}) = f(a_1, \ldots, a_k)
\]

as required.

(2) We use induction on \( k \). For \( k = 1 \), assume that \( f(a) < a \). Then there exists \( c = \min\{y \in \mathbb{N} \mid f(y) < y\} \) and so, by condition (C), \( a \geq c \) implies \( f(a) = f(c) = c - 1 \).

Assume now that \( k > 1 \) and (ii) holds for smaller values of \( k \). Let \( b = \min\{y \in \mathbb{N} \mid f(a_1, \ldots, a_{k-1}, y) < y\} \).

Then \( b \leq a_k \). Define \( g \) as above. We have

\[
g(a_1, \ldots, a_{k-1}) = f(a_1, \ldots, a_{k-1}, b) = b - 1 = f(a_1, \ldots, a_k)
\]

by condition (C). By the induction hypothesis, there exists \( c \leq a_1, \ldots, a_{k-1} \) such that \( g(a_1, \ldots, a_{k-1}) = g(c, \ldots, c) = c - 1 \). Thus \( c - 1 = g(a_1, \ldots, a_{k-1}) = b - 1 \) and so \( b = c \). Since \( b \leq a_k \), we get \( c \leq a_1, \ldots, a_k \). Thus \( f(a_1, \ldots, a_k) = c - 1 = g(c, \ldots, c) = f(c, \ldots, c) \) as required.

Theorem 6.4 Let \( f : \mathbb{N}^k \to \mathbb{N} \) be a mapping. Then the following conditions are equivalent:

1. \( f \) is A-hereditarily continuous;
2. every slice function of \( f \) is either extensive or truncated.

Proof. (1) \( \Rightarrow \) (2). Let \( g \) be the slice function of \( f \) defined by \( g(x) = f(a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_k) \) and let \( a_j = \min\{x \in \mathbb{N} \mid g(x) < x\} \). If \( x > a_j \), then one gets by Theorem 6.2

\[
a_j = (a_1, \ldots, a_k) \wedge (a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_k) \leq g(a_j) \wedge g(x).
\]
and since $g(a_j) < a_j$, it follows that $g(a_j) = g(x)$.

Let $z = g(a_j)$. It remains to prove that $z = a_j - 1$. Since $z < a_j$, we have $z + 1 \leq a_j$. Suppose that $z + 1 < a_j$. By Theorem 6.2, one has
\[ z+1 = (a_1, \ldots, a_k) \land (a_1, \ldots, a_{j-1}, z+1, a_{j+1}, \ldots, a_k) \leq g(a_j) \land g(z+1) = z \land g(z+1), \]
hence $g(z+1) = z < z + 1$. Since $z + 1 < a_j$, this contradicts the minimality of $a_j$. Thus $z + 1 = a_j$ and (C) holds.

(2) $\Rightarrow$ (1). We use induction on $k$. For $k = 1$, assume that $f$ satisfies condition (C) and let $u, v \in \mathbb{N}$ be distinct. By Theorem 6.2, we must prove that $u \land v \leq f(u) \land f(v)$. Let $Y = \{ y \in \mathbb{N} \mid f(y) < y \}$. If $u, v \notin Y$, the claim follows. Hence we may assume that $Y \neq \emptyset$ and $b = \min Y$. If $u, v \in Y$, then $f(u) = b - 1 = f(v)$ by condition (C) and so $u \land v \leq \infty = f(u) \land f(v)$. Finally, assume that $u \in Y$ and $v \notin Y$. Then $f(u) = b - 1$. Without loss of generality, we may assume that $f(v) \neq b - 1$. Hence $v < b$ and so $v \leq (f(u) \land f(v))$. Thus $u \land v \leq f(u) \land f(v)$ and the result holds for $k = 1$.

Assume now that $k > 1$ and the theorem holds for smaller values of $k$. Assume that $f : \mathbb{N}^k \to \mathbb{N}$ satisfies condition (C) and let $u, v \in \mathbb{N}^k$ be distinct.

Assume first that $u_i = v_i$ for some $i \in \{1, \ldots, k\}$. Without loss of generality, we may assume that $i = k$. Define $g : \mathbb{N}^{k-1} \to \mathbb{N}$ by $g(y_1, \ldots, y_{k-1}) = f(y_1, \ldots, y_{k-1}, u_k)$. Since $f$ satisfies (C), so does $g$. By the induction hypothesis and Theorem 6.2, we get
\[ u \land v = (u_1, \ldots, u_{k-1}) \land (v_1, \ldots, v_{k-1}) \leq g(u_1, \ldots, u_{k-1}) \land g(v_1, \ldots, v_{k-1}) = f(u) \land f(v) \]
as required.

Hence we may assume that $u_i \neq v_i$ for every $i \in \{1, \ldots, k\}$. Without loss of generality, we may also assume that $u \land v = u_1$. Suppose first that $f(v) < u_1$. Since $u_1 \leq v_1, \ldots, v_k$, we may apply Lemma 6.3(2) and get some $c \leq v_1, \ldots, v_k$ such that $f(v) = f(c, \ldots, c) = c - 1$. Thus $c - 1 < u_1$ and so $c < u_1 \leq u_2, \ldots, u_k$. By Lemma 6.3(1), it follows that $f(u) = f(c, \ldots, c) = f(v)$. Therefore $u \land v \leq f(u) \land f(v)$.

Next we assume that $f(u) < u_1$. Since $u_1 \leq u_2, \ldots, u_k$, we may apply Lemma 6.3(2) and get some $c \leq u_1$ such that $f(u) = f(c, \ldots, c) = c - 1$. Since $v_1, \ldots, v_k \geq u_1 \geq c$, it follows from Lemma 6.3(1) that $f(v) = f(c, \ldots, c) = f(u)$. Therefore $u \land v \leq f(u) \land f(v)$ also in this case.

The final case $f(u), f(v) \geq u_1 = u \land v$ is trivial. \qed

## 7 M-hereditary continuity

**Proposition 7.1** Let $M$ be a monoid and $f : M \to \mathbb{N}$ a mapping. Then the following conditions are equivalent:

1. $f$ is M-hereditarily continuous;
2. $f$ is both G- and A-hereditarily continuous;
3. $f$ is both Ab- and CA-hereditarily continuous.

**Proof.** The equivalence of (1) and (3) follows from [12, Proposition 5.8] and that of (2) and (3) from Proposition 2.5. \qed
Theorem 7.2 Let \( f : \mathbb{N}^k \to \mathbb{N} \) be a mapping. Then \( f \) is \( M \)-hereditarily continuous if and only if:

1. \( \gcd\{u_i - v_i \mid i = 1, \ldots, k\} \) divides \( f(u) - f(v) \) for all \( u, v \in \mathbb{N}^k \),
2. every slice function of \( f \) is either extensive or constant.

Proof. By Proposition 7.1, \( f \) is \( M \)-hereditarily continuous if and only if it is both \( G \)- and \( A \)-hereditarily continuous. Now condition (1) is equivalent to \( G \)-hereditary continuity by Theorem 4.1. By Theorem 6.4, \( A \)-hereditary continuity is equivalent to every slice function of \( f \) being either extensive or truncated. Clearly, every constant function \( f : \mathbb{N} \to \mathbb{N} \) is necessarily truncated. It remains to prove that every truncated slice function must be indeed constant in these circumstances.

Suppose that \( g : \mathbb{N} \to \mathbb{N} \) defined by \( g(x) = f(a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_k) \) is truncated with \( f(x) = m \) for every \( x > m \). Let \( M = \max(\text{Im } f) \) and take \( s < m \) arbitrary. We consider

\[ u = (a_1, \ldots, a_{j-1}, s, a_{j+1}, \ldots, a_k), \quad v = (a_1, \ldots, a_{j-1}, M + s + 1, a_{j+1}, \ldots, a_k). \]

Since

\[ M + 1 = \gcd_{1 \leq i \leq k} (u_i - v_i) | f(u) - f(v) \]

and \( |f(u) - f(v)| \leq M \), it follows that \( f(u) = f(v) \), hence \( g(s) = g(M + s + 1) = m \). Therefore \( g \) is constant as claimed. \( \square \)

Corollary 7.3 Let \( f : \mathbb{N} \to \mathbb{N} \) be a mapping. Then \( f \) is \( M \)-hereditarily continuous if and only if \( f \) is extensive or constant, and \( u - v \) divides \( f(u) - f(v) \) for all \( u, v \in \mathbb{N} \).

We can now adapt the proof of Corollary 3.13 to strengthen it:

Theorem 7.4 There are uncountably many \( M \)-hereditarily continuous functions from \( \mathbb{N} \) to \( \mathbb{N} \).

Proof. By the uniqueness of Mahler expansions, the mapping

\[
(n_r)_{r \in \mathbb{N}} \mapsto \sum_{r \in \mathbb{N}} n_r \ \text{lcm}(1, \ldots, r) \binom{r}{x}
\]

induces an injection \( \theta \) from \((\mathbb{N} \setminus \{0\})^\mathbb{N}\) to \( \mathbb{N}^\mathbb{N} \). Let \( f \in \text{Im } \theta \). By Theorem 4.2, \( f \) is \( G \)-hereditarily continuous. Since \( n_1 \geq 1 \), we have

\[
\sum_{r \in \mathbb{N}} n_r \ \text{lcm}(1, \ldots, r) \binom{r}{x} \geq n_1 \binom{x}{1} \geq x
\]

for every \( x \in \mathbb{N} \) and so \( f \) is extensive and thus \( A \)-hereditarily continuous by Theorem 6.4. Therefore \( f \) is \( M \)-hereditarily continuous by Proposition 7.1. Since \((\mathbb{N} \setminus \{0\})^\mathbb{N}\) is uncountable and \( \theta \) is one-to-one, \( \text{Im } \theta \) is an uncountable set of \( M \)-hereditarily continuous functions from \( \mathbb{N} \) to \( \mathbb{N} \). \( \square \)

We can also settle the case of functions from \( \mathbb{Z}^k \) to \( \mathbb{Z} \).
**Corollary 7.5** A function from $\mathbb{Z}^k$ to $\mathbb{Z}$ is $M$-hereditarily continuous if and only if, for each prime $p$, it is nonexpansive for the $p$-adic norm.

**Proof.** Let $f : \mathbb{Z}^k \to \mathbb{Z}$ be a function and let $V$ denote a subvariety of $M$. Since every quotient of $\mathbb{Z}^k$ is necessarily a group, the pseudo-metrics $d_V$ and $d_{V \cap G}$ coincide in $\mathbb{Z}^k$ (and in particular in $\mathbb{Z}$). It follows that $f$ is $V$-uniformly continuous if and only if it is $V \cap G$-uniformly continuous. Since $V \cap G$ takes all possible values among the subvarieties of $G$, it follows that $f$ is $M$-hereditarily continuous if and only if it is $G$-hereditarily continuous. One can now apply Theorem 4.1 to conclude.  

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**References**


