Algebraic tools for the concatenation product

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Abstract

This paper is a contribution to the algebraic study of the concatenation product. In the first part of the paper, we extend to the ordered case standard algebraic tools related to the concatenation product, like the Schützenberger product and the relational morphisms. We show in a precise way how the ordered Schützenberger product corresponds to polynomial operations on languages. In the second part of the paper, we apply these results to establish a bridge between the three standard concatenation hierarchies, namely the Straubing-Thérien's hierarchy, the Brzozowski's (or dot-depth) hierarchy and the group hierarchy.

1 Introduction

In the seventies, several classification schemes for the rational languages were proposed, based on the alternate use of certain operators (union, complementation, product and star). Some thirty years later, although much progress has been done, several of the original problems are still open. Furthermore, their significance has grown considerably over the years, on account of the successive discoveries of links with other fields, like non commutative algebra [7], finite model theory [44], structural complexity [5] and topology [14, 19, 22].

This paper is a contribution to this theory. Our original motivation was a question on concatenation hierarchies left open in [14]. Roughly speaking, the concatenation hierarchy of a given class of recognizable languages is built by alternating boolean operations (union, intersection, complement) and polynomial operations (union and marked product). For instance, the *Straubing-Thérien hierarchy* [43, 38, 40] is based on the empty and full languages of A^* and the *group hierarchy* is built on the group-languages, the languages recognized by a finite permutation automaton. It can be shown that, if the basis of a concatenation hierarchy is a variety of languages, then every level is a positive variety of languages [3, 4, 28], and therefore

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corresponds to a variety of finite ordered monoids [19]. These varieties are denoted by $\mathbf{V}_n$ for the Straubing-Thérien hierarchy and $\mathbf{G}_n$ for the group hierarchy. It was conjectured in [14] that, for each level n,

$$\mathbf{G}_n = \mathbf{V}_n * \mathbf{G} \tag{1}$$

that is, the variety $\mathbf{G}_n$ is generated by semidirect products of a monoid of $\mathbf{V}_n$ by a group. We will prove a more general result which holds for any hierarchy based on a group variety (such as commutative groups, nilpotent groups, solvable groups, etc.). A similar bridge

$$\mathbf{B}_n = \mathbf{V}_n * \mathbf{LI} \tag{2}$$

between the Brzozowski (or dot-depth) hierarchy $\mathbf{B}_n$ and the Straubing-Thérien hierarchy was established in [40, 30]. Formula (2) is the key step to show that the decidability of the variety $\mathbf{B}_n$ is equivalent to that of $\mathbf{V}_n$ [40, 30]. The hope is that Formula (1) will lead to a similar result for the group hierarchy. This issue is discussed in the last section of the paper.

These results rely on a systematic use of ordered semigroups. As was shown in [20], Eilenberg's variety theory can be extended to classes of languages which are not necessarily closed under complement. However, this new approach requires a complete recasting of most results and tools. In recent articles [28, 27, 29, 30], Pascal Weil and the author have undertaken this rewriting process, which already lead to several new results. This article is a continuation of this task, devoted this time to the concatenation product. Three main algebraic tools are adapted to ordered semigroups: power semigroups, Schützenberger products and Malcev products. Then the ordered version of the wreath product principle [30] is intensively used to obtain commutation rules between the semidirect product and the Schützenberger product. This leads to several new results on the polynomial operations.

A part of the results of this paper were announced in [24].

2 Semigroups and varieties

2.1 Ordered semigroups

All semigroups and monoids considered in this paper are either free or finite. The set of idempotents of a semigroup S is denoted $E(S)$.

A relation $\leq$ on a semigroup S is *stable* if, for every $x, y, z \in S$, $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$. An *ordered semigroup* is a semigroup S equipped with a stable partial order $\leq$ on S. Ordered monoids are defined analogously. The notation $(S, \leq)$ will sometimes be used to emphasize the role of the order relation, otherwise the order will be implicit and the notation S will be used for semigroups as well as for ordered semigroups.

The *order ideal* generated by an element s of an ordered semigroup S is the set

$$\downarrow s = \{x \in S \mid x \leq s\}$$

If S is a semigroup, S^1 denotes the monoid equal to S if S has an identity element and to $S \cup \{1\}$, where 1 is a new element, otherwise. In the latter case, the multiplication on S is extended by setting $s1 = 1s = s$ for every $s \in S^1$. If S is an ordered semigroup without identity, the order on S is extended to an order on S^1 by setting $1 \leq 1$, but no relation of the form $1 \leq s$ or $s \leq 1$ holds for $s \neq 1$.

2.2 Power semigroups

Given a semigroup S, denote by $\mathcal{P}(S)$ the semigroup of subsets of S under the multiplication of subsets, defined, for all $X, Y \subseteq S$ by

$$XY = \{xy \mid x \in X \text{ and } y \in Y\}$$

Then $\mathcal{P}(S)$ is not only a semigroup but also a semiring under union as addition and the product of subsets as multiplication, sometimes called the *power semiring* of S (or *power semigroup* of S, if the mere multiplicative structure is considered).

It is tempting to extend this notion to ordered semigroups, but the appropriate solution is somewhat subtle. Let $(S, \leq)$ be an ordered semigroup. We shall actually define not only one, but three semirings, denoted respectively by $\mathcal{P}^+(S, \leq)$, $\mathcal{P}^-(S, \leq)$ and $\mathcal{P}(S, \leq)$. The first key ingredient is a relation $\leq_+$ defined on $\mathcal{P}(S)$ by setting

$X \leq_+ Y$ if and only if, for all $y \in Y$, there exists $x \in X$ such that $x \leq y$.

It is immediate to see that the relation $\leq_+$ is a preorder and is stable under the two operations of the semiring $\mathcal{P}(S)$. Furthermore if $Y \subseteq X$, then $X \leq_+ Y$. However, it may happen that $X \leq_+ Y$ and $Y \leq_+ X$ for some $X \neq Y$. Denote by $\sim_+$ the equivalence defined by $X \sim_+ Y$ if $X \leq_+ Y$ and $Y \leq_+ X$. Then $\sim_+$ is a semiring congruence and, by a standard construction, $\leq_+$ induces a stable order on the semiring $\mathcal{P}(S)/{\sim_+}$. The underlying ordered semiring (resp. semigroup) will be denoted $\mathcal{P}^+(S, \leq)$.

The semiring (resp. semigroup) $\mathcal{P}^-(S, \leq)$ is the same semiring (resp. semigroup), equipped with the dual order.

We now come to the definition of $\mathcal{P}(S, \leq)$. We introduce another relation on $\mathcal{P}(S)$, denoted by $\leq$, and defined by setting $X \leq Y$ if and only if,

(1) for all $y \in Y$, there exists $x \in X$ such that $x \leq y$,

(2) for all $x \in X$, there exists $y \in Y$ such that $x \leq y$.

It is not difficult to see that $\leq$ is also a stable preorder on the semiring $\mathcal{P}(S)$. The associated semiring congruence $\sim$ is defined by setting $X \sim Y$ if

$X \le Y$ and $Y \le X$. Then again, $\le$ induces a stable order on the semiring $\mathcal{P}(S)/\!\sim$ and the underlying ordered semiring (resp. semigroup) is denoted $\mathcal{P}(S, \le)$.

Example 2.1 Let $(S, =)$ be a semigroup, equipped with the equality as the order relation. Then $X \le_+ Y$ if and only if $Y \subseteq X$, but $X \le Y$ if and only if $X = Y$. Therefore $\mathcal{P}(S, =) = (\mathcal{P}(S), =)$, $\mathcal{P}^+(S, =) = (\mathcal{P}(S), \supseteq)$ and $\mathcal{P}^-(S, =) = (\mathcal{P}(S), \subseteq)$.

Example 2.2 Let S be the ordered monoid $(\{0, a, 1\}, \le)$ in which 1 is the identity, 0 is a zero, $a^2 = a$ and $0 \le a \le 1$.

First, $\{0, 1\} \sim \{0, a, 1\}$. Thus, in $\mathcal{P}(S)$, $\{0, 1\}$ and $\{0, a, 1\}$ should be identified. Similarly, $\{0\} \sim_+ \{0, 1\} \sim_+ \{0, a\} \sim_+ \{0, a, 1\}$ and $\{a\} \sim_+ \{a, 1\}$. Thus $\mathcal{P}^+(S, \le) = \{\emptyset, \{0\}, \{a\}, \{1\}\}$. The orders $\le$ and $\le_+$ are represented in Figure 2.1.

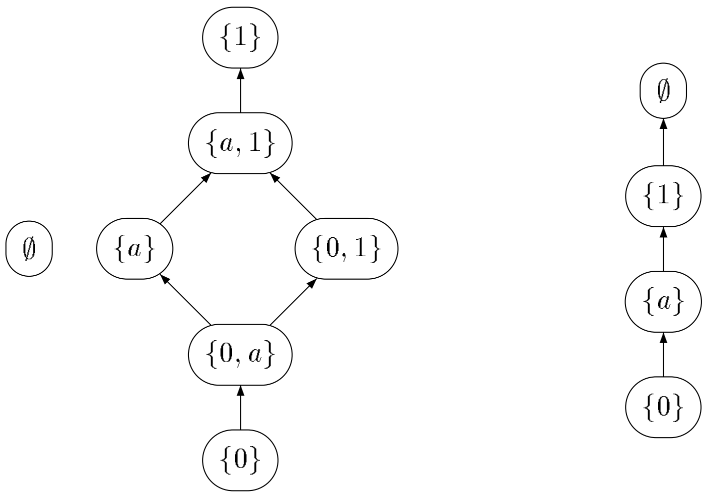

Figure 2.1: The orders $\le$ (on the left) and $\le_+$ (on the right).

The next proposition shows that the operators $\mathcal{P}$, $\mathcal{P}^+$ and $\mathcal{P}^-$ behave nicely.

Proposition 2.1 *Let $(S, \le)$ be an ordered subsemigroup (resp. a quotient, a divisor) of $(T, \le)$. Then $\mathcal{P}(S, \le)$ is an ordered subsemigroup (resp. a quotient, a divisor) of $\mathcal{P}(T, \le)$. A similar result holds for $\mathcal{P}^+$ and $\mathcal{P}^-$.*

Proof. The result is trivial for subsemigroups. Since a divisor is a quotient of a subsemigroup, it suffices to treat the case of a quotient. Let $\pi : (T, \le) \to (S, \le)$ be a surjective morphism. Then π induces a surjective semigroup morphism from $\mathcal{P}^+(T)$ onto $\mathcal{P}^+(S)$, defined by $\pi(X) = \{\pi(x) \mid x \in X\}$. Suppose that $X_1 \le X_2$. We claim that $\pi(X_1) \le \pi(X_2)$. Indeed, let $y_2 \in \pi(X_2)$. Then $y_2 = \pi(x_2)$ for some $x_2 \in X_2$, and since $X_1 \le X_2$, there exists

$x_1 \in X_1$ such that $x_1 \leq x_2$. It follows $\pi(x_1) \leq \pi(x_2)$, proving the claim. Thus π is a morphism of ordered semigroups. $\square$

2.3 Varieties of ordered semigroups

A *variety of semigroups* is a class of semigroups closed under taking subsemigroups, quotients and finite direct products [7]. *Varieties of ordered semigroups* are defined analogously [20]. Varieties of semigroups or ordered semigroups will be denoted by boldface capital letters (e.g. $\mathbf{V}$, $\mathbf{W}$).

Varieties are conveniently defined by identities. For instance, the identity $x \leq 1$ defines the variety of ordered monoids M such that, for all $x \in M$, $x \leq 1$. This variety is denoted $[\![x \leq 1]\!]$. The notation x^ω can be considered as an abbreviation for "the unique idempotent of the subsemigroup generated by x". For instance, the variety $[\![x^\omega y = x^\omega]\!]$ is the variety of semigroups S such that, for each idempotent $e \in S$ and for each $y \in S$, $ey = e$. Precise definitions can be found in the first sections of the survey paper [23]. See also [20, 26] for more specific information.

We illustrate these definitions with a detailed study of the variety of ordered semigroups defined by the identity $x^\omega y x^\omega \leq x^\omega$, denoted $\mathbf{LJ}^+$, which plays a prominent role in this paper. By definition, an ordered semigroup $(S, \leq)$ belongs to $\mathbf{LJ}^+$ if and only if, for every $s \in S$ and $e \in E(S)$, $ese \leq e$.

The notation $\mathbf{LJ}^+$ is worth an explanation. We need to go back to the variety $\mathbf{J}$ of all $\mathcal{J}$-trivial monoids. This variety can be decomposed in a natural way into a "positive part", denoted $\mathbf{J}^+$, and a "negative part", denoted $\mathbf{J}^-$, which are both varieties of ordered monoids. The former is defined by the identity $x \leq 1$ and the latter by the dual identity $1 \leq x$. One can show [28] that, as a variety of ordered monoids, $\mathbf{J} = \mathbf{J}^+ \vee \mathbf{J}^-$. Having explained the origin of $\mathbf{J}^+$, it remains to justify the $\mathbf{L}$ of $\mathbf{LJ}^+$. If $\mathbf{V}$ is a variety of ordered monoids, $\mathbf{LV}$ denotes the variety of ordered semigroups S such that, for every $e \in S$, the local semigroup eSe belongs to $\mathbf{V}$. Thus $\mathbf{LJ}^+$ denotes the variety of ordered monoids whose local semigroups are in $\mathbf{J}^+$.

Proposition 2.2 *Let S be an ordered semigroup and let $e \in E(S)$. Then the ordered semigroup $e(\downarrow e)e$ belongs to $\mathbf{LJ}^+$.*

Proof. Let $R = e(\downarrow e)e$. Let $r \in R$ and $f \in E(R)$. Then $f = ege$ with $g \leq e$ and $r = ese$ with $s \leq e$. It follows $ef = f = fe$ and $frf = fesef = fsf \leq fef = f$. Thus $R \in \mathbf{LJ}^+$. $\square$

2.4 Positive varieties

Let A be a finite alphabet. The free monoid on A is denoted by A^* and the free semigroup by A^+. A language L of A^+ is said to be *recognized* by an ordered semigroup S if there exists a semigroup morphism from A^+ onto S and an order ideal I of S such that $L = \varphi^{-1}(I)$. In this case, we also say that L is recognized by φ. It is easy to see that a language is recognized by a finite ordered semigroup if and only if it is recognized by a finite automaton, and thus is a rational (or regular) language. However, ordered semigroups provide access to a more powerful algebraic machinery than automata do, and that will be required for proving our main result.

A set of languages closed under finite intersection and finite union is called a *positive boolean algebra*. Thus a positive boolean algebra always contains the empty language and the full language A^+ since $\emptyset = \bigcup_{i\in\emptyset} L_i$ and $A^+ = \bigcap_{i\in\emptyset} L_i$. A positive boolean algebra closed under complementation is a *boolean algebra*.

A *class of recognizable languages* is a correspondence $\mathcal{C}$ which associates with each alphabet A a set $\mathcal{C}(A^+)$ of recognizable languages of A^+.

A *positive variety of languages* is a class of recognizable languages $\mathcal{V}$ such that

(1) for every alphabet A, $\mathcal{V}(A^+)$ is a positive boolean algebra,

(2) if $\varphi\colon A^+ \to B^+$ is a morphism of semigroups, $L \in \mathcal{V}(B^+)$ implies $\varphi^{-1}(L) \in \mathcal{V}(A^+)$,

(3) if $L \in \mathcal{V}(A^+)$ and if $a \in A$, then $a^{-1}L$ and La^{-1} are in $\mathcal{V}(A^+)$.

A *variety of languages* is a positive variety closed under complement. Given two positive varieties of languages $\mathcal{V}$ and $\mathcal{W}$, we write $\mathcal{V} \subseteq \mathcal{W}$ if, for each alphabet A, $\mathcal{V}(A^+) \subseteq \mathcal{W}(A^+)$.

If $\mathbf{V}$ is a variety of finite ordered semigroups, we denote by $\mathcal{V}(A^+)$ the set of $\mathbf{V}$-*languages*, that is, languages of A^+ which are recognized by an ordered semigroup of $\mathbf{V}$. Then $\mathcal{V}$ is a positive variety of languages and the correspondence $\mathbf{V} \to \mathcal{V}$ preserves inclusion. In fact, an extension of Eilenberg's variety theorem [20] states that this defines a one-to-one onto correspondence between the varieties of finite ordered semigroups and the positive varieties of languages. In particular, given two varieties of ordered semigroups $\mathbf{V}$ and $\mathbf{W}$, proving the inclusion $\mathbf{V} \subseteq \mathbf{W}$ amounts to showing that every $\mathbf{V}$-language is a $\mathbf{W}$-language.

Similarly, there is a one-to-one onto correspondence between the varieties of finite semigroups and the varieties of languages.

Finally, let us mention an elementary, but useful result. If $\mathbf{V}$ is a variety of finite ordered semigroups, let $\tilde{\mathbf{V}}$ be the *dual variety* of $\mathbf{V}$, that is, the variety of all ordered semigroups of the form $(S, \geq)$, where $(S, \leq)$ is in $\mathbf{V}$. We denote by $\mathcal{V}$ (resp. $\tilde{\mathcal{V}}$) the positive variety corresponding to $\mathbf{V}$ (resp. $\tilde{\mathcal{V}}$). If L is a language of A^*, we denote by L^c its complement in A^*.

Proposition 2.3 *For each alphabet A, $\tilde{\mathcal{V}}(A^*)$ is the set of languages of the form L^c, where L is in $\mathcal{V}(A^*)$.*

3 Algebraic study of the concatenation product

In this section, we introduce the algebraic tools used to study the concatenation product, in their ordered version. Two main tools will be considered: the relational morphims and the Schützenberger products.

3.1 Relational morphisms

The definition of a relational morphism [16] can be easily extended to ordered semigroups. If $(S, \leq)$ and $(T, \leq)$ are ordered semigroups, a *relational morphism* from S to T is a relation $\tau\colon (S, \leq) \to (T, \leq)$, i.e. a mapping from S into $\mathcal{P}(T)$ such that:

(1) $\tau(s)\tau(t) \subseteq \tau(st)$ for all $s, t \in S$,

(2) $\tau(s)$ is non-empty for all $s \in S$.

For a relational morphism between two ordered monoids $(S, \leq)$ and $(T, \leq)$, a third condition is required

(3) $1 \in \tau(1)$

Equivalently, τ is a relation whose graph

$$\mathrm{graph}(\tau) = \{\ (s, t) \in S \times T \mid t \in \tau(s)\ \}$$

is an ordered subsemigroup (resp. submonoid if S and T are monoids) of $S \times T$, with first-coordinate projection surjective onto S.

Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be varieties of ordered semigroups. A relational morphism $\tau : S \to T$ is a $(\mathbf{V}_1, \mathbf{V}_2)$-relational morphism if, for every ordered subsemigroup R of T in $\mathbf{V}_2$, the ordered semigroup $\tau^{-1}(R)$ belongs to $\mathbf{V}_1$. A $(\mathbf{V}, \mathbf{V})$-relational morphism is simply called a $\mathbf{V}$-relational morphism.

Let $\mathbf{W}$ be a variety of ordered semigroups. The class of all ordered semigroups S such that there exists a $(\mathbf{V}_1, \mathbf{V}_2)$-relational morphism $\tau : S \to T$, with $T \in \mathbf{W}$ is a variety of ordered semigroups, denoted $(\mathbf{V}_1, \mathbf{V}_2) \circledM \mathbf{W}$. If $\mathbf{V}_1 = \mathbf{V}$ and if $\mathbf{V}_2$ is the trivial variety, the notation simplifies to $\mathbf{V} \circledM \mathbf{W}$ (this is the Mal'cev product of $\mathbf{V}$ and $\mathbf{W}$). If $\mathbf{V} = \mathbf{V}_1 = \mathbf{V}_2$, we adopt the notation $\mathbf{V}^{-1}\mathbf{W}$, introduced by Straubing in [37].

To illustrate these notions, we give a characterization of $\mathbf{LJ}^+$-relational morphisms.

Proposition 3.1 *Let $\tau : S \to T$ be a relational morphism. The following conditions are equivalent:*

(1) *τ is a $\mathbf{LJ}^+$-relational morphism,*

(2) *for any $e \in E(T)$, $\tau^{-1}(e(\downarrow e)e) \in \mathbf{LJ}^+$,*

(3) *for any $e \in E(T)$, $f \in E(\tau^{-1}(e))$ and $s \in \tau^{-1}(e(\downarrow e)e)$, $fsf \leq f$.*

Proof. Proposition 2.2 shows that (1) implies (2) and (2) implies (3) is trivial. Let us show that (3) implies (1). Assuming (3), let R be an ordered subsemigroup of T such that $R \in \mathbf{LJ}^+$. Let $U = \tau^{-1}(R)$, $s \in U$, $r \in \tau(s) \cap R$ and $f \in E(U)$. Since $\tau(f) \cap R$ is a non empty subsemigroup of T, it contains an idempotent e. Now $ere \leq e$ since $R \in \mathbf{LJ}^+$ and thus $e, ere \in e(\downarrow e)e$. Furthermore $f \in \tau^{-1}(e)$, and since $ere \in \tau(f)\tau(s)\tau(f) \subseteq \tau(fsf)$, $fsf \in \tau^{-1}(ere)$. It follows by (3) that $fsf \leq f$ and thus $U \in \mathbf{LJ}^+$. Therefore, τ is a $\mathbf{LJ}^+$-relational morphism. □

We derive a simple formula on Malcev products of the form $\mathbf{LJ}^+ \circledM \mathbf{V}$, where $\mathbf{V}$ is a variety of semigroups. It is important to note, however, that this result does not extend to a variety of ordered semigroups.

Corollary 3.2 *Let* $\mathbf{V}$ *be a variety of semigroups (resp. monoids). Then the following equalities hold* $(\mathbf{LJ}^+)^{-1}\mathbf{V} = (\mathbf{LJ}^+, \mathbf{LI}) \circledM \mathbf{V} = \mathbf{LJ}^+ \circledM \mathbf{V}$.

Proof. The inclusions $(\mathbf{LJ}^+)^{-1}\mathbf{V} \subseteq (\mathbf{LJ}^+, \mathbf{LI}) \circledM \mathbf{V} \subseteq \mathbf{LJ}^+ \circledM \mathbf{V}$ are clear. If $(S, \leq) \in \mathbf{LJ}^+ \circledM \mathbf{V}$, there exists a relational morphism $\tau : S \to T$ with $T \in \mathbf{V}$ such that, for every $e \in E(T)$, $\tau^{-1}(e) \in \mathbf{LJ}^+$. Since equality is the order on T, it follows that $e = e(\downarrow e)e$ and thus, by Proposition 3.1, τ is a $\mathbf{LJ}^+$-relational morphism. Therefore $S \in (\mathbf{LJ}^+)^{-1}\mathbf{V}$. □

We now come back to a simple syntactic property of the concatenation product. For $1 \leq i \leq n$, let L_i be a recognizable language of A^+, let $\eta_i : A^+ \to S(L_i)$ be its syntactic morphism and let

$$\eta : A^+ \to S(L_1) \times S(L_2) \times \cdots \times S(L_n)$$

be the morphism defined by

$$\eta(u) = (\eta_1(u), \eta_2(u), \ldots, \eta_n(u))$$

Let u_0, u_1, ..., u_n be words of A^* and let $L = u_0 L_1 u_1 \cdots L_n u_n$. Let $\mu : A^+ \to S(L)$ be the syntactic morphism of L. The properties of the relational morphism

$$\tau = \eta \circ \mu^{-1} : S(L) \to S(L_1) \times S(L_2) \times \cdots \times S(L_n)$$

were first studied by Straubing [39] and later in [17]. The next proposition is a slight improvement of the version stated in [28].

Proposition 3.3 *The relational morphism* $\tau : S(L) \to S(L_1) \times S(L_2) \times \cdots \times S(L_n)$ *is a* $\mathbf{LJ}^+$*-relational morphism.*

Proof. Let R be an ordered subsemigroup of $S(L_1) \times S(L_2) \times \cdots \times S(L_n)$ satisfying the identity $x^\omega y x^\omega \leq x^\omega$, and let $x, y \in \eta^{-1}(R)$. Let k be an integer such that $\mu(x^k)$ and $\eta(x^k)$ are idempotent. It suffices to show that for every $u, v \in A^*$, $ux^kv \in L$ implies $ux^kyx^kv \in L$. Let $r = 2(n + |u_0 u_1 \cdots u_n|)$. Then $\eta(x^{rk}) = \eta(x^k)$, and since $ux^kv \in L$, $ux^{rk}v \in L$. Therefore, there is a factorization of the form $ux^{rk}v = u_0w_1u_1 \cdots w_nu_n$, where $w_i \in L_i$ for $0 \le i \le n$. By the choice of r, there exist $1 \le h \le n$ and $0 \le j \le r-2$ such that $w_h = w'_h x^{2k} w''_h$ for some $w'_h, w''_h \in A^*$, $ux^{jk} = u_0w_1 \cdots u_{h-1}w'_h$ and $x^{(r-j-2)k}v = w''_h u_h \cdots w_n u_n$. Now since $\eta_h(x^k)\eta_h(y)\eta_h(x^k) \le \eta_h(x^k) = \eta(x^{2k})$, the condition $w'_h x^{2k} w''_h \in L_h$ implies $w'_h x^k y x^k w''_h \in L_h$. It follows $ux^{(j+1)k}yx^{(r-j-1)k}v \in L$, and hence $ux^kyx^kv \in L$, which concludes the proof. $\square$

There is a similar result for syntactic monoids. Let, for $0 \le i \le n$, L_i be recognizable languages of A^*, let $\eta_i : A^* \to M(L_i)$ be their syntactic morphism and let

$$\eta : A^* \to M(L_0) \times M(L_1) \times \cdots \times M(L_n)$$

be the morphism defined by

$$\eta(u) = (\eta_0(u), \eta_1(u), \ldots, \eta_n(u))$$

Let a_1, a_2, ..., a_n be letters of A and let $L = L_0a_1L_1 \cdots a_nL_n$. Let $\mu : A^* \to M(L)$ be the syntactic morphism of L. Finally, consider the relational morphism

$$\tau = \mu^{-1}\eta : M(L) \to M(L_0) \times M(L_1) \times \cdots \times M(L_n)$$

Proposition 3.4 *The relational morphism $\tau : M(L) \to M(L_1) \times M(L_2) \times \cdots \times M(L_n)$ is a* $\mathbf{LJ}^+$*-relational morphism.*

3.2 Schützenberger product

One of the most useful tools for studying the concatenation product is the *Schützenberger product* of n monoids, which was originally defined by Schützenberger for two monoids [32], and extended by Straubing [38] for any number of monoids. We give an ordered version of this definition.

Let S_1, ..., S_n be ordered semigroups. The product $S_1^1 \times \cdots \times S_n^1$ is an ordered monoid, that we denote by $(M, \le)$. Let k be one of the ordered semirings $\mathcal{P}(M, \le)$, $\mathcal{P}^+(M, \le)$ or $\mathcal{P}^-(M, \le)$. Then $k^{n\times n}$, the semiring of square matrices of size n with entries in k, is also an ordered semiring, the order on which is simply inherited from the order on k: if P and P' are two matrices, $P \le P'$ if and only if for $1 \le i \le j \le n$, $P_{i,j} \le P'_{i,j}$ in k.

For now, let k be the semiring $\mathcal{P}(M, \le)$. The *Schützenberger product* of $S_1, \ldots, S_n$, denoted by $\Diamond_n(S_1, \ldots, S_n)$, is the ordered subsemigroup of the multiplicative ordered semigroup composed of all the matrices P of $k^{n\times n}$ satisfying the three following conditions:

(1) If $i > j$, $P_{i,j} = 0$

(2) If $1 \le i \le n$, $P_{i,i} = \{(1, \ldots, 1, s_i, 1, \ldots, 1)\}$ for some $s_i \in S_i$

(3) If $1 \le i \le j \le n$, $P_{i,j} \subseteq 1 \times \cdots \times 1 \times S_i^1 \times \cdots \times S_j^1 \times 1 \cdots \times 1$.

Condition (1) shows that the matrices of the Schützenberger product are upper triangular, condition (2) enables us to identify the diagonal coefficient $P_{i,i}$ with an element s_i of S_i and condition (3) shows that if $i < j$, $P_{i,j}$ can be identified with a subset of $S_i^1 \times \cdots \times S_j^1$. With this convention, a matrix of $\diamondsuit_3(S_1, S_2, S_3)$ will have the form

$$\begin{pmatrix} s_1 & P_{1,2} & P_{1,3} \\ 0 & s_2 & P_{2,3} \\ 0 & 0 & s_3 \end{pmatrix}$$

with $s_i \in S_i$, $P_{1,2} \subseteq S_1^1 \times S_2^1$, $P_{1,3} \subseteq S_1^1 \times S_2^1 \times S_3^1$ and $P_{2,3} \subseteq S_2^1 \times S_3^1$.

The *positive (resp. negative) Schützenberger product* of $S_1, \ldots, S_n$, denoted by $\diamondsuit_n^+(S_1, \ldots, S_n)$ (resp. $\diamondsuit_n^-(S_1, \ldots, S_n)$) are defined in the same way, by replacing the semiring $\mathcal{P}(M, \le)$ by $\mathcal{P}^+(M, \le)$ (resp $\mathcal{P}^-(M, \le)$).

We first state some elementary properties of the Schützenberger product. Let $S_1, \ldots, S_n$ be ordered semigroups and let S be their (resp. positive, negative) Schützenberger product.

Proposition 3.5 *Each S_i is a quotient of S.*

Proof. Let $\pi_{i,i} : S \to S_i$ the map defined by $\pi_{i,i}(P) = P_{i,i}$. Then $\pi_{i,i}$ is a surjective morphism of ordered semigroups. Thus S_i is a quotient of S. □

Proposition 3.6 *For each sequence $1 \le i_1 < \ldots < i_k \le n$, $\diamondsuit_k^+(S_{i_1}, \ldots, S_{i_k})$ is an ordered subsemigroup of S.*

Proof. Let, for $1 \le i \le n$, e_i be an idempotent of S_i and set

$$M_i = \begin{cases} S_i^1 & \text{if } i \in \{i_1, \ldots, i_k\} \\ e_i & \text{otherwise} \end{cases}$$

Consider the ordered subsemigroup T of S consisting of the matrices P such that:

(1) $P_{i,i} = \{e_i\}$ if $i \notin \{i_1, \ldots, i_k\}$,

(2) $P_{i,j} \subseteq M_i \times \cdots \times M_j$ if $j > i$ and $i, j \in \{i_1, \ldots, i_k\}$,

(3) $P_{i,j} = \emptyset$ if $i \notin \{i_1, \ldots, i_k\}$ or $j \notin \{i_1, \ldots, i_k\}$

For instance, if $n = 3$, $k = 2$, $i_1 = 1$ and $i_2 = 3$, T would consist of the matrices of the form

$$\begin{pmatrix} s_1 & 0 & P_{1,3} \\ 0 & e_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix}$$

with $P_{1,3} \subseteq S_1^1 \times 1 \times S_3^1$.

Now, extract from each matrix $P \in T$ the matrix $\varphi(P)$ obtained by deleting the rows and columns of index not in $\{i_1, \dots, i_k\}$. By construction, φ induces an isomorphism from T onto $\diamondsuit_k^+(S_{i_1}, \dots, S_{i_k})$. $\square$

The Schützenberger product preserves ordered subsemigroups, quotient and division. The proof, which relies on Proposition 2.1, is left to the reader.

Proposition 3.7 *Let, for $1 \le i \le n$, S_i be an ordered subsemigroup (resp. a quotient, a divisor) of T_i. Then $\diamondsuit_n^+(S_1, \dots, S_n)$ is an ordered subsemigroup (resp. a quotient, a divisor) of $\diamondsuit_n^+(T_1, \dots, T_n)$.*

Our next result gives an algebraic characterization of the languages recognized by a Schützenberger product. It is the "ordered version" of a result first proved by Reutenauer [31] for $n = 1$ and by the author [15] in the general case (see also [45] and [28]). We follow the elegant proof given by Simon [34]. We shall state separately the monoid case and the semigroup case and prove only the latter one, which is the most difficult.

Theorem 3.8 *Let M_1, ..., M_n be monoids. A language of A^* is recognized by $\diamondsuit_n^+(M_1, \dots, M_n)$ if and only if it is a positive boolean combination of languages of the form*

$$L_0 a_1 L_1 \cdots a_k L_k \tag{3}$$

where $k \ge 0$, $a_1, \dots, a_k \in A$ and L_j is recognized by M_{i_j} for some sequence $1 \le i_0 < i_1 < \cdots < i_k \le n$.

For the semigroup case, we need a technical definition. Let S be an ordered semigroup and let L be a language of A^*. We say that S recognizes L if S is a monoid and recognizes L or if S is not a monoid and there exists a semigroup morphism $\varphi : A^+ \to S$ such that the monoid morphism from A^* into S^1 induced by φ recognizes L. Equivalently, L is recognized by a monoid morphism $\varphi : A^* \to S^1$ such that $\varphi^{-1}(1) = 1$. This condition is crucial as we shall see in Example 3.1.

Theorem 3.9 *Let S_1, ..., S_n be ordered semigroups. A language of A^+ is recognized by $\diamondsuit_n^+(S_1, \dots, S_n)$ if and only if it is a positive boolean combination of languages recognized by one of the S_i's or of the form*

$$L_0 a_1 L_1 \cdots a_k L_k \tag{4}$$

where $k > 0$, $a_1, \dots, a_k \in A$ and L_j is a language of A^* recognized by S_{i_j} for some sequence $1 \le i_0 < i_1 < \cdots < i_k \le n$.

Proof. Let us show that the condition is sufficient. First, the languages recognized by a given ordered semigroup form a positive boolean algebra.

Next, by Proposition 3.5, every language recognized by some S_i is recognized by $S = \Diamond_n^+(S_1, \dots, S_n)$. Finally, by Proposition 3.6, it suffices to showing that every language L of the form $L = L_1 a_1 L_2 \cdots a_{n-1} L_n$, where $a_1, \dots, a_{n-1} \in A$ and L_i is a language of A^* recognized by S_i, is recognized by S. Let $\varphi_i : A^* \to S_i^1$ be a morphism recognizing L_i and such that $\varphi_i^{-1}(1) = 1$ and let $I_i = \varphi_i(L_i)$. Let $\varphi : A^+ \to S$ be the function defined by

$$\begin{aligned}
\big(\varphi(u)\big)_{i,i} &= \varphi_i(u_i) \\
\big(\varphi(u)\big)_{i,j} &= \{(1, \dots, 1, \varphi_i(u_i), \dots, \varphi_j(u_j), 1, \dots, 1) \mid \\
&\qquad\qquad u_i, \dots, u_j \in A^* \text{ and } u_i a_i u_{i+1} \cdots a_{j-1} u_j = u\}
\end{aligned}$$

It is proved in [15] that φ is actually a morphism. The next lemma shows that L is recognized by φ.

Lemma 3.10 *The set $I = \{P \in \Diamond_n^+(S_1, \dots, S_n) \mid P_{1,n} \cap (I_1 \times \cdots \times I_n) \neq \emptyset\}$ is an ideal order of S and $\varphi^{-1}(I) = L$.*

Proof. Let $P \in I$ and $P' \leq_+ P$. Since $P \in I$, $P_{1,n} \cap (I_1 \times \cdots \times I_n)$ contains some element $m = (m_1, \dots, m_n)$. Now since $P'_{1,n} \leq_+ P_{1,n}$, there exists $m' = (m'_1, \dots, m'_n) \in P'$ such that $m' \leq m$. It follows that $m' \in I_1 \times \cdots \times I_n$ and thus $P' \cap (I_1 \times \cdots \times I_n) \neq \emptyset$, that is, $P' \in I$.
The second part of the lemma follows from the following computation:

$$\begin{aligned}
\varphi^{-1}(I) &= \{u \in A^+ \mid \varphi(u) \in I\} \\
&= \{u \in A^+ \mid \varphi(u)_{1,n} \cap (I_1 \times \cdots \times I_n) \neq \emptyset\} \\
&= \{u \in A^+ \mid u = u_1 a_1 u_2 \cdots a_{n-1} u_n \text{ for some} \\
&\qquad\qquad u_1 \in \varphi^{-1}(I_1), \dots, u_n \in \varphi^{-1}(I_n)\} \\
&= L_1 a_1 L_2 \cdots a_{n-1} L_n = L \qquad \square
\end{aligned}$$

Coming back to the proof of Theorem 3.9, we now prove that the condition is necessary. Let L be a language of A^+ recognized by a morphism $\mu : A^+ \to S$. Set, for $1 \leq i, j \leq n$ and $u \in A^*$, $\mu_{i,j}(u) = \big(\mu(u)\big)_{i,j}$. This defines, for $1 \leq i \leq n$, a semigroup morphism $\mu_{i,i}$ from A^+ into S_i, which can be extended to a monoid morphism from A^* into S_i^1 such that $\mu_{i,i}^{-1}(1) = 1$. The proof is based on the following summation formula, proved in [15, 34]:

Lemma 3.11 *For every word in A^*, and for every $i, j \in \{1, \dots, n\}$,*

$$\mu_{i,j}(u) = \sum \mu_{i_0,i_0}(u_0)\mu_{i_0,i_1}(a_1)\mu_{i_1,i_1}(u_1) \cdots \mu_{i_{k-1},i_k}(a_k)\mu_{i_k,i_k}(u_k)$$

where the sum extends over all $0 \leq k \leq n$, all sequences $i \leq i_0 < i_1 < \cdots < i_k = j$ and all factorisations $u = u_0 a_1 u_1 \cdots a_k u_k$ with $a_i \in A$.

Following Simon [34], we define an *object* as a sequence

$$o = (i_0, m_0, a_1, i_1, \ldots, a_k, i_k, m_k)$$

where $0 \le k \le n$, $1 \le i_0 < i_1 < \cdots < i_k \le n$, $a_i \in A$ and $m_j \in S^1_{i_j}$. Attach to each word $u \in A^+$ the set of objects

$$F(u) = \{(i_0, m_0, a_1, i_1, \ldots, a_k, i_k, m_k) \mid \\ u \in \mu^{-1}_{i_0,i_0}(\downarrow m_0) a_1 \mu^{-1}_{i_1,i_1}(\downarrow m_1) \cdots a_k \mu^{-1}_{i_k,i_k}(\downarrow m_k)\}$$

and define a quasi-order $\le_\diamond$ on A^+ by $u \le_\diamond v$ if and only if

$$F(v) \subseteq F(u) \text{ and } \mu_{i,i}(u) \le \mu_{i,i}(v) \text{ for } 1 \le i \le n$$

Let us verify that $\le_\diamond$ is stable. Let $u, v \in A^+$ with $u \le_\diamond v$ and $a \in A$. Then $F(v) \subseteq F(u)$ and, for $1 \le i \le n$, $\mu_{i,i}(u) \le \mu_{i,i}(v)$. Since $\mu_{i,i}$ is a morphism, it follows $\mu_{i,i}(ua) \le \mu_{i,i}(va)$. Furthermore, if $o = (i_0, m_0, a_1, i_1, \ldots, a_k, i_k, m_k)$ is an object of $F(va)$, there is a factorization $va = v_0 a_1 v_1 a_2 \cdots a_k v_k$ with $v_0 \in \mu^{-1}_{i_0,i_0}(\downarrow m_0)$, $\ldots$, $v_k \in \mu^{-1}_{i_k,i_k}(\downarrow m_k)$. Two cases arise:

If $v_k \ne 1$, set $v_k = v'_k a$, $m'_k = \mu_{i_k,i_k}(v'_k)$ and $o' = (i_0, m_0, a_1, i_1, \ldots, a_k, i_k, m'_k)$. Then $o' \in F(v)$ and thus $o' \in F(u)$. Therefore

$$u \in \mu^{-1}_{i_0,i_0}(\downarrow m_0) a_1 \mu^{-1}_{i_1,i_1}(\downarrow m_1) \cdots a_k \mu^{-1}_{i_k,i_k}(\downarrow m'_k)$$

and since $m'_k \mu_{i_k,i_k}(a) = \mu_{i_k,i_k}(v'a) = \mu_{i_k,i_k}(v) \le m_k$,

$$ua \in \mu^{-1}_{i_0,i_0}(\downarrow m_0) a_1 \mu^{-1}_{i_1,i_1}(\downarrow m_1) \cdots a_k \mu^{-1}_{i_k,i_k}(\downarrow m_k)$$

that is, $o \in F(ua)$.

If $v_k = 1$, then $a = a_k$ and $1 \le m_k$. If $k = 1$, then $v = v_0$ and $o = (i_0, m_0, a, i_1, m_1)$. Now since $\mu_{i_0,i_0}(u) \le \mu_{i_0,i_0}(v) \le m_0$, the factorization $u = u_0 a u_1$, with $u_0 = u$ and $u_1 = 1$, shows that $o \in F(ua)$. If $k > 1$, set $o' = (i_0, m_0, a_1, \ldots, a_{k-1}, i_{k-1}, m_{k-1})$. Then $o' \in F(v) \subseteq F(u)$. Therefore

$$u \in \mu^{-1}_{i_0,i_0}(\downarrow m_0) a_1 \mu^{-1}_{i_1,i_1}(\downarrow m_1) \cdots a_{k-1} \mu^{-1}_{i_{k-1},i_{k-1}}(\downarrow m_{k-1})$$

Now, the factorization $u = u_0 a_1 u_1 a_2 \cdots a_{k-1} u_{k-1} a u_k$, with $u_k = 1$, shows that $o \in F(ua)$.

Thus $F(va) \subseteq F(ua)$ and $ua \le_\diamond va$. A dual proof would show that $au \le_\diamond av$. The next lemma shows that the order induced by μ is coarser than $\le_\diamond$.

Lemma 3.12 *If $u \le_\diamond u'$, then $\mu(u) \le_+ \mu(u')$.*

Proof. Let $u \leq_\diamondsuit u'$. Then, by definition, $\mu_{i,i}(u) \leq \mu_{i,i}(u')$ for $1 \leq i \leq n$. If $i < j$, then $\mu_{i,j}(u) = 0 = \mu_{i,j}(u')$. Suppose now $j > i$ and let $m' \in \mu_{i,j}(u')$. Then, by the summation formula,

$$m' \in m_0\mu_{i_0,i_1}(a_1)m_1 \cdots \mu_{i_{k-1},i_k}(a_k)m_k$$

for some object $o = (i_0, m_0, a_1, i_1, \ldots , a_k, i_k, m_k)$ of $F(u')$. Since $u \leq_\diamondsuit u'$, $F(u') \subseteq F(u)$. Therefore, $u = u_0a_1u_1 \cdots a_ku_k$ with $\mu_{i_0,i_0}(u_0) \leq m_0, \ldots,$ $\mu_{i_k,i_k}(u_k) \leq m_k$. Set

$$m = \mu_{i_0,i_0}(u_0)\mu_{i_0,i_1}(a_1)\mu_{i_1,i_1}(u_1) \cdots \mu_{i_{k-1},i_k}(a_k)\mu_{i_k,i_k}(u_k)$$

Then $m \leq m'$ and by the summation formula, $m \in \mu_{i,j}(u)$. We have shown that, for every m' in $\mu_{i,j}(u')$, there exists $m \leq m'$ in $\mu_{i,j}(u)$, that is, $\mu_{i,j}(u) \leq_+ \mu_{i,j}(u')$. Thus $\mu(u) \leq_+ \mu(u')$. $\square$

We are now ready to complete the proof of Theorem 3.9. By Lemma 3.12, every language recognized by S is an order ideal for $\leq_\diamondsuit$. Now, given $u \in A^+$, the principal ideal generated by u is given by the formula

$$\downarrow u = \bigcap_{1 \leq i \leq n} \mu_{i,i}^{-1}(\downarrow \mu_{i,i}(u)) \bigcap_{o \in F(u)} \mu_{i_0,i_0}^{-1}(\downarrow m_0)a_1\mu_{i_1,i_1}^{-1}(\downarrow m_1) \cdots a_k\mu_{i_k,i_k}^{-1}(\downarrow m_k)$$

Since $\leq_\diamondsuit$ has a finite index, every order ideal is a finite union of principal order ideals, which concludes the proof. $\square$

The next example shows that, in Theorem 3.9, the condition "L_j is recognized by S_{i_j}" cannot be replaced by "L_j is recognized by $S^1_{i_j}$".

Example 3.1 Consider the semigroups $S_1 = \{a, 0\}$ with $aa = 0a = 00 = 0a = 0$ and $S_2 = \{1\}$. Let $A = \{a, b\}$, and let L_1, L_2 and L be the languages of A^* defined by $L_1 = b^*ab^*$, $L_2 = A^*$ and $L = L_1aL_2$. Then L_1 is recognized by the monoid S^1_1 (but not by the semigroup S^1_1, since the morphism $\varphi : A^* \to S^1_1$ that recognizes φ needs to map b onto 1). Now, L is not recognized by $S = \diamondsuit_2(S_1, S_2)$. Indeed, a simple computation (done for instance in [7] in the monoid case) shows that, since $S_2 = \{1\}$, the semigroup S is a semidirect product of an idempotent and commutative monoid by a nilpotent semigroup. It follows by [2] that S satisfies the identities $x^\omega y^2 = x^\omega y$ and $x^\omega yz = x^\omega zy$. However, the syntactic semigroup of L is the three element semigroup $S^1_1 = \{1, a, 0\}$, which does not satisfy the first identity (take $x = 1$ and $y = a$). It follows that S^1_1 does not divide S and thus S cannot recognize L.

As a direct application of Proposition 2.3, the languages recognized by $\diamondsuit^-_n(S_1, \ldots , S_n)$ are of the form L^c (the complement of L) where L is recognized by $\diamondsuit^+_n(S_1, \ldots , S_n)$. The description of the languages recognized by $\diamondsuit_n(S_1, \ldots , S_n)$ is more involved, although the proof, which is omitted, is quite similar.

Theorem 3.13 *Let $S_1, \dots, S_n$ be ordered semigroups. A language of A^+ is recognized by $\Diamond_n(S_1, \dots, S_n)$ if and only if it is a positive boolean combination of languages recognized by one of the S_i's or of the form*

$$L_0 a_1 L_1 \cdots a_k L_k \text{ or } (L_0^c a_1 L_1^c \cdots a_k L_k^c)^c \tag{5}$$

where $k > 0$, $a_1, \dots, a_k \in A$ and L_j is a language of A^* recognized by S_{i_j} for some sequence $1 \leq i_0 < i_1 < \cdots < i_k \leq n$.

4 Semidirect products

In this section, we recall the definition of the semidirect product of ordered semigroups, its extension to varieties and we prove several commutation rules involving the Schützenberger product and the semidirect product of varieties.

4.1 Semidirect products of ordered semigroups

Let S and T be ordered semigroups. We write the product in S additively to provide a more transparent notation, but it is not meant to suggest that S is commutative. A *left action* of T on S is a map $(t, s) \mapsto t{\cdot}s$ from $T^1 \times S$ into S such that, for all $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$,

(1) $(t_1 t_2){\cdot}s = t_1(t_2{\cdot}s)$

(2) $t{\cdot}(s_1 + s_2) = t{\cdot}s_1 + t{\cdot}s_2$

(3) $1{\cdot}s = s$

(4) if $s \leq s'$ then $t{\cdot}s \leq t{\cdot}s'$

(5) if $t \leq t'$ then $t{\cdot}s \leq t'{\cdot}s$

If S is a monoid with identity 0, the action is *unitary* if it satisfies, for all $t \in T$,

(6) $t{\cdot}0 = 0$

The *semidirect product* of S and T (with respect to the given action) is the ordered semigroup $S * T$ defined on $S \times T$ by the multiplication

$$(s, t)(s', t') = (s + t{\cdot}s', tt')$$

and the product order:

$$(s, t) \leq (s', t') \quad \text{if and only if } s \leq s' \text{ and } t \leq t'$$

Given two varieties of ordered semigroups (resp. monoids) $\mathbf{V}$ and $\mathbf{W}$, denote by $\mathbf{V} * \mathbf{W}$ the variety of ordered semigroups (resp. monoids) generated by the semidirect products $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$.

The *wreath product* is closely related to the semidirect product. The wreath product $S \circ T$ of two ordered semigroups S and T is the semidirect product $S^{T^1} * T$ defined by the action of T on S^{T^1} given by

$$(t \cdot f)(t') = f(t't)$$

for $f : T^1 \to S$ and $t, t' \in T^1$. In particular, the multiplication in $S \circ T$ is given by

$$(f_1, t_1)(f_2, t_2) = (f, t_1 t_2) \text{ where } f(t) = f_1(t) + f_2(tt_1) \text{ for all } t \in T^1$$

and the order on $S \circ T$ is given by

$$(f_1, t_1) \leq (f_2, t_2) \text{ if and only if } t_1 \leq t_2 \text{ and } f_1(t) \leq f_2(t) \text{ for all } t \in T$$

It is shown in [29] that the semigroups of the variety $\mathbf{V} * \mathbf{W}$ are the divisors of the wreath products of the form $S \circ T$, where $S \in \mathbf{V}$ and $T \in \mathbf{W}$.

4.2 The wreath product principle

The wreath product principle was first stated by Straubing [36]. It provides a description of the languages recognized by the wreath product of two semigroups. It was extended to ordered semigroups in [30]. The version that will be used in this paper lies somewhere inbetween, since it corresponds to the wreath product of an ordered semigroup by a semigroup. In order to keep this paper self-contained, we reformulate the wreath product principle in this particular case.

Let T be a semigroup and let $\varphi : A^+ \to T$ be a semigroup morphism. We extend φ to a monoid morphism from A^* into T^1 by setting $\varphi(1) = 1$. Let $B_T = T^1 \times A$ and let $\sigma_\varphi : A^+ \to B_T^+$ be the sequential function associated with φ, defined by

$$\sigma_\varphi(a_1 \cdots a_n) = (1, a_1)(\varphi(a_1), a_2) \cdots (\varphi(a_1 \cdots a_{n-1}), a_n)$$

The wreath product principle can be stated as follows:

Proposition 4.1 *Let S be an ordered semigroup and let T be a semigroup. Every language of A^+ recognized by $S \circ T$ is a finite union of languages of the form $U \cap \sigma_\varphi^{-1}(V)$, where $\varphi : A^+ \to T$ is a semigroup morphism, U is a language of A^+ recognized by φ and V is a language of B_T^+ recognized by S.*

There is also a variety version of the wreath product principle:

Proposition 4.2 *Let $\mathbf{V}$ be a variety of ordered semigroups, $\mathbf{W}$ a variety of semigroups and $\mathcal{U}$ the positive variety associated with $\mathbf{V} * \mathbf{W}$. Then, for every alphabet A, $\mathcal{U}(A^+)$ is the smallest positive variety containing $\mathcal{W}(A^+)$ and the languages of the form $\sigma_\varphi^{-1}(V)$, where σ_φ is the sequential function associated with a morphism $\varphi : A^+ \to T$, with $T \in \mathbf{W}$ and $V \in \mathcal{V}(B_T^+)$.*

We shall use the wreath product principle in connection with Theorem 3.9 to obtain commutation rules between the Schützenberger product and the semidirect product of varieties. This will lead us to consider expressions of the form $\sigma_\varphi^{-1}(V)$, where V is a language of B_T^+ of the form $L_0(t_1, a_1)L_1 \cdots (t_k, a_k)L_k$, where (t_1, a_1), ..., (t_k, a_k) are elements of B_T and L_0, ..., L_k are languages of B_T^*.

Define, for each $t \in T^1$, a morphism $\lambda_t : B_T^+ \to B_T^+$ by setting $\lambda_t(s, a) = (ts, a)$. Then for each $u, v \in A^*$ and $a \in A$:

$$\sigma_\varphi(uav) = \sigma_\varphi(u)(\varphi(u), a)\lambda_{\varphi(ua)}(\sigma_\varphi(v)) \tag{6}$$

Let $t_{k+1} \in T$. Setting $s_0 = 1$ and, for $1 \le j \le k$, $s_j = t_j\varphi(a_j)$, the following formula holds

Lemma 4.3 *(Inversion formula)*

$$\sigma_\varphi^{-1}\big(L_0(t_1, a_1)L_1 \cdots (t_k, a_k)L_k\big) \cap \varphi^{-1}(t_{k+1}) = K_0a_1K_1 \cdots a_kK_k$$

where, for $1 \le j \le k$, $K_j = \sigma_\varphi^{-1}(\lambda_{s_j}^{-1}(L_j)) \cap \varphi^{-1}(s_j^{-1}t_{j+1})$.

Proof. Denote respectively by L and R the left and the right hand sides of the formula. If $u \in L$, then

$$\sigma_\varphi(u) = v_0(t_1, a_1)v_1(t_2, a_2) \cdots (t_k, a_k)v_k$$

with $v_j \in L_j$. Let $u = u_0a_1u_1 \cdots a_ku_k$, with $|u_j| = |v_j|$ for $0 \le j \le k$. Then

$$\begin{aligned}\sigma_\varphi(u) = {}& \underbrace{\sigma_\varphi(u_0)}_{v_0}\,\underbrace{(\varphi(u_0), a_1)}_{(t_1,a_1)}\,\underbrace{\lambda_{\varphi(u_0a_1)}(\sigma_\varphi(u_1))}_{v_1}\ \cdots \\ & \underbrace{(\varphi(u_0a_1 \cdots u_{k-1}), a_k)}_{(t_k,a_k)}\,\underbrace{\lambda_{\varphi(u_0a_1u_1\cdots u_{k-1}a_k)}(\sigma_\varphi(u_k))}_{v_k}\end{aligned}$$

It follows

$$\sigma_\varphi(u_0) \in L_0, \lambda_{\varphi(u_0a_1)}(\sigma_\varphi(u_1)) \in L_1, \ldots, \lambda_{\varphi(u_0a_1u_1\cdots u_{k-1}a_k)}(\sigma_\varphi(u_k)) \in L_k$$

and $(\varphi(u_0), a_1) = (t_1, a_1)$, ..., $(\varphi(u_0a_1 \cdots u_{k-1}), a_k) = (t_k, a_k)$. These conditions, added to the condition $\varphi(u) = t_{k+1}$, can be rewritten as

$$s_j\varphi(u_j) = t_{j+1} \text{ and } \lambda_{s_j}(\sigma_\varphi(u_j)) \in L_j \text{ for } 0 \le j \le k$$

and thus, are equivalent to $u_j \in K_j$, for $0 \le j \le k$. Thus $u \in R$.

In the opposite direction, let $u \in R$. Then $u = u_0a_1u_1 \cdots a_ku_k$ with $u_0 \in K_0$, ..., $u_k \in K_k$. It follows $s_j\varphi(u_j) = t_{j+1}$, for $0 \le j \le k$. Let us show that $\varphi(u_0a_1 \cdots a_ju_j) = t_{j+1}$. Indeed, for $j = 0$, $\varphi(u_0) = s_0\varphi(u_0) = t_1$, and, by induction,

$$\varphi(u_0a_1 \cdots a_ju_j) = t_j\varphi(a_ju_j) = t_j\varphi(a_j)\varphi(u_j) = s_j\varphi(u_j) = t_{j+1}$$

Now, by formula (6):

$$\sigma_\varphi(u) = \sigma_\varphi(u_0)(t_1, a_1)\lambda_{s_1}(\sigma_\varphi(u_1))(t_2, a_2) \cdots (t_k, a_k)\lambda_{s_k}(\sigma_\varphi(u_k))$$

Furthermore, by the definition of K_j, $\sigma_\varphi(u_j) \in L_j$ and thus $u \in L$, concluding the proof. $\square$

4.3 Commutation rules

We are now ready to establish our first commutation rule.

Theorem 4.4 *Let $\mathbf{V}_1, \ldots, \mathbf{V}_n$ be varieties of ordered monoids and let $\mathbf{W}$ be a variety of semigroups (resp. monoids). Then the following inclusions hold*

(1) $\Diamond_n^+(\mathbf{V}_1, \cdots, \mathbf{V}_n) * \mathbf{W} \subseteq \Diamond_n^+(\mathbf{V}_1 * \mathbf{W}, \cdots, \mathbf{V}_n * \mathbf{W})$
(2) $\Diamond_n^-(\mathbf{V}_1, \cdots, \mathbf{V}_n) * \mathbf{W} \subseteq \Diamond_n^-(\mathbf{V}_1 * \mathbf{W}, \cdots, \mathbf{V}_n * \mathbf{W})$
(3) $\Diamond_n(\mathbf{V}_1, \cdots, \mathbf{V}_n) * \mathbf{W} \subseteq \Diamond_n(\mathbf{V}_1 * \mathbf{W}, \cdots, \mathbf{V}_n * \mathbf{W})$

Proof. We give the proof of (1) when $\mathbf{W}$ is a variety of semigroups, which is the most difficult case. The other cases are similar.

Let $\mathbf{U} = \Diamond_n^+(\mathbf{V}_1, \cdots, \mathbf{V}_n) * \mathbf{W}$ and let $\mathbf{V} = \Diamond_n^+(\mathbf{V}_1 * \mathbf{W}, \cdots, \mathbf{V}_n * \mathbf{W})$. By the variety theorem, it suffices to prove that the $\mathbf{U}$-languages are $\mathbf{V}$-languages. By Theorem 4.2, every $\mathbf{U}$-language is a positive boolean combination of $\mathbf{W}$-languages and of languages of the form $\sigma_\varphi^{-1}(L)$, where $\varphi : A^+ \to T$ is a morphism from A^+ into some semigroup $T \in \mathbf{W}$, σ_φ is the sequential function associated with φ and L is a language of B_T^+ recognized by a semigroup of $\Diamond_n^+(\mathbf{V}_1, \cdots, \mathbf{V}_n)$. Since $\mathbf{W} \subseteq \mathbf{V}$, the $\mathbf{W}$-languages are $\mathbf{V}$-languages. Now, by Theorem 3.9, L is a positive boolean combination of languages of the form

$$L_0(t_1, a_1)L_1(t_2, a_2) \cdots (t_k, a_k)L_k \tag{7}$$

where $k \geq 0$, $(t_i, a_i) \in B_T$, $1 \leq i_0 < \cdots < i_k \leq n$ and L_j is a language of B_T^* recognized by a semigroup of $\mathbf{V}_{i_j}$. Since boolean operations commute with σ_φ^{-1}, it suffices to verify that $\sigma_\varphi^{-1}(L)$ is a $\mathbf{V}$-language when L is of the form (7). Observing that

$$\sigma_\varphi^{-1}(L) = \bigcup_{t_{k+1} \in T} \left(\sigma_\varphi^{-1}(L) \cap \varphi^{-1}(t_{k+1})\right)$$

Lemma 4.3 shows that $\sigma_\varphi^{-1}(L) \cap \varphi^{-1}(t_{k+1}) = K_0a_1K_1 \cdots a_kK_k$, where, for $1 \leq j \leq k$,

$$K_j = \sigma_\varphi^{-1}(\lambda_{s_j}^{-1}(L_j)) \cap \varphi^{-1}(s_j^{-1}t_{j+1})$$

Finally, L_j is recognized by a semigroup of $\mathbf{V}_{i_j}$ and since λ_{s_j} is length preserving, $\lambda_{s_j}^{-1}(L_j)$ has the same property. Similarly, since σ_φ is length preserving, it follows from [30, Theorem 3.2] that $\sigma_\varphi^{-1}(\lambda_{s_j}^{-1}(L_j))$ is recognized by a semigroup of $\mathbf{V}_{i_j} * \mathbf{W}$. Since $\varphi^{-1}(s_j^{-1}t_{j+1})$, which is recognized by T, is a $\mathbf{W}$-language, K_j is also recognized by a semigroup of $\mathbf{V}_{i_j} * \mathbf{W}$ and by Theorem 3.9, L is a $\mathbf{V}$-language. $\square$

The inclusion stated in Theorem 4.4 can be strict. For instance, take $n = 2$, $\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{I}$ and $\mathbf{W} = \mathbf{Nil}$. Then $\diamondsuit_2(\mathbf{I}, \mathbf{I}) = \mathbf{J_1}$ and it is shown in [2] that $\diamondsuit_2(\mathbf{V}_1, \mathbf{V}_2) * \mathbf{W} = \mathbf{J_1} * \mathbf{Nil} = [\![x^\omega y^2 = x^\omega y, x^\omega yz = x^\omega zy]\!]$, but $\diamondsuit_2(\mathbf{V}_1 * \mathbf{W}, \mathbf{V}_2 * \mathbf{W}) = \diamondsuit_2(\mathbf{Nil}, \mathbf{Nil}) = [\![x^\omega y^2 t^\omega = x^\omega y t^\omega, x^\omega yzt^\omega = x^\omega zyt^\omega]\!]$.

However, there are two important special cases for which Theorem 4.4 can be improved. The first case is when $\mathbf{W}$ is the variety $\mathbf{LI}$ of locally trivial semigroups. The second case is when $\mathbf{W}$ is a variety of groups. We now consider these two cases separately.

The varieties of the form $\mathbf{V} * \mathbf{LI}$ were studied in detail in [40], and, in the ordered case, in [30]. Let us briefly recall the main characterization of these varieties.

Let A be an alphabet. For each $k \geq 0$, let $C_k = A^k$. Then each word u of length k of A^* defines a letter of C_k, denoted $[u]$ to avoid any confusion. Let $\sigma_k : A^+ \to C_k^*$ be the function defined on $A^{k-1}A^*$ by

$$\sigma_k(a_1a_2 \cdots a_n) = \begin{cases} 1 & \text{if } n = k-1 \\ [a_1 \cdots a_k][a_2 \cdots a_{k+1}] \cdots [a_{n-k+1} \cdots a_n] & \text{if } n \geq k \end{cases}$$

Thus σ_k "spells" the factors of length k of u. The following result is extracted from [30, Corollary 4.23].

Proposition 4.5 *Let $\mathbf{V}$ be a non-trivial variety of ordered monoids and let $\mathcal{V}$ be the corresponding positive variety. Then, for every language L of A^+, the following conditions are equivalent:*

(1) *L is a finite union of languages of the form $\{u\}$, with $u \in A^+$ or $pA^* \cap \sigma_k^{-1}(K) \cap A^*s$ where $p, s \in A^{k-1}$ and $K \in \mathcal{V}(C_k^*)$ for some $k > 0$,*

(2) *$S(L) \in \mathbf{V} * \mathbf{LI}$,*

We shall need a technical result, which appeared as [30, Lemma 5.5].

Lemma 4.6 *Let, for $0 \leq i \leq r$, u_i be a word of A^* of length $\geq k-1$ and let $p_i = p_{k-1}(u_i)$, $s_{i+1} = s_{k-1}(u_i)$ and $u_i = p_iu_i'$. Let, for $1 \leq i \leq r$, K_i be* *a language of* C_k^*, *and let*

$$H_i = \{x \in A^* \mid \sigma_k(s_i x) \in K_i \text{ and } s_{k-1}(s_i x) = p_i\}$$

Then the following equality holds:

$$u_0 H_1 u_1' \cdots H_r u_r' = p_0 A^* \cap \sigma_k^{-1}[\sigma_k(u_0) K_1 \sigma_k(u_1) \cdots \sigma_k(u_{r-1}) K_r \sigma_k(u_r)] \cap A^* s_{r+1}$$

We can now state our second commutation property.

Theorem 4.7 *Let* $\mathbf{V}_1, \ldots, \mathbf{V}_n$ *be varieties of ordered monoids. Then the following formulas hold*

(1) $\Diamond_n^+(\mathbf{V}_1, \cdots, \mathbf{V}_n) * \mathbf{LI} = \Diamond_n^+(\mathbf{V}_1 * \mathbf{LI}, \cdots, \mathbf{V}_n * \mathbf{LI})$

(2) $\Diamond_n^-(\mathbf{V}_1, \cdots, \mathbf{V}_n) * \mathbf{LI} = \Diamond_n^-(\mathbf{V}_1 * \mathbf{LI}, \cdots, \mathbf{V}_n * \mathbf{LI})$

(3) $\Diamond_n(\mathbf{V}_1, \cdots, \mathbf{V}_n) * \mathbf{LI} = \Diamond_n(\mathbf{V}_1 * \mathbf{LI}, \cdots, \mathbf{V}_n * \mathbf{LI})$

Proof. We keep the same notation as in the proof of Theorem 4.4, with $\mathbf{W} = \mathbf{LI}$. This theorem already gives the inclusion $\mathbf{U} \subseteq \mathbf{V}$. Therefore, it suffices to show that each $\mathbf{V}$-language is a $\mathbf{U}$-language. This part of the proof is similar to the second part of the proof of [30, Lemma 5.6] and is reproduced here for the convenience of the reader.

Let K be a $\mathbf{V}$-language of A^+. Since every semigroup in $\mathbf{V}_i * \mathbf{LI}$ divides a wreath product of the form $M_i \circ S_i$, with $M_i \in \mathbf{V}_i$ and $S_i \in \mathbf{LI}$, $\mathbf{V}$ is generated by the ordered semigroups of the form $\Diamond_n^+(M_1 \circ S_1, \ldots, M_n \circ S_n)$, where $M_1 \in \mathbf{V}_1, \ldots, M_n \in \mathbf{V}_n$ and $S_1 \in \mathbf{LI}, \ldots, S_n \in \mathbf{LI}$. Therefore, we may assume that K is recognized by an ordered semigroup of this type. By Theorem 3.9, K is a positive boolean combination of languages either recognized by one of the ordered semigroups $M_i \circ S_i$ or of the form

$$K_0 a_1 K_1 \cdots a_r K_r$$

where $k > 0$, $a_1, \cdots a_r \in A$, and K_j is recognized by $(M_{i_j} \circ S_{i_j})$ for some sequence $1 \leq i_0 < i_1 < \cdots < i_r \leq n$. Using the expression of the K_j's given by Proposition 4.5, K can be written as a finite union of languages of the form

$$L = x_0 L_1 x_1 \cdots L_r x_r$$

where $x_0, \ldots, x_r \in A^*$ and $L_i = s_i A^* \cap \sigma_k^{-1}(K_i) \cap A^* p_i$ with $p_i, s_i \in A^{k-1}$ and $K_j \in \mathcal{V}_{i_j}(C_k^*)$ for some $k > 0$. Setting $u_0 = x_0 s_1$, $u_r = p_r x_r$ and, for $1 \leq i \leq r$, $u_i' = x_i s_{i+1}$, $u_i = p_i u_i'$ and $H_i = s_i^{-1} L_i$, we have

$$L = u_0 H_1 u_1' H_2 \cdots H_r u_r'$$

Furthermore, for $1 \leq i \leq r$,

$$H_i = \{x \in A^* \mid s_i x \in L_i\} = \{x \in A^* \mid \sigma_k(s_i x) \in K_i \text{ and } s_{k-1}(s_i x) = p_i\}$$

Applying Lemma 4.6 gives

$$L = p_0 A^* \cap \sigma_k^{-1}[\sigma_k(u_0)K_1\sigma_k(u_1)\cdots\sigma_k(u_{r-1})K_r\sigma_k(u_r)] \cap A^* s_{r+1}$$

where $p_0 = p_{k-1}(u_0)$ and $s_{r+1} = s_{k-1}(u_r)$. It follows that L is a U-language. □

Theorem 4.8 *Let* $\mathbf{V}_1, \ldots, \mathbf{V}_n$ *be varieties of ordered monoids and let* $\mathbf{H}$ *be a variety of groups. Then the following formulas hold*

(1) $\diamondsuit_n^+(\mathbf{V}_1, \cdots, \mathbf{V}_n) * \mathbf{H} = \diamondsuit_n^+(\mathbf{V}_1 * \mathbf{H}, \cdots, \mathbf{V}_n * \mathbf{H})$

(2) $\diamondsuit_n^-(\mathbf{V}_1, \cdots, \mathbf{V}_n) * \mathbf{H} = \diamondsuit_n^-(\mathbf{V}_1 * \mathbf{H}, \cdots, \mathbf{V}_n * \mathbf{H})$

(3) $\diamondsuit_n(\mathbf{V}_1, \cdots, \mathbf{V}_n) * \mathbf{H} = \diamondsuit_n(\mathbf{V}_1 * \mathbf{H}, \cdots, \mathbf{V}_n * \mathbf{H})$

Proof. We keep the notations of the proof of Theorem 4.4, with $\mathbf{W} = \mathbf{H}$. This theorem already gives the inclusion $\mathbf{U} \subseteq \mathbf{V}$. Therefore, it suffices to show that each $\mathbf{V}$-language is a $\mathbf{U}$-language.

Let K be a $\mathbf{V}$-language. Since every monoid in $\mathbf{V}_i * \mathbf{H}$ divides a wreath product of the form $M_i \circ G_i$, with $M_i \in \mathbf{V}_i$ and $G_i \in \mathbf{H}$, $\mathbf{V}$ is generated by the ordered monoids of the form $\diamondsuit_n^+(M_1 \circ G_1, \ldots, M_n \circ G_n)$, where $M_1 \in \mathbf{V}_1$, $\ldots$, $M_n \in \mathbf{V}_n$ and $G_1 \in \mathbf{H}$, $\ldots$, $G_n \in \mathbf{H}$. Therefore, we may assume that K is recognized by an ordered monoid of this type. Let $G = G_1 \times \cdots \times G_n$. Then $G \in \mathbf{H}$, each G_i is a quotient of G, and, by Proposition 3.7, $\diamondsuit_n^+(M_1 \circ G_1, \ldots, M_n \circ G_n)$ is a quotient of $\diamondsuit_n^+(M_1 \circ G, \ldots, M_n \circ G)$. Thus K is also recognized by the latter ordered monoid, and, by Theorem 3.8, K is a positive boolean combination of languages of the form

$$K_0 a_1 K_1 \cdots a_k K_k$$

where $k \geq 0$, $a_1, \cdots a_k \in A$, and K_j is recognized by $M_{i_j} \circ G$ for some sequence $1 \leq i_0 < i_1 < \cdots < i_k \leq n$. Now, by Proposition 4.1, K_j is a finite union of languages of the form $\sigma_\varphi^{-1}(L_j) \cap \varphi^{-1}(g_j)$ where $\varphi : A^* \to G$ is a morphism, $g_j \in G$, $\sigma_\varphi : A^* \to (G \times A)^*$ is the sequential function associated with φ and L_j is recognized by M_{i_j}. Using distributivity of product over union, we may thus suppose that $K_j = \sigma_\varphi^{-1}(L_j) \cap \varphi^{-1}(g_j)$ for $0 \leq j \leq k$. Set $n_0 = 1$, $m_1 = g_0$ and, for $1 \leq j \leq k$, $n_j = m_j\varphi(a_j)$ and $m_{j+1} = n_j g_j$.

Two special features of groups will be used now. First, if $g, h \in G$, the set $g^{-1}h$, computed in the monoid sense, is equal to $\{g^{-1}h\}$, where this time g^{-1} denotes the inverse of g as a group element. Next, each function λ_g is a bijection, and $\lambda_g^{-1} = \lambda_{g^{-1}}$. With these observations in mind, one gets

$$K_j = \sigma_\varphi^{-1}\Big(\lambda_{n_j}^{-1}\big(\lambda_{n_j^{-1}}^{-1}(L_j)\big)\Big) \cap \varphi^{-1}(n_j^{-1}m_{j+1})$$

whence, by Lemma 4.3,

$$K = \sigma_\varphi^{-1}\big(L_0'(m_1, a_1)L_1'(m_2, a_2)\cdots(m_k, a_k)L_k'\big) \cap \varphi^{-1}(m_{k+1})$$

where $L'_j = \lambda^{-1}_{n_j^{-1}}(L_j)$. Now, L'_j is recognized by M_{i_j}, and by Theorem 3.8, $L'_0(m_1, a_1)L'_1(m_2, a_2)\cdots(m_k, a_k)L'_k$ is recognized by $\Diamond^+_n(M_1, \ldots, M_n)$. It follows, by [30, Theorem 3.2], that K is a **U**-language. $\square$

Corollary 4.9 *Let* $\mathbf{V}_1, \ldots, \mathbf{V}_n$ *be varieties of ordered monoids and let* $\mathbf{H}$ *be a variety of groups. Then the following formulas hold*

(1) $\Diamond^+_n(\mathbf{V}_1, \cdots, \mathbf{V}_n) * \mathbf{LH} = \Diamond^+_n(\mathbf{V}_1 * \mathbf{LH}, \cdots, \mathbf{V}_n * \mathbf{LH})$

(2) $\Diamond^-_n(\mathbf{V}_1, \cdots, \mathbf{V}_n) * \mathbf{LH} = \Diamond^-_n(\mathbf{V}_1 * \mathbf{LH}, \cdots, \mathbf{V}_n * \mathbf{LH})$

(3) $\Diamond_n(\mathbf{V}_1, \cdots, \mathbf{V}_n) * \mathbf{LH} = \Diamond_n(\mathbf{V}_1 * \mathbf{LH}, \cdots, \mathbf{V}_n * \mathbf{LH})$

Proof. This follows from Theorems 4.8 and 4.7, since $\mathbf{LH} = \mathbf{H} * \mathbf{LI}$. $\square$

5 Polynomial closure

Let $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_n$ be positive $*$-varieties of languages. For each alphabet A, let $\mathrm{Pol}_n(\mathcal{V}_1, \ldots, \mathcal{V}_n)(A^*)$ be the positive boolean algebra generated by the languages of the form

$$L_0 a_1 L_1 \cdots a_k L_k$$

where $k \geq 0$, $a_1, \cdots a_k \in A$ and $L_j \in \mathcal{V}_{i_j}(A^*)$ for some sequence $1 \leq i_0 < i_1 < \cdots < i_k \leq n$.

If $\mathbf{V}_1, \ldots, \mathbf{V}_n$ are varieties of ordered semigroups, we denote by

$$\Diamond^+_n(\mathbf{V}_1, \ldots, \mathbf{V}_n)$$

the variety generated by the ordered semigroups $\Diamond^+_n(S_1, \ldots, S_n)$ with $S_1 \in \mathbf{V}_1, \ldots, S_n \in \mathbf{V}_n$. There is an analoguous definition for varieties of ordered monoids.

Theorem 5.1 *Let* $\mathbf{V}_1, \ldots, \mathbf{V}_n$ *be varieties of ordered monoids and* $\mathcal{V}_1, \ldots, \mathcal{V}_n$ *the corresponding positive varieties. Then the positive variety corresponding to* $\Diamond^+_n(\mathbf{V}_1, \ldots, \mathbf{V}_n)$ *is* $\mathrm{Pol}_n(\mathcal{V}_1, \ldots, \mathcal{V}_n)$.

Proof. This is an immediate consequence of Theorem 3.8. $\square$

The positive variety corresponding to $\Diamond^-_n(\mathbf{V}_1, \ldots, \mathbf{V}_n)$ is of course the dual of $\mathrm{Pol}_n(\mathcal{V}_1, \ldots, \mathcal{V}_n)$. Finally, the positive variety corresponding to $\Diamond^-_n(\mathbf{V}_1, \ldots, \mathbf{V}_n)$ is described by 3.13. If $\mathbf{V}_1, \ldots, \mathbf{V}_n$ are varieties of semigroups (resp. monoids), we obtain the following corollary, first stated in [31] for $n = 2$ and [15] for the general case

Corollary 5.2 *Let $\mathbf{V}_1, \ldots, \mathbf{V}_n$ be varieties of ordered monoids and $\mathcal{V}_1, \ldots, \mathcal{V}_n$ the corresponding positive varieties. Then the variety of languages corresponding to $\Diamond_n(\mathbf{V}_1, \ldots, \mathbf{V}_n)$ is $\mathrm{BPol}_n(\mathcal{V}_1, \ldots, \mathcal{V}_n)$.*

The definition of Pol_n is slightly different for +-varieties. Let $\mathcal{V}_1$, $\mathcal{V}_2$, $\ldots$, $\mathcal{V}_n$ be positive +-varieties of languages. For each alphabet A,

$$\mathrm{Pol}_n(\mathcal{V}_1, \ldots, \mathcal{V}_n)(A^+)$$

is the positive boolean algebra generated by the languages of the form

$$u_0 L_1 u_1 L_2 \cdots L_k u_k$$

where $k \geq 0$, u_0, u_1, $\ldots$, u_k are words of A^* such that $u_0 u_1 \cdots u_k \neq 1$ and, for $L_j \in \mathcal{V}_{i_j}(A^+)$ for some sequence $1 \leq i_0 < i_1 < \cdots < i_k \leq n$. Theorem 3.9 does not suffice, in general, to describe the languages of $\mathrm{Pol}_n(\mathcal{V}_1, \ldots, \mathcal{V}_n)(A^+)$, but it suffices when the varieties $\mathcal{V}_i$ are closed under the operations $L \to uL$ and $L \to Lu$ (where L is a language and u is a word). It is shown in [40] that a variety of langagues is closed under the operation $L \to Lu$ if and only if the corresponding variety of semigroups $\mathbf{V}$ is closed under the operation $\mathbf{V} \to \mathbf{D} * \mathbf{V}$ and this result has been extended to positive varieties in [30]. Of course, a dual statement holds for the operation $L \to uL$. We note in particular that the positive varieties corresponding to varieties of ordered semigroups of the form $\mathbf{V} * \mathbf{LI}$ are closed under the operations $L \to uL$ and $L \to Lu$.

Theorem 5.3 *Let $\mathbf{V}_1, \ldots, \mathbf{V}_n$ be varieties of ordered semigroups and $\mathcal{V}_1, \ldots, \mathcal{V}_n$ the corresponding positive varieties. If $\mathcal{V}_1, \ldots, \mathcal{V}_n$ are closed under the operations $L \to Lu$ and $L \to uL$, then the positive variety corresponding to $\Diamond_n^+(\mathbf{V}_1, \ldots, \mathbf{V}_n)$ is $\mathrm{Pol}_n(\mathcal{V}_1, \ldots, \mathcal{V}_n)$.*

Proof. Let L be a finite union of languages of the form $u_0 L_1 u_1 L_2 \cdots L_k u_k$, where u_0, u_1, $\ldots$, u_n are words of A^* and $L_j \in \mathcal{V}_{i_j}(A^*)$ for some sequence $1 \leq i_0 < i_1 < \cdots < i_k \leq n$. Observing that, for a language K of A^+ and a letter a of A,

$$K = \bigcup_{a \in A} a(a^{-1}K)$$

we may assume that the words u_1, $\ldots$, u_{n-1} are non-empty. Setting $u_i = a_i u_i'$, where $a_i \in A$, we obtain

$$u_0 L_1 u_1 L_2 \cdots L_k u_k = (u_0 L_1) a_1 (u_1' L_2) \cdots a_{k-1} (u_{k-1}' L_k u_k)$$

Now $u_0 L_1 \in \mathcal{V}_{i_1}(A^+)$, $(u_1' L_2) \in \mathcal{V}_{i_2}(A^+)$ and $u_{k-1}' L_k u_k \in \mathcal{V}_{i_k}(A^+)$. Thus, by Theorem 3.9, L is a $\Diamond_n^+(\mathbf{V}_1, \ldots, \mathbf{V}_n)$-language.

Conversely, let L be a $\Diamond_n^+(\mathbf{V}_1, \ldots, \mathbf{V}_n)$-language. By Theorem 3.9, L is is a positive boolean combination of languages recognized by one of the S_i's or of the form $L_0a_1L_1\cdots a_kL_k$ where $k>0$, $a_1,\ldots,a_k\in A$ and L_j is a language of A^* recognized by S_{i_j} for some sequence $1\le i_0<i_1<\cdots<i_k\le n$. By the definition of a language of A^* recognized by an ordered semigroup, L_j is either a language of A^+ recognized by S_{i_j} or of the form $L'_j\cup\{1\}$ where L'_j is a language of A^+ recognized by S_{i_j}. Now, using distributivity of the concatenation product over union, L can be rewritten as a finite union of languages the form $u_0L_1u_1L_2\cdots L_ku_k$, where u_0, u_1, $\ldots$, u_n are words of A^* and $L_j\in\mathcal{V}_{i_j}(A^*)$ for some sequence $1\le i_0<i_1<\cdots<i_k\le n$ and thus $L\in\mathrm{Pol}_n(\mathcal{V}_1,\ldots,\mathcal{V}_n)$. □

If $\mathcal{V}$ is a positive variety, the *polynomial closure* of $\mathcal{V}$ is the positive variety Pol $\mathcal{V}$ defined as follows

$$\mathrm{Pol}\ \mathcal{V}=\bigcup_{n>0}\mathrm{Pol}_n(\mathcal{V},\mathcal{V},\ldots,\mathcal{V})$$

We also denote by BPol $\mathcal{V}$ the boolean closure of Pol $\mathcal{V}$.

The algebraic characterization of the polynomial closure given in [28] was only proved when $\mathcal{V}$ is a variety of languages. Nevertheless, it still holds for positive varieties.

Theorem 5.4 *Let* $\mathbf{V}$ *be a variety of finite ordered monoids (resp. semigroups) and let* $\mathcal{V}$ *be the associated positive variety. Then* Pol $\mathcal{V}$ *is a positive variety and the associated variety of finite ordered monoids (resp. semigroups) is the Mal'cev product* $(\mathbf{LJ}^+)^{-1}\mathbf{V}$.

Proof. We only patch the proof of [28] by indicating the corrections to be done. The relational morphism τ, defined at the bottom of page 397, should have the following property: if an ordered subsemigroup R of V satisfies the identity $x^\omega yx^\omega\le x^\omega$, then the ordered semigroup $R\tau^{-1}$ also satisfies this identity.

Formula (5.1), on page 398, should be rewritten as

$$L=\bigcup u_0(\downarrow e_1)\mu^{-1}u_1(\downarrow e_2)\mu^{-1}u_2\ \cdots\ (\downarrow e_k)\mu^{-1}u_k$$

where the union is taken over the sequences $(e_1,e_2,\ldots,e_k)$ of idempotents of V such that $k\le K$, $|u_0u_1u_2\ \cdots\ u_k|\le K$ and $u_0(\downarrow e_1)\mu^{-1}u_1(\downarrow e_2)\mu^{-1}u_2\ \cdots$ $(\downarrow e_k)\mu^{-1}u_k\subseteq L$.

Lemma 5.6 of [28] and its proof should be modified as follows:

Lemma Let $x\in A^+$ such that $d(x)=(x_1,\ldots,x_n)$ with $n\ge 3$ and let (f,e) be an idempotent of $S\times V$ such that $x_1\delta=\ldots=x_n\delta=(f,e)$. Then, for all $u,v\in A^*$ such that $uxv\in L$, the language $ux_1(\downarrow e)\mu^{-1}x_nv$ is contained in L.

Proof. Since $x=x_1x_2\ldots x_n$, it follows $x\mu=x_1\mu\cdots x_n\mu=e$. We claim that the semigroup $R=e(\downarrow e)e$ satisfies the identity $x^\omega yx^\omega\le x^\omega$. Indeed, let $f, s \in R$, with f idempotent. Then $f = ege$ for some $g \leq e$ and $s = ete$ for some $t \leq e$. Thus $fe = f = ef$ and $fsf = f(ete)f = ftf \leq fef = f$, proving the claim. Thus $R\tau^{-1}$ satisfies the identity $x^\omega y x^\omega \leq x^\omega$.

Let now $y \in (\downarrow e)\mu^{-1}$. Then $x_1, x_n, x_1 y x_n \in R\mu^{-1}$ and hence $(x_1 y x_n)\eta = f(y\eta)f \leq f = x\eta$, etc. □

The rest of the proof is unchanged, except that all occurrences of the form $e\mu^{-1}$ should be changed to $(\downarrow e)\mu^{-1}$.

Theorems 5.1 and 5.3 lead to a characterization of the polynomial closure of a positive variety in terms of the Schützenberger product. Combining these results with Theorems 4.7 and 4.8, Corollary 4.9 and Theorem 5.4, we obtain the following commutation rules:

Theorem 5.5 *Let* $\mathbf{V}$ *be a variety of finite ordered monoids and let* $\mathbf{H}$ *be a variety of groups. Then the following equalities hold:*

(1) $(\mathbf{LJ}^+)^{-1}(\mathbf{V} * \mathbf{LI}) = ((\mathbf{LJ}^+)^{-1}\mathbf{V}) * \mathbf{LI})$,

(2) $(\mathbf{LJ}^+)^{-1}(\mathbf{V} * \mathbf{H}) = ((\mathbf{LJ}^+)^{-1}\mathbf{V}) * \mathbf{H})$,

(3) $(\mathbf{LJ}^+)^{-1}(\mathbf{V} * \mathbf{LH}) = ((\mathbf{LJ}^+)^{-1}\mathbf{V}) * \mathbf{LH})$.

6 Concatenation hierarchies

By alternating the use of the polynomial closure and of the boolean closure one can obtain hierarchies of recognizable languages. Let $\mathcal{U}$ be a variety of languages. The concatenation hierarchy of basis $\mathcal{U}$ is the hierarchy of classes of languages defined as follows.

(1) $\mathcal{U}_0 = \mathcal{U}$,

(2) for every integer $n \geq 0$, $\mathcal{U}_{n+1/2} = \mathrm{Pol}\,\mathbf{U}_n$,

(3) for every integer $n \geq 0$, $\mathcal{U}_n = \mathrm{BPol}\,\mathbf{U}_n$.

Theorem 5.4 shows that each $\mathcal{U}_{n+1/2}$ is a positive variety of languages. Furthermore the boolean closure of a positive variety of languages is a variety of languages. Therefore, each $\mathcal{U}_n$ is a variety of languages.

The associated varieties of monoids and ordered monoids (resp. semigroups and ordered semigroups) are denoted $\mathbf{U}_n$ and $\mathbf{U}_{n+1/2}$. Theorem 5.4 shows that, for every integer $n \geq 0$,

$$\mathbf{U}_{n+1/2} = (\mathbf{LJ}^+)^{-1}\mathbf{U}_n$$

The hierarchy obtained by starting with the trivial variety of monoids is called the *Straubing-Thérien hierarchy*. The corresponding varieties are denoted $\mathbf{V}_n$. Thus $\mathbf{V}_0 = \mathbf{I}$, and it is known that $\mathbf{V}_{1/2} = \mathbf{LJ}^+$ and $\mathbf{V}_1 = \mathbf{J}$. These first levels are decidable varieties. The variety $\mathbf{V}_{3/2}$ is also known to be decidable [28], but the decidability of the other levels is still an open problem.

Other concatenation hierarchies have been considered so far in the literature. The first one, introduced by Brzozowski [6] and called the *dot-depth hierarchy*, is the hierarchy $\mathbf{B}_n$ of positive $+$-varieties whose basis is the trivial variety. Given a group variety $\mathbf{H}$, on can also consider the hierarchy whose basis is the variety of languages corresponding to $\mathbf{H}$ (see [14]).

The main open problem about these hierarchies is to decide, given a rational language, whether it belongs to the n-th level of a given hierarchy. For the Straubing-Thérien and the Brzozowski hierarchies, the problem has been solved positively for $n \leq \frac{3}{2}$ [33, 3, 4, 12, 13, 28, 8, 30] and for the group hierarchy, for $n \leq 1$ [14, 11]. It is still open for the other values of n, although some partial results for the levels 2 and 5/2 of the Straubing-Thérien hierarchy are known [25, 41, 42, 28, 46, 10]. A logical approach is also possible : it amounts to deciding whether a first order formula of Büchi's sequential calculus is equivalent to a Σ_n-formula on finite words models. See [44, 18, 21] for more details.

A bridge between the Straubing-Thérien hierarchy and the Brzozowski hierarchy was built in [40] and [30]. The results of the previous sections lead to a similar bridge with the group hierarchies. Both results are summarized in the next theorem:

Theorem 6.1 *Let* $\mathbf{H}$ *be a variety of groups. For each half-integer* $n > 0$, *the following relations holds*

$$\mathbf{B}_n = \mathbf{V}_n * \mathbf{LI}, \quad \mathbf{H}_n = \mathbf{V}_n * \mathbf{H}$$

One important consequence of Theorem 6.1 is that a given level of the Brzozowski hierarchy is decidable if and only if the corresponding level of the Straubing-Thérien hierarchy is decidable, but the proof requires the so-called delay theorem (see [40] and [30]). It is tempting to conjecture a similar result for the group hierarchies, but no such general result is known, even if $\mathbf{H}$ is the variety of all groups, which is the most important case.

Does this result reduce the study of the group hierarchy to that of Straubing-Thérien's? Yes and no. Formally, our result doesn't suffice to reduce the decidability problem of $\mathbf{G}_n$ to that of $\mathbf{V}_n$. However, a recent result of Almeida and Steinberg [1] gives a reduction of the decidability problem of $\mathbf{G}_n$ to a strong property of $\mathbf{V}_n$. More precisely, Almeida and Steinberg showed that if the variety of finite categories $\mathbf{gV}_n$ generated by $\mathbf{V}_n$ has a recursively enumerable basis of (pseudo)identities, then the decidability of $\mathbf{V}_n$ implies that of $\mathbf{G}_n$. Of course, even more algebra is required to use (and even state !) this result, but it is rather satisfactory for the following reason: although the decidability of $\mathbf{V}_n$ is still an open problem for $n \geq 2$, recent conjectures tend to indicate that a good knowledge of the identities of $\mathbf{gV}_n$ will be required to prove the decidability of $\mathbf{V}_n$. In other words, it is expected that the proof of the decidability of $\mathbf{V}_n$ will require the knowledge of the identities of $\mathbf{gV}_n$, giving in turn the decidability of $\mathbf{G}_n$.

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