Local languages and the Berry-Sethi algorithm

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Abstract

One of the basic tasks in compiler construction, document processing, hypertext software and similar projects is the efficient construction of a finite automaton from a given rational (regular) expression. The aim of the present paper is to give an exposition and a formal proof of the background of the algorithm of Berry and Sethi relating the computation involved to a well-known family of recognizable languages, the local languages.

1 Introduction

One of the basic tasks in compiler construction, document processing, hypertext software and similar projects is the efficient construction of a finite automaton from a given rational (regular) expression. There exist a great variety of algorithms for this. An impressive account has been given recently by Watson [11]. For several reasons, the algorithm of Berry and Sethi [2] is of particular interest (see [4, 5] for a discussion). The aim of the present paper is to give an exposition and a formal proof of the background for this algorithm by relating the computation involved to a well-known family of recognizable languages, the local languages.

Local languages were studied in some detail in [10], see also [7]. These languages are very easy to define, and they are exactly the languages recognized by a special family of automata also called Glushkov automata. The main result used in Berry-Sethi's algorithm is that every language denoted by a linear rational expression can be recognized by a Glushkov automaton. We give a short proof of this, by showing that every language denoted by a linear rational expression is local. Observe however that the inclusion is strict.

The development of efficient algorithms is an important issue (see [8, 5, 13]) but we are not concerned with this problem in this paper. Our goal is rather to provide a simple formal proof of the correctness of the algorithm.

In the topic of transducing a regular expression to an automaton, the terminology is not yet uniform. Thus, linear expressions are called restricted in [11]. Also, what we denote by P and S is frequently written *First* and *Last*. The set of factors of length 2 of a language (or of the language denoted by an expression) that we write F for short is sometimes written *Follow*.

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A first presentation of the relation between Berry-Sethi's algorithm and local languages appeared in [3].

2 Local Languages

Given a language $L \subset A^*$, define

$$\begin{aligned} P(L) &= \{a \in A \mid aA^* \cap L \neq \emptyset\} \\ S(L) &= \{a \in A \mid A^*a \cap L \neq \emptyset\} \\ F(L) &= \{x \in A^2 \mid A^*xA^* \cap L \neq \emptyset\} \\ N(L) &= A^2 \setminus F(L) \end{aligned}$$

By definition, $P(L)$ is the set of first letters of words in L and $F(L)$ is the set of factors (subwords) of length 2 of words in L. Clearly, for every language, one has

$$L \setminus \{1\} \subset (P(L)A^* \cap A^*S(L)) \setminus A^*N(L)A^*$$

A language L is called *local* if equality holds. More precisely, a language $L \subset A^*$ is said to be *local* if there exist two subsets P and S of A and a subset N of A^2 such that [1]

$$L \setminus \{1\} = (PA^* \cap A^*S) \setminus A^*NA^*$$

For example, if $A = \{a, b, c\}$, the language

$$(abc)^* = \{1\} \cup [(aA^* \cap A^*c) \setminus A^*\{aa, ac, ba, bb, cb, cc\}A^*]$$

is local. The terminology "local" can be explained as follows: in order to know whether a given word is in L, it suffices to verify that its first letter is in P, its last letter is in S, and all its factors of length 2 are not in N. Thus, membership in L can be checked by scanning the word through a window of size 2. Conversely, if a language L is local, it is easy to recover the parameters P, S and N. Indeed P (respectively S) is the set of all first (last) letters of the words of L and N is the set of words of length 2 that are not factors of any word in L.

One can easily find a deterministic automaton recognizing a local language given the parameters P, S and N. We consider the following type of automata which, as we shall see, characterize local languages: a deterministic (but not necessarily complete) automaton $\mathcal{A} = (Q, A, ., i, T)$ is said to be *local* if, for every letter a, the set $\{q.a \mid q \in Q\}$ contains at most one element. A deterministic automaton is said to be *standard* if it contains no transition arriving on the initial state.

Proposition 2.1 *Let $L = (PA^* \cap A^*S) \setminus A^*NA^*$ be a local language. Then L is recognized by the standard local automaton $\mathcal{A}$ having $A \cup \{1\}$ as set of states, 1 as initial state, S as set of final states and whose transitions are given by the rules $1.a = a$ if $a \in P$ and $a.b = b$ if $ab \notin N$.*

Proof. Let indeed $u = a_1 \cdots a_n$ be a word accepted by $\mathcal{A}$. Then there is a successful path

$$1 \xrightarrow{a_1} a_1 \xrightarrow{a_2} a_2 \quad \cdots \quad a_{n-1} \xrightarrow{a_n} a_n$$

[1] P stands for prefix, S for suffix, and N for non-factor.

Consequently, the end of the path, a_n, is a final state and thus $a_n \in S$. Similarly, since there is a transition $1 \xrightarrow{a_1} a_1$, one has necessarily $a_1 \in P$. Finally, for $1 \leq j \leq n-1$, there is a transition $a_j \xrightarrow{a_{j+1}} a_{j+1}$, and thus $a_j a_{j+1} \notin N$. It follows that $u \in L$.

Conversely, if $u = a_1 \cdots a_n \in L$, it follows $a_1 \in P$, $a_n \in S$ and, for $1 \leq j \leq n$, $a_j a_{j+1} \notin N$. Therefore $1 \xrightarrow{a_1} a_1 \xrightarrow{a_1} a_1 \quad \cdots \quad a_{n-1} \xrightarrow{a_n} a_n$ is a successful path of $\mathcal{A}$ and $\mathcal{A}$ accepts u. Consequently the language recognized by $\mathcal{A}$ is L.

If the local language contains the empty word, the previous construction can be applied, by taking $S \cup \{1\}$ as set of final states. This completes the proof. $\square$

Proposition 2.2 *Let $L \subset A^*$ be a rational language. The following conditions are equivalent:*

(1) *L is a local language,*

(2) *L is recognized by a local automaton.*

(3) *L is recognized by a standard local automaton.*

Proof. (1) implies (3) by proposition 2.1. (3) implies (2) is trivial.
(2) implies (1). Let $\mathcal{A} = (Q, A, , ., i, T)$ be a local automaton that recognizes a language L. Set

$$\begin{aligned}
P &= \{a \in A \mid i.a \text{ is defined}\},\\
S &= \{a \in A \mid \text{ there exists } q \in Q \text{ such that } q.a \in T\},\\
N &= \{x \in A^2 \mid x \text{ is the label of no path in } \mathcal{A}\ \}\\
K &= (PA^* \cap A^*S) \setminus A^*NA^*.
\end{aligned}$$

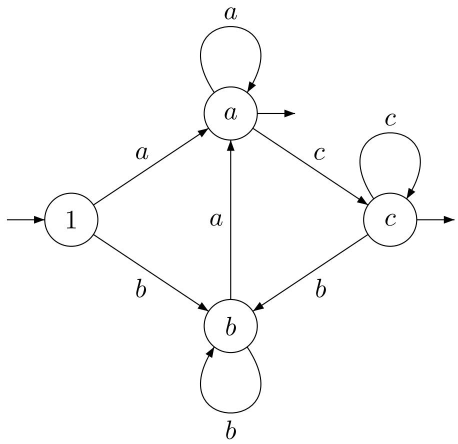
Let $u = a_1 \cdots a_n$ be a non-empty word of L. Then u is the label of a successful path

$$c : i = q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \quad \cdots \quad q_{n-1} \xrightarrow{a_n} q_n$$

In particular, $a_1 \in P$, $q_n \in T$ and thus $a_n \in S$, and for $1 \leq j \leq n-1$, one has $a_j a_{j+1} \notin N$. Consequently $u \in K$, and thus $L \setminus \{1\}$ is contained in K.

Conversely, let $u = a_1 \cdots a_n$ be a non-empty word of K and set $q_0 = i$. By assumption, $a_1 \in P$, $a_n \in S$ and, for $1 \leq j \leq n-1$, $a_j a_{j+1} \notin N$. Since $a_1 \in P$, $q_0.a_1$ is defined. Set $q_0.a_1 = q_1$. We show by induction that there exists a sequence of states q_j $(0 \leq j \leq n)$ such that $a_1 \cdots a_j$ is the label of a path $q_0 \longrightarrow q_1 \longrightarrow \cdots \longrightarrow q_j$ of $\mathcal{A}$. Indeed, since $a_j a_{j+1} \notin N$, $a_j a_{j+1}$ is the label of some path $p \xrightarrow{a_j} q \xrightarrow{a_{j+1}} r$. But since the automaton $\mathcal{A}$ is a local, $q_{j-1}.a_j = p.a_j$, that is $q = q_j$ and thus q_{j+1} is defined as $q_{j+1} = r$. Finally, since $a_n \in S$, it follows that $q_n \in T$. Consequently $q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \quad \cdots \quad q_{n-1} \xrightarrow{a_n} q_n$ is a successful path of $\mathcal{A}$ and its label u is recognized by $\mathcal{A}$. $\square$

Example 2.1 Let $A = \{a, b, c\}$, $P = \{a, b\}$, $S = \{a, c\}$ and $N = \{ab, bc, ca\}$. Then the language $L = (PA^* \cap A^*S) \setminus A^*NA^*$ is recognized by the automaton represented below.



Local languages are stable under various operations:

Proposition 2.3 *Let A_1 and A_2 be two disjoint subsets of the alphabet A, and let $L_1 \subset A_1^*$ and $L_2 \subset A_2^*$ be two local languages. Then the languages $L_1 \cup L_2$ and L_1L_2 are also local languages.*

Proof. Let $\mathcal{A}_1 = (Q_1, A_1, E_1, i_1, T_1)$ and $\mathcal{A}_2 = (Q_2, A_2, E_2, i_2, T_2)$ be standard local automata recognizing L_1 and L_2 respectively. Then $L_1 \cup L_2$ is recognized by the local automaton (Q, A, E, i, T) where

$$
\begin{aligned}
Q &= (Q_1 \setminus \{i_1\}) \cup (Q_2 \setminus \{i_2\}) \cup \{i\} \quad (i \text{ is a new state})\\
E &= \{(q, a, q') \mid (q, a, q') \in E_1 \cup E_2,\ q \neq i_1,\ q \neq i_2\}\\
&\quad \cup \{(i, a, q) \mid (i_1, a, q) \in E_1 \text{ or } (i_2, a, q) \in E_2\}\\
T &= \begin{cases} T_1 \cup T_2 & \text{if } i_1 \notin T_1 \text{ and } i_2 \notin T_2\\ T_1 \setminus \{i_1\}) \cup (T_2 \setminus \{i_2\}) \cup \{i\} & \text{otherwise} \end{cases}
\end{aligned}
$$

For the product, set $\mathcal{A} = (Q, A, E, I, T)$, with

$$
\begin{aligned}
Q &= (Q_1 \cup Q_2) \setminus \{i_2\}\\
E &= E_1 \cup \{(q, a, q') \in E_2 \mid q \neq i_2\} \cup \{(q_1, a, q_2) \mid q_1 \in T_1 \text{ and } (i_2, a, q_2) \in E_2\}\\
I &= I_1\\
T &= \begin{cases} T_2 & \text{if } i_2 \notin T_2,\\ T_1 \cup (T_2 \setminus \{i\}) & \text{if } i_2 \in T_2 \text{ (that is if } 1 \in L_2). \end{cases}
\end{aligned}
$$

By construction, $\mathcal{A}$ is a local automaton and it is easy to verify that it recognizes L_1L_2. □

Proposition 2.4 *Let L be a local language. Then the language L^* is also a local language.*

Proof. Let $\mathcal{A} = (Q, A, E, i, T)$ be a standard local automaton recognizing L. Consider the automaton $\mathcal{A}' = (Q, A, E', i, T \cup \{i\})$, with

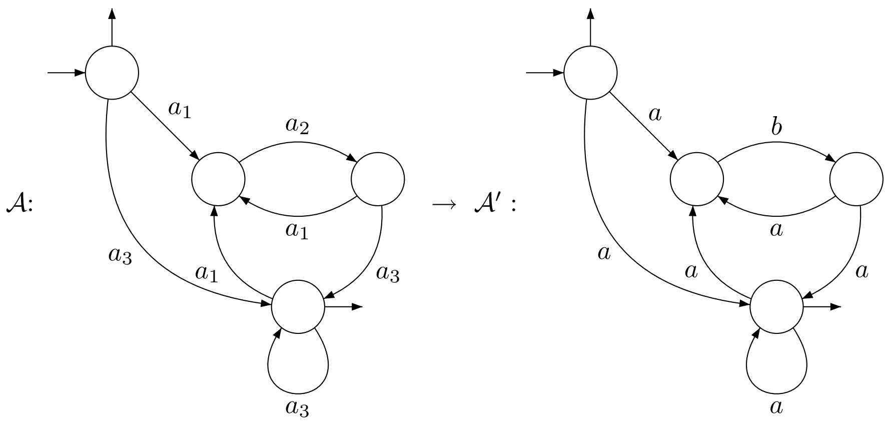
$$E' = E \cup \{(q, a, q') \mid q \in T \text{ and } (i, a, q') \in E\}$$

Then $\mathcal{A}'$ is local and recognizes L^*. □

3 Berry-Sethi Algorithm

Berry and Sethi proposed an algorithm to find a non-deterministic automaton recognizing a given rational expression. For any rational expression e, we denote by $L(e)$ the language that e represents.

We say that a rational expression is *linear* if every letter a has at most one occurrence in the expression (in Watson [11], it is called *restricted*). For example, the expression $[a_1a_2(a_3a_4)^* \cup (a_5a_6)^*a_7]^*$ is linear. One can linearize any rational expression by replacing all the letters that occur in it by distinct symbols. For example, the above expression is a linearization of the expression $e = [ab(ba)^* \cup (ac)^*b]^*$. Now, given an automaton that recognizes the language $L(e')$ of a linearized version e' of a rational expression e, it is easy to obtain an automaton for the language $L(e)$, by replacing letters of e' by the corresponding letters of e. For instance, if $\mathcal{A}$ is the automaton represented below (which recognizes the language $[(a_1a_2)^*a_3]^*$), one obtains, by replacing a_1 and a_3 by a and a_2 by b, the (non-deterministic) automaton $\mathcal{A}'$, which recognizes $[(ab)^*a]^*$.



Therefore it suffices to be able to compute an automaton for each linear expression.

Proposition 3.1 *For every linear expression e, the language $L(e)$ is local.*

Proof. The proof is by induction on the formation rules of linear expressions. First, the languages represented by 0, 1 and a, for $a \in A$, are local languages. Next, by proposition 2.4, if e represents a local language, then e^* represents also a local language. Let now be e and e' two linear expressions and suppose that the expression $(e \cup e')$ is linear. Let B (respectively B') the set of letters occurring in e (e'). Since $(e \cup e')$ is linear, the sets B and B' are disjoint, and the local language $L(e)$ ($L(e')$) is contained in B^* (B'^*). By proposition 2.3, the languages $L(e \cup e')$ and $L(ee')$ are also local. $\square$

Observe that the converse does not hold: for instance, the language $(ab)^*a$ is local but is not denoted by a linear expression.

We have seen in the previous section an algorithm to compute a deterministic automaton recognizing a given local language L. It suffices to test whether the empty word belongs to L and to compute the sets

$$P(L) = \{a \in A \mid aA^* \cap L \neq \emptyset\},$$
$$S(L) = \{a \in A \mid A^*a \cap L \neq \emptyset\},$$
$$F(L) = \{x \in A^2 \mid A^*xA^* \cap L \neq \emptyset\}.$$

But this can be easily done given a rational expression (linear or not) representing the language, by making use of the following well-known recursive procedures. First, we compute $\Lambda(e) = \{1\} \cap L(e)$ as follows:

$$\begin{aligned}
\Lambda(0) &= \emptyset;\\
\Lambda(1) &= \{1\};\\
\Lambda(a) &= \emptyset \text{ for all } a \in A;\\
\Lambda(e \cup e') &= \Lambda(e) \cup \Lambda(e');\\
\Lambda(e.e') &= \Lambda(e) \cap \Lambda(e');\\
\Lambda(e^*) &= \{1\};
\end{aligned}$$

Next,

$$\begin{array}{ll}
\mathrm{P}(0) = \emptyset; & \mathrm{S}(0) = \emptyset;\\
\mathrm{P}(1) = \emptyset; & \mathrm{S}(1) = \emptyset;\\
\mathrm{P}(a) = \{a\} \text{ for all } a \in A; & \mathrm{S}(a) = \{a\} \text{ for all } a \in A;\\
\mathrm{P}(e \cup e') = \mathrm{P}(e) \cup \mathrm{P}(e'); & \mathrm{S}(e \cup e') = \mathrm{S}(e) \cup \mathrm{S}(e');\\
\mathrm{P}(e.e') = \mathrm{P}(e) \cup \Lambda(e)\mathrm{P}(e') & \mathrm{S}(e.e') = \mathrm{S}(e') \cup \mathrm{S}(e)\Lambda(e');\\
\mathrm{P}(e^*) = \mathrm{P}(e); & \mathrm{S}(e^*) = \mathrm{S}(e);
\end{array}$$

$$\begin{array}{l}
\mathrm{F}(0) = \emptyset;\\
\mathrm{F}(1) = \emptyset;\\
\mathrm{F}(a) = \emptyset \text{ for all } a \in A;\\
\mathrm{F}(e \cup e') = \mathrm{F}(e) \cup \mathrm{F}(e');\\
\mathrm{F}(e.e') = \mathrm{F}(e) \cup \mathrm{F}(e') \cup \mathrm{S}(e)\mathrm{P}(e');\\
\mathrm{F}(e^*) = \mathrm{F}(e) \cup \mathrm{S}(e)\mathrm{P}(e);
\end{array}$$

To sum up, given a rational expression e, Berry-Sethi algorithm produces a non-deterministic automaton as follows:

(1) Compute a linear version e' of e and memorize the encoding of letters.

(2) Compute recursively the sets $P(e')$, $S(e')$ and $F(e')$.

(3) Compute a deterministic automaton $\mathcal{A}'$ recognizing e'.

(4) Decode the letters of e' to compute a non-deterministic automaton recognizing e.

4 Final remark

Observe that Berry and Sethi's given an unusual proof of a well-known result, namely that every rational language is the homomorphic image of a local language.

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