Actions, Wreath Products of C-varieties and Concatenation Product

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Final version

Abstract

The framework of $\mathcal{C}$-varieties, introduced by the third author, extends the scope of Eilenberg's variety theory to new classes of languages. In this paper, we first define $\mathcal{C}$-varieties of actions, which are closely related to automata, and prove their equivalence with the original definition of $\mathcal{C}$-varieties of stamps. Next, we complete the study of the wreath product initiated by Ésik and Ito by extending its definition to $\mathcal{C}$-varieties in two different ways, which are proved to be equivalent. We also state an extension of the wreath product principle, a standard tool of language theory. Finally, our main result generalizes to $\mathcal{C}$-varieties the algebraic characterization of the closure under product of a variety of languages.

Through the work of Eilenberg [3] and Schützenberger [13], the theory of varieties of finite semigroups and monoids emerged as an essential tool in the study of the algebra underlying families of regular languages. The current literature on the subject (see [10, 2] for a comprehensive bibliography) attests to the richness of this theory and the diversity of its applications in an increasing number of research fields including automata theory and formal languages but also model theory and logic, circuit complexity, communication complexity, discrete dynamical systems, etc. However, some important families of languages arising from open problems in language theory (the generalized star height problem), logic and circuit complexity [17], do not form varieties of languages in the sense originally described by Eilenberg. To study these new varieties of languages, Straubing [18] recently introduced the notion of $\mathcal{C}$-varieties. A similar notion was introduced independently by Ésik and Ito [6]. The formal definition of a $\mathcal{C}$-variety of languages is quite similar to Eilenberg's except that it only requires closure under inverse images of morphisms belonging to some natural class $\mathcal{C}$. (In the important applications, this class $\mathcal{C}$ is typically either the class of all length-preserving morphisms, or of all length-multiplying morphisms. In contrast, the theory developed by Eilenberg requires closure under inverse images of

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arbitrary morphisms, or all non-erasing morphisms.) In place of the finite semigroups and monoids of the original theory, Straubing considers *stamps*, which are surjective morphisms from a free monoid onto a finite monoid. This new approach permits an algebraic study of families of languages that could not be treated in Eilenberg's original framework. Examples include languages occurring in circuit complexity, temporal logic [5, 6, 7], and languages of generalized star height $\leqslant n$ for a given n.

The original tools developed for the restricted theory must now be extended to this new setting. Early papers by Kunc [8], Pin-Straubing [11], and Ésik-Ito [6] have already shown the way by generalizing the equational theory for $\mathcal{C}$-varieties, the Mal'cev product and the cascade product. The present paper is a further contribution to the theory. We first define $\mathcal{C}$-varieties of actions, which are closely related to automata, and prove their equivalence with the original definition of $\mathcal{C}$-varieties of stamps. Next, we complete the study of the wreath product initiated by Ésik and Ito by extending its definition to $\mathcal{C}$-varieties in two different ways, which are proved to be equivalent. We also state an extension of the wreath product principle, a standard tool of language theory. Finally, our main result generalizes to $\mathcal{C}$-varieties the algebraic characterization of the closure under product of a variety of languages: if $\mathbf{V}$ is the $\mathcal{C}$-variety of stamps associated with a $\mathcal{C}$-variety of languages $\mathcal{V}$, then the variety of stamps associated with the closure of $\mathcal{V}$ under concatenation product is the Mal'cev product $\mathbf{A} \circledM \mathbf{V}$.

Throughout the paper, all monoids are either finite or free. In particular, "variety of monoids" will mean variety of finite monoids.

1 $\mathcal{C}$-varieties

In this paper, $\mathcal{C}$ denotes a class of morphisms between finitely generated free monoids that is closed under composition and contains all length-preserving morphisms. Examples include the classes of all *length-preserving* morphisms (morphisms for which the image of each letter is a letter), of all *length-multiplying* morphisms (morphisms such that, for some integer k, the length of the image of a word is k times the length of the word), all *non-erasing* morphisms (morphisms for which the image of each letter is a nonempty word), all *length-decreasing* morphisms (morphisms for which the image of each letter is either a letter or the empty word) and all morphisms.

We now define successively $\mathcal{C}$-varieties of stamps, of actions and of languages.

1.1 Stamps

We briefly recall the definitions introduced in [18, 11]. Similar, but slightly more restricted notions, were also introduced in [5, 7].

A *stamp* is a morphism from a finitely generated free monoid onto a finite monoid. A stamp $\varphi : A^* \to M$ is said to be *trivial* if M is the trivial monoid. A $\mathcal{C}$*-morphism* from a stamp $\varphi : A^* \to M$ to a stamp $\psi : B^* \to N$ is a pair (f, α), where $f : A^* \to B^*$ is in $\mathcal{C}$, $\alpha : M \to N$ is a monoid morphism, and $\psi \circ f = \alpha \circ \varphi$. If f and α are both bijections then the pair (f, α) is an *isomorphism*. Note that this implies f is length-preserving and thus the notion of isomorphism does not depend upon the class $\mathcal{C}$. In the remainder of this paper, we do not distinguish between isomorphic stamps.

A stamp $\varphi : A^* \to M$ *$\mathcal{C}$-divides* a stamp $\psi : B^* \to N$ if there is a pair (f, η) (called a $\mathcal{C}$-division), where $f : A^* \to B^*$ is in $\mathcal{C}$, $\eta : N \to M$ is a partial surjective monoid morphism, and $\varphi = \eta \circ \psi \circ f$. If f is the identity on A^*, the pair (f, η) is simply called a division. Note that $\mathcal{C}$-division is transitive [18] but *not* antisymmetric. However, if φ $\mathcal{C}$-divides ψ and ψ $\mathcal{C}$-divides φ, then the finite monoids $\mathrm{Im}(\varphi)$ and $\mathrm{Im}(\psi)$ are isomorphic.

The *product* of two stamps $\varphi_1 : A^* \to M_1$ and $\varphi_2 : A^* \to M_2$ is the stamp φ with domain A^* defined by $\varphi(a) = (\varphi_1(a), \varphi_2(a))$. The image of φ is a submonoid of $M_1 \times M_2$.

A *$\mathcal{C}$-variety of stamps* is a class of stamps containing the trivial stamps and closed under $\mathcal{C}$-division and finite products. When $\mathcal{C}$ is the class of all (resp. length-preserving, length-multiplying, non-erasing, length-decreasing) morphisms, we use the term *all-variety* (resp. *lp-variety*, *lm-variety*, *ne-variety*, *ld-variety*).

As was mentioned before, Eilenberg's varieties can be considered as a particular case of $\mathcal{C}$-varieties. Indeed, given a variety of monoids $\mathbf{V}$, the class of stamps whose range is in $\mathbf{V}$ is an *all*-variety of stamps, and any *all*-variety of stamps is of this form [18]. A similar observation holds for varieties of semigroups and *ne*-varieties.

1.2 Actions

1.2.1 Definitions

Let P be a finite non-empty set. Recall that a transformation on P is a function $u : p \mapsto p\cdot u$ from P into itself. The product of two transformations u and v is the transformation uv defined by $p\cdot(uv) = (p\cdot u)\cdot v$. We denote by $\mathfrak{T}(P)$ the monoid of all transformations on the set P.

Let A be an alphabet. A (*right*) *action* of A on P is a map $P \times A \to P$, denoted $(p, a) \mapsto p\cdot a$. An action of A on P is usually denoted by $(P, A, \cdot)$, but the symbol $\cdot$ is often omitted, in the same way as operation symbols are omitted in group or semigroup theory. An *identity action* on the alphabet A is an action (P, A) such that $p\cdot a = p$ for each p in P and a in A. Such an action is denoted by $I_P(A)$.

We now detail the connections between stamps and actions. We first associate a stamp with an action (P, A) as follows. Extend recursively the action of A on P to a map $P \times A^* \to P$ by setting, for all $p \in P$, $p\cdot 1 = p$ and for all u in A^* and a in A, $p\cdot ua = (p\cdot u)\cdot a$. Then the function $\mu : A^* \to \mathfrak{T}(P)$ which maps the word u onto the transformation $p \mapsto p\cdot u$ defines the *stamp associated with the action* (P, A) and is denoted by $\mathrm{Stp}(P, A)$. The set $\mu(A^*)$ is called the *transformation monoid* of the action (P, A).

Conversely, given a stamp $\varphi : A^* \to M$, define an action (M, A) by setting, for each $m \in M$ and $a \in A$, $m\cdot a = m\varphi(a)$. This action is called the *action associated with the stamp* φ, and is denoted by $\mathrm{Act}(\varphi)$.

The *product* of two actions (P_1, A) and (P_2, A) (denoted by $(P_1, A)\times(P_2, A)$) is the action $(P_1 \times P_2, A)$ defined by $(p_1, p_2)\cdot a = (p_1 \cdot a, p_2\cdot a)$. This corresponds to the notion of a direct product of automata, see [6]. A $\mathcal{C}$-morphism from an action (P, A) into an action (Q, B) is a pair (f, η) where $f \colon A^* \to B^*$ is in $\mathcal{C}$ and $\eta : P \to Q$ is a function satisfying, for each $p \in P$ and $a \in A$, $\eta(p\cdot a) = \eta(p)\cdot f(a)$.

An action (P, A) *$\mathcal{C}$-divides* an action (Q, B) if there is a pair (f, η) (called a *$\mathcal{C}$-division*) where $f\colon A^* \to B^*$ is in $\mathcal{C}$ and $\eta : Q \to P$ is a surjective partial function such that for each $q \in \mathrm{Dom}(\eta)$ and each $a \in A$, $\eta(q)\cdot a = \eta(q\cdot f(a))$. When $A = B$ and f is the identity on A^*, the pair (f, η) is simply called a *division*. The notion of $\mathcal{C}$-division generalizes the definition of division of transformation monoids and captures the intuitive notion of simulating one automaton by another. The class $\mathcal{C}$ enters the picture in the manner in which letters of the divisor action are encoded by words in the divided action. It is easy to see that $\mathcal{C}$-division of actions is transitive. We will need the following straightforward lemma in the sequel.

Lemma 1.1 *Let $(P_i, A), (Q_i, A)$ be actions such that (P_i, A) divides (Q_i, A) , for $i = 1, 2$. Then $(P_1, A) \times (P_2, A)$ divides $(Q_1, A) \times (Q_2, A)$.*

Although it is clear that for any stamp φ, $\mathrm{Stp} \circ \mathrm{Act}(\varphi) = \varphi$, the action $\mathrm{Act} \circ \mathrm{Stp}(P, A)$ differs in general from (P, A). However the following result holds.

Lemma 1.2

(1) *The action* $\mathrm{Act} \circ \mathrm{Stp}(P, A)$ *divides the product* $(P, A)^{|P|}$.

(2) *The action* (P, A) *divides the product* $I_P(A) \times \mathrm{Act} \circ \mathrm{Stp}(P, A)$.

Proof. Let $(M, A) = \mathrm{Act} \circ \mathrm{Stp}(P, A)$.

(1) The natural embedding of M into P^P respects actions of A.

(2) Let $\eta : P \times M \to P$ be the map defined by $\eta(p, m) = p\cdot m$. This map is onto since $p\cdot 1 = p$, and for every $a \in A$, $p \in P$ and $m \in M$, $\eta(p, m)\cdot a = (p\cdot m)\cdot a = p\cdot(m\cdot a) = \eta(p, m\cdot a) = \eta((p, m)\cdot a)$. Thus the pair (Id_{A^*}, η) is a division from (P, A) into $I_P(A) \times (M, A)$. $\square$

1.2.2 $\mathcal{C}$-varieties of actions

Let $\mathbf{V}$ be a $\mathcal{C}$-variety of stamps, and let $\mathbf{V}_{\mathrm{act}}$ be the collection of all actions whose underlying stamp belongs to $\mathbf{V}$. We call $\mathbf{V}_{\mathrm{act}}$ a $\mathcal{C}$-variety of actions. The proof of the following result is trivial.

Proposition 1.3 *The mapping $\mathbf{V} \mapsto \mathbf{V}_{\mathrm{act}}$ is one-to-one.*

The inverse of this mapping assigns to a $\mathcal{C}$-variety of actions $\mathbf{W}$ the collection of stamps $\{\mathrm{Stp}(P, A) \mid (P, A) \in \mathbf{W}\}$.

As an example, given a variety of monoids $\mathbf{V}$, the class of actions whose transformation monoid is in $\mathbf{V}$ is an *all*-variety of actions [18]. When $\mathcal{C}$ is the class of *lp*-morphisms, $\mathcal{C}$-varieties of actions correspond to the q-varieties of [6]. Length-preserving and length-decreasing varieties of actions closed under the cascade product also appeared in [4].

The main result of the section is:

Theorem 1.4 *A collection of actions is a $\mathcal{C}$-variety if and only if it contains all identity actions and is closed under product and $\mathcal{C}$-division.*

Proof. We will use the following auxiliary lemmas:

Lemma 1.5 *Let* (P_1, A) *and* (P_2, A) *be two actions. Then* $\mathrm{Stp}((P_1, A) \times (P_2, A)) = \mathrm{Stp}(P_1, A) \times \mathrm{Stp}(P_2, A)$.

The proof is trivial.

Lemma 1.6

(1) *If an action* (P, A) $\mathcal{C}$*-divides an action* (Q, B)*, then* $\mathrm{Stp}(P, A)$ $\mathcal{C}$*-divides* $\mathrm{Stp}(Q, B)$.

(2) *If a stamp* φ $\mathcal{C}$*-divides a stamp* ψ*, then* $\mathrm{Act}(\varphi)$ $\mathcal{C}$*-divides* $\mathrm{Act}(\psi)$.

Proof. (1) Let (f, η) be a $\mathcal{C}$-division from an action (P, A) into an action (Q, B). Let $\mu = \mathrm{Stp}(P, A)$ and $\nu = \mathrm{Stp}(Q, B)$. We claim that, for every $u, v \in A^*$, $\nu \circ f(u) = \nu \circ f(v)$ implies $\mu(u) = \mu(v)$. Indeed, let $p \in P$ and let $q \in Q$ be such that $\eta(q) = p$. If $\nu \circ f(u) = \nu \circ f(v)$, then in particular $q \cdot f(u) = q \cdot f(v)$, whence $\eta(q \cdot f(u)) = \eta(q \cdot f(v))$. Now $\eta(q \cdot f(u)) = \eta(q) \cdot u = p \cdot u$ and similarly, $\eta(q \cdot f(v)) = p \cdot v$. It follows that, for each $p \in P$, $p \cdot u = p \cdot v$ and thus $\mu(u) = \mu(v)$, which proves the claim. Consequently, there is a surjective morphism $\rho : \mathrm{Im}(\nu \circ f) \to \mathrm{Im}(\mu)$ such that $\rho \circ \nu \circ f = \mu$, and μ $\mathcal{C}$-divides ν.

(2) Let (f, η) be a $\mathcal{C}$-division from a stamp φ into a stamp ψ. Then one verifies easily that (f, η) is also a $\mathcal{C}$-division from $\mathrm{Act}(\varphi)$ into $\mathrm{Act}(\psi)$. $\square$

We return to the proof of Theorem 1.4. By Lemmas 1.5 and 1.6, it is immediate that a $\mathcal{C}$-variety of actions is closed under $\mathcal{C}$-division and product. Further, it is clear that for any identity action $I_P(A)$, $\mathrm{Stp}(I_P(A))$ is a trivial stamp. Therefore, a $\mathcal{C}$-variety of actions contains all identity actions. Conversely, let $\mathcal{A}$ be a collection of actions containing all identity actions and closed under $\mathcal{C}$-division and product. Let $\mathbf{V} = \{\mathrm{Stp}(Q, A) \mid (Q, A) \in \mathcal{A}\}$. We show that $\mathbf{V}$ is a $\mathcal{C}$-variety of stamps and that $\mathcal{A} = \mathbf{V}_{\mathrm{act}}$. First, by Lemma 1.5, $\mathbf{V}$ is closed under product. Now, let ψ be a stamp in $\mathbf{V}$ and let φ be a stamp $\mathcal{C}$-dividing ψ. By construction of $\mathbf{V}$, $\psi = \mathrm{Stp}(Q, A)$, for some (Q, A) in $\mathcal{A}$. By Lemma 1.2, the action $\mathrm{Act}(\psi) = \mathrm{Act} \circ \mathrm{Stp}(Q, A)$ divides $(Q, A)^{|Q|}$ and is thus in $\mathcal{A}$. By Lemma 1.6, $\mathrm{Act}(\varphi)$ $\mathcal{C}$-divides $\mathrm{Act}(\psi)$. Thus, $\mathrm{Act}(\varphi)$ is in $\mathcal{A}$ and $\mathrm{Stp} \circ \mathrm{Act}(\varphi) = \varphi$ is in $\mathbf{V}$. Therefore, $\mathbf{V}$ is a $\mathcal{C}$-variety of stamps. By definition of $\mathbf{V}$, it is immediate that $\mathcal{A} \subseteq \mathbf{V}_{\mathrm{act}}$. Conversely, let $(P, A) \in \mathbf{V}_{\mathrm{act}}$. Then $\mathrm{Stp}(P, A)$ is in $\mathbf{V}$, which means that $\mathrm{Stp}(P, A) = \mathrm{Stp}(Q, A)$, for some action (Q, A) in $\mathcal{A}$. Therefore, by Lemma 1.2, $\mathrm{Act} \circ \mathrm{Stp}(P, A)$ divides $(Q, A)^{|Q|}$ and is thus in $\mathcal{A}$. By Lemma 1.2, (P, A) divides $I_P(A) \times \mathrm{Act} \circ \mathrm{Stp}(P, A)$ and since $\mathcal{A}$ contains all identity actions, (P, A) is in $\mathcal{A}$. Thus, $\mathcal{A} = \mathbf{V}_{\mathrm{act}}$, and $\mathcal{A}$ is a $\mathcal{C}$-variety of actions. $\square$

Note that the condition on identity actions is truly necessary. Indeed, there exist collections of actions closed under product and $\mathcal{C}$-division that are *not* $\mathcal{C}$-varieties. For example, let $\mathcal{A}$ be the collection of actions (P, A) such that for any p, p' in P and a in A, there exists $n > 0$ such that $p \cdot a^n = p \cdot a^{n+1} = p' \cdot a^n = p' \cdot a^{n+1}$. Then, one can verify that $\mathcal{A}$ contains $I_1(A)$, is closed under product and lp-division, and does not contain $I_2(A)$. Since $\mathrm{Stp}(I_1(A)) = \mathrm{Stp}(I_2(A)) = \varphi : A^* \to \{1\}$, $\mathcal{A}$ is not a lp-variety of actions.

1.3 $\mathcal{C}$-varieties of languages

A language $L \subseteq A^*$ is *recognized by a stamp* $\varphi : A^* \to M$ if and only if there exists a set $I \subseteq M$ such that $L = \varphi^{-1}(I)$. Similarly, a language $L \subseteq A^*$ is *recognized by an action* (P, A) if and only if there exist an *initial state* $p_0 \in P$ and a set of *final states* $F \subseteq P$, such that $L = \{u \in A^* \mid p_0 \cdot u \in F\}$.

Given a language $L \subseteq A^*$, let (Q_L, A) be the action induced by the complete minimal automaton of L. It follows from standard automata theory that an action (Q, A) recognizes L if and only if (Q_L, A) divides (Q, A). We shall also use the following straightforward lemma.

Lemma 1.7 *If an action (P_1, A) divides an action (P_2, A), any language recognized by (P_1, A) is recognized by (P_2, A).*

The stamp associated with the action (Q_L, A) is called the *syntactic stamp* (or *syntactic morphism*) of L. In particular, a language is recognized by a stamp φ if and only if its syntactic stamp divides φ.

A *class of recognizable languages* $\mathcal{V}$ assigns to each finite alphabet A a set $\mathcal{V}(A^*)$ of recognizable languages of A^*. Given a $\mathcal{C}$-variety of stamps $\mathbf{V}$, the class $\mathcal{V}$ of languages recognized by some stamp in $\mathbf{V}$ is called a *$\mathcal{C}$-variety of languages*. Note that a language is in $\mathcal{V}$ if and only if its syntactic stamp is in $\mathbf{V}$. It is shown in [18] that a class $\mathcal{V}$ of languages is a $\mathcal{C}$-variety of languages if and only if it satisfies

(1) for every alphabet A, $\mathcal{V}(A^*)$ is a Boolean algebra, that is, is closed under finite union, finite intersection and complement,

(2) if $L \in \mathcal{V}(A^*)$ and $a \in A$ then $a^{-1}L$ and La^{-1} are in $\mathcal{V}(A^*)$,

(3) if $\varphi\colon A^* \to B^*$ is a morphism in $\mathcal{C}$, $L \in \mathcal{V}(B^*)$ implies $\varphi^{-1}(L) \in \mathcal{V}(A^*)$.

Moreover, the correspondence $\mathbf{V} \to \mathcal{V}$ is one-to-one.

We now state an analogous result for $\mathcal{C}$-varieties of actions. A similar result was proved by Ésik for *lp*-varieties (see [6, Theorem 2.9] and [7, Theorem 8.5]).

Theorem 1.8 *Let $\mathbf{V}$ be a $\mathcal{C}$-variety of stamps, and let $\mathcal{V}$ consist of all the languages recognized by actions in $\mathbf{V}_{\mathrm{act}}$. Then $\mathcal{V}$ is the $\mathcal{C}$-variety of languages corresponding to $\mathbf{V}$.*

Proof. The result follows almost immediately from the properties of the syntactic stamp recalled above. $\square$

2 Wreath products

2.1 Sequential products of actions

The sequential product of actions corresponds to the notion of cascade product of automata in the work of Ésik and Ito [6].

The *sequential product* of two actions $(P, Q \times A)$ and (Q, A) is the action $(P \times Q, A)$ defined by $(p, q) \cdot a = (p \cdot (q, a), q \cdot a)$ and denoted by $(P, Q \times A) \circ (Q, A)$. Observe that for a word $u = a_1 \cdots a_n$,

$$(p, q) \cdot u = \big(p \cdot (q, a_1)(q \cdot a_1, a_2) \cdots (q \cdot a_1 \cdots a_{n-1}, a_n), q \cdot u\big).$$

We now state some basic properties of the cascade product in terms of actions. The straightforward proofs are omitted.

Proposition 2.1 *For any actions $(P, Q \times R \times A)$, $(Q, R \times A)$ and (R, A), the sequential product $\big((P, Q \times R \times A) \circ (Q, R \times A)\big) \circ (R, A)$ is isomorphic to $(P, Q \times R \times A) \circ \big((Q, R \times A) \circ (R, A)\big)$.*

Proposition 2.2 *Let S_i be a sequential product $(P_i, Q_i \times A) \circ (Q_i, A)$ for $i = 1, 2$. Define an action $U = (P_1 \times P_2, Q_1 \times Q_2 \times A)$ by setting $(p_1, p_2) \cdot (q_1, q_2, a) = (p_1 \cdot (q_1, a), p_2 \cdot (q_2, a))$. Then $U \circ ((Q_1, A) \times (Q_2, A))$ is isomorphic to $S_1 \times S_2$. Moreover, U can be obtained from the actions $(P_i, Q_i \times A)$ by lp-divisions and product.*

The next proposition shows that the sequential product preserves division.

Proposition 2.3 *Let $B = Q \times A$ and suppose that the action (P_1, B) divides the action (P_2, B). Then for any action (Q, A), $(P_1, B) \circ (Q, A)$ divides $(P_2, B) \circ (Q, A)$.*

Proof. Let the pair (Id_{B^*}, η) be a division from (P_1, B) into (P_2, B). Let (Q, A) be an action. Define the partial function $\tilde{\eta} : P_2 \times Q \to P_1 \times Q$ by setting, for each (p, q) in $\mathrm{Dom}(\eta) \times Q$, $\tilde{\eta}(p, q) = (\eta(p), q)$. One verifies that the pair $(Id_{A^*}, \tilde{\eta})$ is a division from $(P_1, B) \circ (Q, A)$ into $(P_2, B) \circ (Q, A)$. $\square$

2.2 Wreath product of $\mathcal{C}$-varieties of actions

Let $\mathbf{V}, \mathbf{W}$ be two $\mathcal{C}$-varieties of actions. A $(\mathbf{V}, \mathbf{W})$-sequential product is an action of the form $(P, Q \times A) \circ (Q, A)$ with $(P, Q \times A)$ in $\mathbf{V}$ and (Q, A) in $\mathbf{W}$. We define $\mathbf{V} * \mathbf{W}$ to be the class of all actions that $\mathcal{C}$-divide a $(\mathbf{V}, \mathbf{W})$-sequential product. The class $\mathbf{V} * \mathbf{W}$ is called the *wreath product* of the $\mathcal{C}$-varieties of actions $\mathbf{V}$ and $\mathbf{W}$.

To avoid technical difficulties, we restrict ourselves to some specific classes of morphims, that nevertheless include all classical examples. A class of morphisms $\mathcal{C}$ is said to be *convenient* if it is closed under composition, contains all length-preserving morphisms and satisfies that membership of a morphism $f : A^* \to B^*$ in $\mathcal{C}$ depends only on the set of integers $\{|f(a)| \mid a \in A\}$. We did not find any natural example of nonconvenient classes. A rather artificial example of a nonconvenient class is the class of all morphisms $f : A^* \to B^*$ such that, for each letter a in A, there exists b in B such that $f(a) \in b^+$.

Although the definition of the wreath product depends on the class $\mathcal{C}$, the following proposition shows that one can write $\mathbf{V} * \mathbf{W}$ without referring to $\mathcal{C}$, provided $\mathcal{C}$ is convenient. Indeed, in this case $\mathbf{V} * \mathbf{W}$ appears to be the class of all actions that *lp*-divide a $(\mathbf{V}, \mathbf{W})$-sequential product.

Proposition 2.4 *Let $\mathcal{C}$ be a convenient class of morphisms and let $\mathbf{V}$ and $\mathbf{W}$ be two $\mathcal{C}$-varieties of actions. An action (P, A) is in $\mathbf{V} * \mathbf{W}$ if and only if there exist a $(\mathbf{V}, \mathbf{W})$-sequential product $(T, Q \times A) \circ (Q, A)$ and a division from (P, A) into $(T, Q \times A) \circ (Q, A)$.*

Proof. Let (P, A) be an action in $\mathbf{V} * \mathbf{W}$. By definition, there exist a $(\mathbf{V}, \mathbf{W})$-sequential product $(T, Q \times B) \circ (Q, B)$ and a $\mathcal{C}$-division (f, η) from (P, A) into $(T, Q \times B) \circ (Q, B)$. Define a morphism $g\colon (Q \times A)^* \to (Q \times B)^*$ by $g(q, a) = 1$ if $f(a) = 1$ and

$$g(q, a) = (q, b_1)(q\cdot b_1, b_2) \cdots (q\cdot b_1 \cdots b_{k-1}, b_k)$$

when $f(a) = b_1 \cdots b_k$. The morphism g is in $\mathcal{C}$ because f is in $\mathcal{C}$ and $\mathcal{C}$ is convenient. Define an action $(T, Q \times A)$ by setting

$$t\cdot (q, a) = t\cdot g(q, a).$$

The pair (g, Id_T) is a $\mathcal{C}$-division from $(T, Q \times A)$ into $(T, Q \times B)$ and thus $(T, Q \times A)$ is in $\mathbf{V}$. In the same way, define an action (Q, A) by setting

$$q\cdot a = q\cdot f(a).$$

This action is in $\mathbf{W}$ since (f, Id_Q) is a $\mathcal{C}$-division from (Q, A) into (Q, B). Consider now the $(\mathbf{V}, \mathbf{W})$-sequential product $S = (T, Q \times A) \circ (Q, A)$. Let us show that the pair (Id_{A^*}, η) is a division from (P, A) into S. Indeed, for each $(t, q) \in \mathrm{Dom}(\eta)$ and $a \in A$,

$$\begin{aligned}\eta((t, q)\cdot a) &= \eta(t\cdot (q, a), q\cdot a) = \eta(t\cdot (q, b_1) \cdots (q\cdot b_1 \cdots b_{k-1}, b_k), q\cdot f(a)) \\ &= \eta((t, q)\cdot f(a)) = \eta(t, q)\cdot a\end{aligned}$$

where $b_1 \cdots b_k = f(a)$. $\square$

Theorem 2.5 *If $\mathcal{C}$ is a convenient class of morphims and $\mathbf{V}$, $\mathbf{W}$ are $\mathcal{C}$-varieties of actions, then $\mathbf{V} * \mathbf{W}$ is a $\mathcal{C}$-variety of actions.*

Proof. It is immediate that $\mathbf{V} * \mathbf{W}$ contains $\mathbf{W}$ and thus all identity actions. Closure of $\mathbf{V} * \mathbf{W}$ under $\mathcal{C}$-division is given in the definition itself. Thus, we only verify that $\mathbf{V} * \mathbf{W}$ is closed under product. To this end, let (P_1, A) and (P_2, A) be two arbitrary actions in $\mathbf{V} * \mathbf{W}$. By Proposition 2.4, there exist two divisions (Id_{A^*}, η_i) for $i = 1, 2$, from (P_i, A) into a $(\mathbf{V}, \mathbf{W})$-sequential product S_i of the form $(T_i, Q_i \times A) \circ (Q_i, A)$. By Proposition 2.2, the product $S_1 \times S_2$ is in $\mathbf{V} * \mathbf{W}$. By Lemma 1.1, the product $(P_1, A) \times (P_2, A)$ divides $S_1 \times S_2$. This shows that $(P_1, A) \times (P_2, A)$ is in $\mathbf{V} * \mathbf{W}$, which completes the proof. $\square$

2.3 Wreath product of $\mathcal{C}$-varieties of stamps

Our definition of the wreath product of $\mathcal{C}$-varieties of stamps is not as easy to manipulate as the wreath product of $\mathcal{C}$-varieties of actions. Notice though that we require the same restriction on convenient sets of morphisms as in the case of actions.

In the case of monoids, recall that the wreath product $N \circ K$ of two monoids N and K is defined on the set $N^K \times K$ by the following product:

$$(f_1, k_1)(f_2, k_2) = (f, k_1k_2) \ , \text{ with } f(k) = f_1(k)f_2(kk_1).$$

Let $\mathcal{C}$ be a convenient class of morphisms and let $\mathbf{V}, \mathbf{W}$ be two $\mathcal{C}$-varieties of stamps. A $(\mathbf{V}, \mathbf{W})$-product stamp is a stamp $\varphi : A^* \to M$ such that:

(1) M is a submonoid of a wreath product $N \circ K$, where N and K are finite monoids.
(2) Let $\pi : N \circ K \to K$ be the canonical projection morphism. Then the stamp $\pi \circ \varphi : A^* \to \pi(M)$ is in $\mathbf{W}$.
(3) For a in A, we can write $\varphi(a) = (f_a, \pi \circ \varphi(a))$ where f_a is in N^K. We now treat $K \times A$ as a finite alphabet and we define a stamp $\Phi : (K \times A)^* \to \mathrm{Im}(\Phi) \subseteq N$ by $\Phi(k, a) = f_a(k)$. We require Φ to be in $\mathbf{V}$.

We define $\mathbf{V} * \mathbf{W}$ to be the class of all stamps that $\mathcal{C}$-divide a $(\mathbf{V}, \mathbf{W})$-product stamp. The class $\mathbf{V} * \mathbf{W}$ is called the *wreath product* of the $\mathcal{C}$-varieties of stamps $\mathbf{V}$ and $\mathbf{W}$. We will need the following technical lemma, whose proof is straightforward.

Lemma 2.6 *Let $\varphi\colon A^* \to M \subseteq N \circ K$ be a $(\mathbf{V}, \mathbf{W})$-product stamp defined with the notations used in the above definition. Then, for each k in K, the stamp $\Phi_k : (\pi(M) \times A)^* \to \mathrm{Im}(\Phi_k) \subseteq N$, defined by $\Phi_k(x, a) = f_a(kx)$, is in $\mathbf{V}$.*

We are now ready to state the main result of this section.

Theorem 2.7 *Let $\mathcal{C}$ be a convenient class of morphisms, and (R, B) an action. Let $\mathbf{V}$ and $\mathbf{W}$ be $\mathcal{C}$-varieties of stamps. Then $(R, B) \in \mathbf{V}_{\mathrm{act}} * \mathbf{W}_{\mathrm{act}}$ if and only if $\mathrm{Stp}(R, B) \in \mathbf{V} * \mathbf{W}$.*

Proof. Let $(R, B) \in \mathbf{V}_{\mathrm{act}} * \mathbf{W}_{\mathrm{act}}$. Then, (R, B) $\mathcal{C}$-divides a $(\mathbf{V}_{\mathrm{act}}, \mathbf{W}_{\mathrm{act}})$-sequential product $(P, Q \times A) \circ (Q, A)$. Denote by $\rho : A^* \to X \subseteq \mathcal{T}(P \times Q)$ the stamp $\mathrm{Stp}((P, Q \times A) \circ (Q, A))$. We show that ρ is in $\mathbf{V} * \mathbf{W}$.

Denote by $\psi : A^* \to K$ the stamp $\mathrm{Stp}(Q, A)$. Note that K is a submonoid of $\mathfrak{T}(Q)$. Let $N = \mathfrak{T}(P)^Q$, and for each a in A, let $f_a : K \to N$ be the following function: for each k in K, $f_a(k)$ is the function $Q \to \mathfrak{T}(P)$ defined, for each q in Q and p in P, by

$$p f_a(k)(q) = p\cdot (qk, a). \tag{1}$$

In the latter formula, qk is an element of Q and $p\cdot (qk, a)$ denotes the result of the action of the letter (qk, a) of $Q \times A$ on the element p of P.

Define a stamp $\varphi : A^* \to M \subseteq N \circ K$ by setting, for each a in A, $\varphi(a) = (f_a, \psi(a))$ and let $\pi : N \circ K \to K$ be the natural projection morphism.

Lemma 2.8 *The stamp φ is a $(\mathbf{V}, \mathbf{W})$-product stamp.*

Proof. It suffices to verify the properties defining a $(\mathbf{V}, \mathbf{W})$-product stamp:
(1) M is a submonoid of $N \circ K$,
(2) the stamp $\pi \circ \varphi$ is in $\mathbf{W}$,
(3) the stamp Φ is in $\mathbf{V}$.

The first property is trivial. The second one follows from the equality $\pi \circ \varphi = \psi$. Note also that $\pi(M) = \pi \circ \varphi(A^*) = \psi(A^*) = K$. For the third one, observe that Φ is the product of the stamps γ_q from $(K \times A)^*$ onto a subset of $\mathfrak{T}(P)$ defined by setting, for each q in Q,

$$\gamma_q(k, a) = f_a(k)(q).$$

Thus it suffices to show that each stamp γ_q is in $\mathbf{V}$. Such a stamp is associated with the action $(P, K \times A)_q$, defined by setting, for each p in P and (k, a) in $(K \times A)$,

$$p\cdot (k, a) = \gamma_q(k, a)(p) = pf_a(k)(q).$$

Proving that γ_q is in $\mathbf{V}$ amounts to showing that $(P, K \times A)_q$ is in $\mathbf{V}_{\mathrm{act}}$. Let $g_q : (K \times A)^* \to (Q \times A)^*$ be the length-preserving morphism defined by $g_q(k, a) = (qk, a)$. Then, the pair (g_q, Id_P) is a $\mathcal{C}$-division from the action $(P, K \times A)_q$ into $(P, Q \times A)$. Now since $(P, Q \times A)$ is in $\mathbf{V}_{\mathrm{act}}$, $(P, K \times A)_q$ is in $\mathbf{V}_{\mathrm{act}}$ as well, and thus γ_q is in $\mathbf{V}$. It follows that Φ is in $\mathbf{V}$ and φ is a $(\mathbf{V}, \mathbf{W})$-product stamp. □

We claim that the stamp ρ $\mathcal{C}$-divides φ. We first need the following lemma.

Lemma 2.9 *For any words u, v in A^*, $\varphi(u) = \varphi(v)$ implies $\rho(u) = \rho(v)$.*

Proof. Let $u = a_1 \cdots a_n$. Then

$$\varphi(u) = (f_{a_1}, \psi(a_1)) \cdots (f_{a_n}, \psi(a_n)) = (f_u, \psi(u)) \tag{2}$$

where, for each k in K, $f_u(k)$ is a map from Q into $\mathfrak{T}(P)$ which satisfies

$$f_u(k) = f_{a_1}(k) f_{a_2}(k\psi(a_1)) \cdots f_{a_n}(k\psi(a_1 \cdots a_{n-1})). \tag{3}$$

Since ψ is the stamp associated with the action (Q, A), it follows from (1) that, for $1 \leqslant i \leqslant n$ and for each $p \in P$ and $q \in Q$,

$$pf_{a_i}(\psi(a_1 \cdots a_{i-1}))(q) = p\cdot (q\cdot a_1 \cdots a_{i-1}, a_i).$$

Equation (3) becomes in particular,

$$f_u(1) = f_{a_1}(1) f_{a_2}(\psi(a_1)) \cdots f_{a_n}(\psi(a_1 \cdots a_{n-1}))$$

whence

$$pf_u(1)(q) = p\cdot (q, a_1)(q\cdot a_1, a_2) \cdots (q\cdot a_1 \cdots a_{n-1}, a_n). \tag{4}$$

Now, by definition of the sequential product $(P \times Q, A) = (P, Q \times A) \circ (Q, A)$,

$$(p, q)\cdot u = \big(p\cdot (q, a_1) \cdots (q\cdot a_1 \cdots a_{n-1}, a_n), q\cdot u\big) = (pf_u(1)(q), q\cdot \psi(u)). \tag{5}$$

It follows from (2) that if $\varphi(u) = \varphi(v)$, then $f_u = f_v$ and $\psi(u) = \psi(v)$. Thus, by (5), $(p, q)\cdot u = (p, q)\cdot v$ for each (p, q) in $P \times Q$, that is, $\rho(u) = \rho(v)$. □

Now, by Lemma 2.9, there exists a surjective morphism $\alpha : M \to X$ such that $\alpha \circ \varphi = \rho$, so that (Id_{A^*}, α) is a division from ρ into φ. Thus, ρ is in $\mathbf{V} * \mathbf{W}$ and since by Lemma 1.6, $\mathrm{Stp}(R, B)$ $\mathcal{C}$-divides ρ, $\mathrm{Stp}(R, B)$ is in $\mathbf{V} * \mathbf{W}$ as well.

Conversely, let (Q, B) be an action such that $\mathrm{Stp}(Q, B) \in \mathbf{V} * \mathbf{W}$. Then, $\mathrm{Stp}(Q, B)$ $\mathcal{C}$-divides a $(\mathbf{V}, \mathbf{W})$-product stamp $\varphi : A^* \to M \subseteq N \circ K$. Let π be the natural projection from $N \circ K$ into K. We can write for each word u, $\varphi(u) = (f_u, \pi \circ \varphi(u))$, with $f_u : K \to N$. By definition, the map $\pi \circ \varphi$ is in $\mathbf{W}$ and by Lemma 2.6, the stamp $\Phi_k : (\pi(M) \times A)^* \to I_k \subseteq N$, defined by $\Phi_k(x, a) = f_a(kx)$, is in $\mathbf{V}$ for each $k \in K$. In order to show that the action $(M, A) = \mathrm{Act}(\varphi)$

is in $\mathbf{V}_{\text{act}} * \mathbf{W}_{\text{act}}$, we will verify that $\text{Act}(\varphi)$ $\mathcal{C}$-divides $(L, \pi(M) \times A) \circ \text{Act}(\pi \circ \varphi)$, where $(L, \pi(M) \times A)$ denotes the product $\prod_{k \in K} \text{Act}(\Phi_k)$. Notice that L is a subset of N^K. The action $(L, \pi(M) \times A)$ satisfies, for each $\ell = (\ell_k)_{k \in K} \in L, x \in \pi(M)$ and $a \in A$,

$$\ell \cdot (x, a) = \big(\ell_k \ \Phi_k(x, a)\big)_{k \in K} = \big(\ell_k \ f_a(kx)\big)_{k \in K}.$$

The action $\big(\prod_{k \in K} \text{Act}(\Phi_k)\big) \circ \text{Act}(\pi \circ \varphi)$ is of the form $(L \times \pi(M), A)$ and satisfies $(\ell, x) \cdot a = (\ell \cdot (x, a), x \cdot a)$. For the sake of conciseness, given a word $u = a_1 \cdots a_n$ in A^* and x in $\pi(M)$, we denote by $\Phi_k(x, u)$ the element

$$\Phi_k((x, a_1)(x\pi \circ \varphi(a_1), a_2) \cdots (x\pi \circ \varphi(a_1 \cdots a_{n-1}), a_n)).$$

Then we have for each word u and each k in K:

$$\Phi_k(1, u) = f_{a_1}(k) f_{a_2}(k\pi \circ \varphi(a_1)) \cdots f_{a_n}(k\pi \circ \varphi(a_1 \cdots a_{n_1})) = f_u(k)$$

Now let $R = \{\big((\Phi_k(1, u))_{k \in K}, \pi \circ \varphi(u)\big) \mid u \in A^*\}$: R is a subset of $L \times \pi(M)$. Moreover, it follows from the above remark that $R = \{\big(f_u, \pi \circ \varphi(u)\big) \mid u \in A^*\} = M$. Consider a partial function $\eta : L \times \pi(M) \to M$, with domain $R = M$, as the identity on its domain. This function maps onto M, and the pair (Id_{A^*}, η) is a trivial division from the action $\text{Act}(\varphi)$ into the action $\big(\prod_{k \in K} \text{Act}(\Phi_k)\big) \circ \text{Act}(\pi \circ \varphi)$.

Now, for each k in K, $\text{Stp} \circ \text{Act}(\Phi_k) = \Phi_k$ is in $\mathbf{V}$. Thus, by Lemma 1.5, $\text{Stp}(\prod_{k \in K} \text{Act}(\Phi_k)) = \prod_{k \in K} \text{Stp} \circ \text{Act}(\Phi_k)$ is in $\mathbf{V}$ and $\prod_{k \in K} \text{Act}(\Phi_k)$ is in $\mathbf{V}_{\text{act}}$. One also verifies that $\text{Act}(\pi \circ \varphi)$ is in $\mathbf{W}_{\text{act}}$. Finally, $\text{Act}(\varphi)$ $\mathcal{C}$-divides a $(\mathbf{V}_{\text{act}}, \mathbf{W}_{\text{act}})$-sequential product, and thus it is in $\mathbf{V}_{\text{act}} * \mathbf{W}_{\text{act}}$. By Lemma 1.6, $\text{Act} \circ \text{Stp}(Q, B)$ $\mathcal{C}$-divides $\text{Act}(\varphi)$. Moreover, by Lemma 1.2, (Q, B) $\mathcal{C}$-divides $I_Q(B) \times (\text{Act} \circ \text{Stp}(Q, B))$. Thus, (Q, B) is in $\mathbf{V}_{\text{act}} * \mathbf{W}_{\text{act}}$ as well. □

As a consequence, we get by Proposition 1.3:

Theorem 2.10 $\mathbf{V} * \mathbf{W}$ *is the* $\mathcal{C}$*-variety of stamps corresponding to* $\mathbf{V}_{\text{act}} * \mathbf{W}_{\text{act}}$.

3 The Wreath Product Principle for $\mathcal{C}$-varieties

The wreath product principle (WPP for short) gives a description of languages recognized by an action of $\mathbf{V} * \mathbf{W}$. Results of this section are based on similar results for automata, transducers and monoids [15, 12] as well as extensions of these results to *lp*-varieties [6]. Proposition 2.4 enables us to readily extend the results of [6] to all $\mathcal{C}$-varieties, when $\mathcal{C}$ is convenient. Thus, we shall assume from now that $\mathcal{C}$ is a convenient class. We will state the WPP in terms of actions rather than stamps since it will be used this way in the sequel.

Recall that a *(pure sequential) transducer* is a 6-tuple $\mathcal{T} = (Q, A, B, q_0, \cdot, *)$ where $\mathcal{A} = (Q, A, q_0, \cdot)$ is a complete deterministic finite automaton, B is a finite alphabet called the *output alphabet*, and $(q, a) \mapsto q * a \in B^*$ is called the *output function*. This output function can be extended to a function from $Q \times A^*$ to B^* by setting $q * 1 = 1$ and, for each word u, each letter a and each state q,

$$q * (ua) = (q * u)((q \cdot u) * a).$$

The function *realized* by the transducer $\mathcal{T}$ is the function $\sigma : A^* \to B^*$ defined by $\sigma(u) = q_0 * u$. The *input action* of $\mathcal{T}$ is the action (Q, A) defining the transitions of $\mathcal{T}$. The *output morphism* of $\mathcal{T}$ is simply the morphism from $(Q \times A)^*$ into B^* which maps every letter (q, a) of $Q \times A$ onto the word $q*a$ in B^*. The transducer $\mathcal{T}$ is a *$\mathcal{C}$-transducer* if its output morphism belongs to $\mathcal{C}$. A *$\mathcal{C}$-sequential function* is a function that can be realized by a $\mathcal{C}$-transducer. Notice that if $\mathcal{C} = lp$, an lp-transducer is just a Mealy automaton. The following proposition illustrates the natural links between sequential products and $\mathcal{C}$-sequential functions.

Proposition 3.1 *Let $\mathbf{V}$ and $\mathbf{W}$ be two $\mathcal{C}$-varieties of actions. Let $\mathcal{V}$ (resp. $\mathcal{U}$) be the $\mathcal{C}$-variety of languages associated with $\mathbf{V}$ (resp. $\mathbf{V} * \mathbf{W}$). Then, if L is a language of $\mathcal{V}(B^*)$ and $\sigma\colon A^* \to B^*$ is a $\mathcal{C}$-sequential function realized by a transducer whose input action is in $\mathbf{W}$, then $\sigma^{-1}(L)$ is in $\mathcal{U}(A^*)$.*

We now focus on some specific lp-sequential functions in order to state the WPP. Given an action (Q, A) and q_0 in Q, we define the function $\sigma_{q_0}\colon A^* \to (Q \times A)^*$ by setting

$$\sigma_{q_0}(a_1 \cdots a_n) = (q_0, a_1)(q_0 \cdot a_1, a_2) \cdots (q_0 \cdot a_1 \cdots a_{n-1}, a_n)$$

The function σ_{q_0} is realized by a Mealy automaton with initial state q_0, input action (Q, A), output function defined by $q * a = (q, a)$ and all states final. A sequential function σ is said to be *associated with* (Q, A) if $\sigma = \sigma_q$ for some q in Q. We now state the WPP in terms of actions.

Theorem 3.2 (WPP) *Let $L \subseteq A^*$ be a language recognized by an action of the form $(P, Q \times A) \circ (Q, A)$. Then L is a finite union of languages of the form $W \cap \sigma^{-1}(V)$, where $W \subseteq A^*$ is recognized by (Q, A), σ is a $\mathcal{C}$-sequential function associated with the action (Q, A) and $V \subseteq (Q \times A)^*$ is recognized by $(P, Q \times A)$.*

Recall that a *positive Boolean algebra* on A^* is a set of languages of A^* that is closed under finite intersection and finite union. We can now state the WPP in terms of varieties.

Proposition 3.3 *Let $\mathbf{V}, \mathbf{W}$ be two $\mathcal{C}$-varieties of stamps and let $\mathcal{U}$ be the $\mathcal{C}$-variety of languages associated with $\mathbf{V} * \mathbf{W}$. For each alphabet A,*

(1) *$\mathcal{U}(A^*)$ is the smallest positive Boolean algebra containing $\mathcal{W}(A^*)$ and the languages of the form $\sigma^{-1}(V)$, where σ is the $\mathcal{C}$-sequential function associated with an action (Q, A) in $\mathbf{W}$ and V is in $\mathcal{V}\big((Q \times A)^*\big)$.*

(2) *Each language in $\mathcal{U}(A^*)$ is a finite union of languages of the form $W \cap \sigma^{-1}(V)$ where W is in $\mathcal{W}(A^*)$, σ is the $\mathcal{C}$-sequential function associated with an action (Q, A) in $\mathbf{W}$ and V is in $\mathcal{V}\big((Q \times A)^*\big)$.*

4 Closure under concatenation product

In view of Eilenberg's variety theorem, one may expect some relationship between operators on languages (of a combinatorial nature) and operators on varieties of stamps (of an algebraic nature). Several such results have been proved in the setting of Eilenberg's varieties. In particular, the third author gave in

[14] an algebraic counterpart to the closure of languages under concatenation product .

In the present section we extend this result to $\mathcal{C}$-varieties. The algebraic part makes use of the Mal'cev product, an operation that was extended to varieties of stamps in [11]. Let $\mathbf{A}$ be the variety of all aperiodic monoids. A relational morphism $\tau : M \to N$ is said to be *aperiodic* if, for every idempotent e of N, the semigroup $\tau^{-1}(e)$ is aperiodic. It is well-known that τ is aperiodic if and only if, for every aperiodic subsemigroup T of N, the semigroup $\tau^{-1}(T)$ is also aperiodic. It follows in particular that the composition of two aperiodic relational morphisms is aperiodic.

Given a $\mathcal{C}$-variety of stamps $\mathbf{V}$, we denote by $\mathbf{A} \textcircled{m} \mathbf{V}$ the class of all stamps $\varphi : A^* \to M$ for which there exists a stamp $\psi : A^* \to N$ of $\mathbf{V}$ such that the relational morphism $\psi \circ \varphi^{-1}$ is aperiodic. It is proved in [11] that $\mathbf{A} \textcircled{m} \mathbf{V}$ is a $\mathcal{C}$-variety of stamps, called the *Mal'cev product* of $\mathbf{A}$ and $\mathbf{V}$. Note that $\mathbf{A} \textcircled{m} (\mathbf{A} \textcircled{m} \mathbf{V}) = \mathbf{A} \textcircled{m} \mathbf{V}$.

A collection $\mathcal{L}$ of languages of A^* is *closed under marked product* if, for all $L_0, \ldots, L_n \in \mathcal{L}$ and $a_1, \ldots, a_n \in A$, the language $L_0 a_1 L_1 \cdots a_n L_n$ belongs to $\mathcal{L}$. We are now ready to state the main result of this section.

Theorem 4.1 *Let $\mathbf{V}$ be a $\mathcal{C}$-variety of stamps and let $\mathcal{V}$ be the associated $\mathcal{C}$-variety of languages. For each alphabet A, let $\overline{\mathcal{V}}(A^*)$ be the smallest Boolean algebra of languages containing $\mathcal{V}(A^*)$ and closed under marked product. Then $\overline{\mathcal{V}}$ is a $\mathcal{C}$-variety of languages and the associated $\mathcal{C}$-variety of stamps is $\mathbf{A} \textcircled{m} \mathbf{V}$.*

The proof given below is adapted from the one given in [14]. The first part consists in expressing the Mal'cev product $\mathbf{A} \textcircled{m} \mathbf{V}$ as a product of varieties (Theorem 4.2). The main argument of the second part (Theorem 4.8) states that the $\mathcal{C}$-variety of languages corresponding to $\mathbf{A} * \mathbf{V}$ is contained in $\overline{\mathcal{V}}$. Its proof relies on the wreath product principle on the one hand, and on the Krohn-Rhodes theorem for aperiodic monoids on the other hand.

To any monoid M is associated the *reverse monoid* M^r, whose underlying set is M and whose multiplication (denoted by $\circ$) is defined by $x \circ y = yx$. Given a stamp $\varphi : A^* \to M$, we denote by $\varphi^r : A^* \to M^r$ its *reverse stamp*, defined, for all $a \in A$, by $\varphi^r(a) = \varphi(a)$ — so that, for every word $u = a_1 \cdots a_n$, $\varphi^r(u) = \varphi^r(a_1) \circ \cdots \circ \varphi^r(a_n) = \varphi(a_n) \cdots \varphi(a_1)$. By extension, if $\mathbf{V}$ is a variety of stamps, we denote by $\mathbf{V}^r$ the variety of all stamps φ^r, where $\varphi \in \mathbf{V}$. Finally, we set $\mathbf{V} *_r \mathbf{A} = (\mathbf{A} * \mathbf{V}^r)^r$.

Theorem 4.2 *The equality $\mathbf{A} \textcircled{m} \mathbf{V} = \mathbf{A} * (\mathbf{V} *_r \mathbf{A})$ holds for any $\mathcal{C}$-variety of stamps $\mathbf{V}$.*

We first establish the inclusion of the product into the Mal'cev product.

Proposition 4.3 *The inclusion $\mathbf{A} * \mathbf{V} \subseteq \mathbf{A} \textcircled{m} \mathbf{V}$ holds for any $\mathcal{C}$-variety of stamps $\mathbf{V}$.*

Proof. Let ψ be a stamp of $\mathbf{A} * \mathbf{V}$. By definition, ψ $\mathcal{C}$-divides an $(\mathbf{A}, \mathbf{V})$-product $\varphi : A^* \to M$. In particular,

(1) M is a submonoid of a wreath product $N \circ K$.

(2) Let $\pi : N \circ K \to K$ be the canonical projection morphism and let $\alpha = \pi \circ \varphi$. Then the stamp $\alpha : A^* \to \pi(M)$ is in $\mathbf{V}$.

(3) For each u in A^*, set $\varphi(u) = (f_u, \alpha(u))$, where f_u is in N^K, and let $\Phi : (K \times A)^* \to \mathrm{Im}(\Phi) \subseteq N$ be the stamp defined by $\Phi(k, a) = f_a(k)$ for each letter a in A. Then $\mathrm{Im}(\Phi)$ is an aperiodic monoid.

We claim that the morphism $\pi : M \to \pi(M)$ is aperiodic. Let e be an idempotent of $\pi(M)$ and let s be an element of $\pi^{-1}(e) \cap M$. It suffices to show that $s^n = s^{n+1}$ for some $n > 0$. For each $k \in K$, define the sequential function σ_k from A^* into $(K \times A)^*$ by

$$\sigma_k(a_1 \cdots a_n) = (k, a_1)(k\alpha(a_1), a_2) \cdots (k\alpha(a_1 \cdots a_{n-1}), a_n).$$

Let u be a word of A^* such that $s = \varphi(u)$. Then $\varphi(u) = (f_u, \alpha(u))$ where, for each $k \in K$, $f_u(k) = \Phi(\sigma_k(u))$. Similarly, for each $n > 0$, $s^n = \varphi(u^n) = (f_{u^n}, e)$, where, for each $k \in K$,

$$f_{u^n}(k) = \Phi\big(\sigma_k(u)\sigma_{ke}(u)^{n-1}\big) = \Phi(\sigma_k(u))\Phi(\sigma_{ke}(u))^{n-1}.$$

Now, since $\mathrm{Im}(\Phi)$ is aperiodic, $\Phi(\sigma_{ke}(u))^{n-1} = \Phi(\sigma_{ke}(u))^n$ for some $n > 0$. It follows that $f_{u^{n+1}} = f_{u^n}$ and $s^n = s^{n+1}$, which proves the claim. Now since α is in $\mathbf{V}$, the stamps φ and ψ belong to $\mathbf{A} ⓜ \mathbf{V}$. □

It is now easy to prove the first half of Theorem 4.2.

Corollary 4.4 *The inclusion $\mathbf{A} * (\mathbf{V} *_r \mathbf{A}) \subseteq \mathbf{A} ⓜ \mathbf{V}$ holds for any $\mathcal{C}$-variety of stamps $\mathbf{V}$.*

Proof. We first claim that $(\mathbf{A} ⓜ \mathbf{V}^r)^r = \mathbf{A} ⓜ \mathbf{V}$. Indeed, if $\varphi : A^* \to M$ is a stamp of $(\mathbf{A} ⓜ \mathbf{V}^r)^r$, then φ^r belongs to $\mathbf{A} ⓜ \mathbf{V}^r$ and there exists a stamp $\psi : A^* \to N$ in $\mathbf{V}^r$ such that the relational morphism $\psi \circ (\varphi^r)^{-1}$ is aperiodic. Thus the graph of $\psi \circ (\varphi^r)^{-1}$ is an aperiodic subsemigroup S of $N \times M^r$. It follows that S^r, which is the graph of the relational morphism $\psi^r \circ \varphi^{-1}$ is also aperiodic. Thus $\psi^r \circ \varphi^{-1}$ is an aperiodic relational morphism and since ψ^r belongs to $\mathbf{V}$, φ belongs to $\mathbf{A} ⓜ \mathbf{V}$. Thus $(\mathbf{A} ⓜ \mathbf{V}^r)^r \subseteq \mathbf{A} ⓜ \mathbf{V}$. Applying this relation to $\mathbf{V}^r$, we get $(\mathbf{A} ⓜ \mathbf{V})^r \subseteq \mathbf{A} ⓜ \mathbf{V}^r$, whence $\mathbf{A} ⓜ \mathbf{V} \subseteq (\mathbf{A} ⓜ \mathbf{V}^r)^r$, which proves the claim.

The inclusion $\mathbf{A} * \mathbf{V} \subseteq \mathbf{A} ⓜ \mathbf{V}$ follows from Proposition 4.3. Applying this result to $\mathbf{V} *_r \mathbf{A}$, we get $\mathbf{A} * (\mathbf{V} *_r \mathbf{A}) \subseteq \mathbf{A} ⓜ (\mathbf{V} *_r \mathbf{A})$. Furthermore, $\mathbf{V} *_r \mathbf{A} = (\mathbf{A} * \mathbf{V}^r)^r \subseteq (\mathbf{A} ⓜ \mathbf{V}^r)^r = \mathbf{A} ⓜ \mathbf{V}$. Thus $\mathbf{A} * (\mathbf{V} *_r \mathbf{A}) \subseteq \mathbf{A} ⓜ (\mathbf{A} ⓜ \mathbf{V}) = \mathbf{A} ⓜ \mathbf{V}$. □

To complete the proof of Theorem 4.2, it remains to establish the opposite inclusion:

Proposition 4.5 *The inclusion $\mathbf{A} ⓜ \mathbf{V} \subseteq \mathbf{A} * (\mathbf{V} *_r \mathbf{A})$ holds for any $\mathcal{C}$-variety of stamps $\mathbf{V}$.*

Proof. The proof relies on properties of two classical constructions of semigroup theory, the derived semigroup and the Rhodes expansion. We briefly review these constructions and state their main properties. Given a semigroup S, we shall denote by S^I the monoid obtained by adjoining a new identity element to S (even if S already has an identity)

Let $\gamma : M \to N$ be a morphism. The *derived semigroup* D of γ is defined as follows:

$$D = \{(n, m, n') \mid n' = n\gamma(m)\} \cup \{0\}$$

with multiplication given by

$$(n_1, m_1, n'_1)(n_2, m_2, n'_2) = \begin{cases} (n_1, m_1m_2, n'_2) & \text{if } n'_1 = n_2 \\ 0 & \text{otherwise} \end{cases}$$

We now give a weak version of [3, Proposition 8.1, p. 356]. The original result was stated for relational morphisms between semigroups. Our version deals with monoid morphisms.

Lemma 4.6 *Let $\gamma : M \to N$ be a monoid morphism and let D be its derived semigroup. Then M divides $D^I \circ N$. Further, if γ is aperiodic and if the right stabilisers of the elements of N are aperiodic, then D^I is aperiodic.*

Proof. Let $\alpha : M \to D^I \circ N$ be the map defined by $\alpha(m) = (f, \gamma(m))$, where f is the map from N to D^I defined by $f(n) = (n, m, n\gamma(m))$. Then α is clearly injective. Further, if $\alpha(m_1) = (f_1, n_1)$ and $\alpha(m_2) = (f_2, n_2)$, then $\alpha(m_1)\alpha(m_2) = (f, n_1n_2)$, where $f : N \to D^I$ is defined by

$$f(n) = f_1(n)f_2(nn_1) = (n, m_1, nn_1)(nn_1, m_2, nn_1n_2) = (n, m_1m_2, nn_1n_2).$$

This shows that α is an injective semigroup morphism. Since the identity element of $D^I \circ N$ does not belong to the range of α, one can extend α to an injective monoid morphism from M^I to $D^I \circ N$ by setting $\alpha(I) = 1$. Now the map π from M^I onto M defined by $\pi(m) = m$ if $m \in M$ and $\pi(I) = 1$ is a morphism. Thus M divides $D^I \circ N$.

Let x be an element of D. We claim that, under the assumptions of the second part of the lemma, the subsemigroup $\langle x\rangle$ generated by x is aperiodic. Let $x = (n, u, n')$. If $n \neq n'$, then $x^2 = 0$ and the result is trivial. If $n = n'$, then $\langle x\rangle$ is isomorphic to the subsemigroup M_n of M defined by $M_n = \{m \in M \mid n\gamma(m) = n\}$. Now, γ induces a morphism from M_n onto N_n, the right stabiliser of n in N. Since N_n and γ are aperiodic, so are M_n and $\langle x\rangle$. Thus D, and hence D^I, are aperiodic. □

Let us now recall the definition of the relations $\leqslant_{\mathcal{L}}$ and $\mathcal{L}$ on a monoid M. Given two elements s and t of M, we write $s \leqslant_{\mathcal{L}} t$ if $s = xt$ for some x in M and $s \mathrel{\mathcal{L}} t$ if $s \leqslant_{\mathcal{L}} t$ and $t \leqslant_{\mathcal{L}} s$. Finally we write $<_{\mathcal{L}}$ for $s \leqslant_{\mathcal{L}} t$ and not $s \mathrel{\mathcal{L}} t$.

Let M be a monoid. The reduction ρ of an $\leqslant_{\mathcal{L}}$-chain $(s_n, s_{n-1}, \ldots, s_1)$ of M is defined inductively as follows:

(1) $$\rho(s_1) = (s_1)$$

(2) $$\rho(s_n, \ldots, s_1) = \begin{cases} \rho(s_n, s_{n-2}, \ldots, s_1) & \text{if } s_n \mathrel{\mathcal{L}} s_{n-1} \\ (s_n, \rho(s_{n-1}, \ldots, s_1)) & \text{if } s_n <_{\mathcal{L}} s_{n-1}. \end{cases}$$

In other words, $\rho(s_n, \ldots, s_1)$ is the $<_{\mathcal{L}}$-chain obtained from $(s_n, \ldots, s_1)$ by removing from right to left all the terms s_i such that $s_{i+1} \mathrel{\mathcal{L}} s_i$. Consider the set

$L(M)$ of all $<_{\mathcal{L}}$-chains of M. One can verify that the following operation makes $L(M)$ a semigroup

$$(s_n, \ldots, s_1)(t_m, \ldots, t_1) = \rho(s_n t_m, s_{n-1} t_m, \ldots, s_1 t_m, t_m, t_{m-1}, \ldots, t_1),$$

called the *Rhodes expansion* of M. Note that $L(M)$ is not in general a monoid. The Rhodes expansion enjoys the following properties (see [3, 14]):

(1) The map $\eta_M : L(M) \to M$ defined by

$$\eta_M(s_n, \ldots, s_1) = s_n$$

is a surjective aperiodic morphism from $L(M)$ onto M.

(2) If $\gamma : M \to N$ is a surjective morphism, the morphism $L(\gamma) : L(M) \to L(N)$ defined by

$$L(\gamma)(s_n, \ldots, s_1) = \rho(\gamma(s_n), \ldots, \gamma(s_1))$$

is surjective and satisfies $\gamma \circ \eta_M = \eta_N \circ L(\gamma)$. Further, if γ is aperiodic, so is $L(\gamma)$.

(3) The right stabilisers of the elements of $L(M)$ are aperiodic semigroups.

(4) The monoid $L(M)^r$ divides a wreath product of the form $T \circ M^r$, where T is an aperiodic monoid.

(5) If $\gamma : M \to N$ is an aperiodic morphism, then M divides $U \circ L(N)$ for some aperiodic monoid U.

Our objective is now to extend this construction to stamps. Let $\varphi : A^* \to M$ be a stamp. We define a semigroup morphism $\widehat{\varphi} : A^+ \to L(M)$ by setting, for each $a \in A$,

$$\widehat{\varphi}(a) = (\varphi(a)).$$

This morphism extends to a monoid morphism $\widehat{\varphi} : A^* \to L(M)^I$. Setting $\widehat{M} = \widehat{\varphi}(A^*)$, we now have a stamp $\widehat{\varphi} : A^* \to \widehat{M}$. Furthermore, it follows from (3) that the right stabilisers of the elements of $\widehat{M}$ are aperiodic monoids.

Similarly, η_M extends naturally to a morphism from $L(M)^I$ onto M by setting $\eta_M(I) = 1$. The restriction of η_M to $\widehat{M}$ satisfies $\eta_M \circ \widehat{\varphi} = \varphi$ and is still surjective and aperiodic.

Finally, let $\gamma : M \to N$ be a surjective morphism and let $\psi = \gamma \circ \varphi$. The morphism $L(\gamma)$ can be first extended to a morphism from $L(M)^I$ into $L(N)^I$. Furthermore, for every letter $a \in A$,

$$L(\gamma)\big[\widehat{\varphi}(a)\big] = L(\gamma)\big[(\varphi(a))\big] = (\gamma \circ \varphi(a)) = (\psi(a)) = \widehat{\psi}(a)$$

It follows that $L(\gamma) \circ \widehat{\varphi} = \widehat{\psi}$ and hence $L(\gamma)$ induces a surjective map $\widehat{\gamma}$ from $\widehat{M}$ onto $\widehat{N}$, where $\widehat{N} = \widehat{\psi}(A^*)$, such that $\widehat{\gamma} \circ \widehat{\varphi} = \widehat{\psi}$. Further, by (2), if γ is aperiodic, so is $\widehat{\gamma}$.

Let $\varphi : A^* \to M$ be a stamp of $\mathbf{A} ⓜ \mathbf{V}$. By definition, there exists a stamp $\psi : A^* \to N$ in $\mathbf{V}$ such that the relational morphism $\psi \circ \varphi^{-1}$ is aperiodic. Let $\theta : A^* \to R$ be the product of φ and ψ and let $\pi_M : R \to M$ and $\pi_N : R \to N$ be the natural projections. The situation is illustrated in Figure 4.1

$$
\begin{tikzcd}
& A^* \arrow[ddl, "\varphi"'] \arrow[d, "\theta"] \arrow[ddr, "\psi"] & \\
& R \subseteq M \times N \arrow[dl, "\pi_M"] \arrow[dr, "\pi_N"'] & \\
M \arrow[rr, "\pi_N \circ \pi_M^{-1}"] & & N
\end{tikzcd}
$$

Figure 4.1: A relational morphism

When we apply the Rhodes expansion to all the monoids and morphisms of Figure 4.1, we obtain Figure 4.2:

$$
\begin{tikzcd}
& A^* \arrow[ddl, "\widehat{\varphi}"'] \arrow[d, "\widehat{\theta}"] \arrow[ddr, "\widehat{\psi}"] & \\
& \widehat{R} \arrow[dl, "\widehat{\pi}_M"] \arrow[dr, "\widehat{\pi}_N"'] & \\
\widehat{M} & & \widehat{N}
\end{tikzcd}
$$

Figure 4.2: Rhodes expansion applied to Figure 4.1

A standard result on relational morphisms (see, for instance [9, Proposition 5.5, p. 69]) states that $\pi_N \circ \pi_M^{-1}$ is aperiodic if and only if π_N is aperiodic. It follows that the morphism $\widehat{\pi}_N : \widehat{R} \to \widehat{N}$ is also aperiodic. Since the right stabilisers of $\widehat{N}$ are aperiodic semigroups, it follows from Lemma 4.6 that $\widehat{R}$ divides a wreath product of the form $U \circ \widehat{N}$, where U is aperiodic.

In order to transpose this result to stamps, we now consider a more general situation. Let $\varphi : A^* \to M$ be a stamp and let $\beta : M \to N$ be a surjective morphism. Suppose that M divides a wreath product $U \circ N$ for some aperiodic monoid U. Then there is a surjective morphism $\alpha : K \to M$ where K is a submonoid of $U \circ N$. Denote by $\pi : K \to N$ the restriction to K of the natural projection $U \circ N \to N$.

Proposition 4.7 *Assume that $\pi = \beta \circ \alpha$. If $\beta \circ \varphi$ belongs to $\mathbf{V}$, then φ belongs to $\mathbf{A} * \mathbf{V}$.*

Proof. Since α is surjective, there exists a morphism $\kappa : A^* \to K$ such that $\alpha \circ \kappa = \varphi$. The situation is summarized in the following diagram

$$\begin{tikzcd}
A^* \arrow[r, "\kappa"] \arrow[dr, "\varphi"'] & K \subseteq U \circ N \arrow[r, "\pi"] \arrow[d, "\alpha"] & N \\
& M \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Let $R = \kappa(A^*)$. The restriction of α to R is still surjective. For each $a \in A$, let $\kappa(a) = (f_a, \pi \circ \kappa(a)) = (f_a, \beta \circ \alpha \circ \kappa(a)) = (f_a, \beta \circ \varphi(a))$.

Let $\Phi : (N \times A)^* \to \mathrm{Im}\,\Phi \subseteq U$ be the stamp defined by $\Phi(n, a) = f_a(n)$. Since U is aperiodic, Φ belongs to $\mathbf{A}$ and thus κ is an $(\mathbf{A}, \mathbf{V})$-product stamp. The result follows since (Id_{A^*}, α_R) is a division from φ to κ. $\square$

We have seen that $\widehat{R}$ divides $U \circ \widehat{N}$. We thus have the following diagram:

$$\begin{tikzcd}
A^* \arrow[r, "\kappa"] \arrow[dr, "\widehat{\theta}"'] & K \subseteq U \circ \widehat{N} \arrow[r, "\pi"] \arrow[d, "\alpha"] & \widehat{N} \\
& \widehat{R} \arrow[ur, "\widehat{\pi}_N"'] &
\end{tikzcd}$$

Let $\mathbf{W}$ be the $\mathcal{C}$-variety of stamps generated by $\widehat{\psi}$. The stamp $\widehat{\theta} \circ \widehat{\pi}_N$, which is equal to $\widehat{\psi}$, is in $\mathbf{W}$. Furthermore, the morphism $\widehat{\pi}_N$ projects $\widehat{R}$ onto $\widehat{N}$. Thus by Proposition 4.7, $\widehat{\theta}$ belongs to $\mathbf{A} * \mathbf{W}$. Since φ divides $\widehat{\varphi}$, which itself divides $\widehat{\theta}$, φ belongs to $\mathbf{A} * \mathbf{W}$ as well.

Similarly, Property (4)(or rather, its proof, see [14, p. 322]) can be used to show that $\widehat{N}^r$ divides a wreath product of the form $T \circ N^r$, with T aperiodic. Furthermore, this division satisfies the requirement of Proposition 4.7. It follows that $\mathbf{W}$ is contained in $\mathbf{V} *_r \mathbf{A}$ and thus φ belongs to $\mathbf{A} * (\mathbf{V} *_r \mathbf{A})$. $\square$

The second part of the proof of Theorem 4.1 consists of proving that $\overline{\mathcal{V}}$ is the $\mathcal{C}$-variety of languages associated with $\mathbf{A} \circledM \mathbf{V}$.

The proof in one direction is easy. Let $L = L_0 a_1 L_1 \cdots a_n L_n$ be a marked product of languages of A^*. Let $\gamma : A^* \to M$ (respectively $\eta_i : A^* \to M_i$) be the syntactic morphism of L (respectively L_i, for $i = 0, \ldots, n$) and let η be the product of the stamps $\eta_1, \ldots, \eta_n$. It is well-known that the relational morphism $\eta \circ \gamma^{-1}$ is aperiodic [16]. In particular, if η belongs to $\mathbf{A} \circledM \mathbf{V}$, then γ belongs to $\mathbf{A} \circledM (\mathbf{A} \circledM \mathbf{V})$, which is equal to $\mathbf{A} \circledM \mathbf{V}$. It follows that the $\mathcal{C}$-variety of languages associated with $\mathbf{A} \circledM \mathbf{V}$ contains $\overline{\mathcal{V}}$.

We claim that the proof of the opposite direction reduces to establishing Theorem 4.8 below. Indeed, this theorem implies that the languages corresponding to $\mathbf{V} *_r \mathbf{A}$ are in $\overline{\mathcal{V}}$ and that the languages corresponding to $\mathbf{A} * (\mathbf{V} *_r \mathbf{A})$ are in $\overline{\overline{\mathcal{V}}}$. Since $\overline{\overline{\mathcal{V}}} = \overline{\mathcal{V}}$, and since $\mathbf{A} * (\mathbf{V} *_r \mathbf{A})$ is equal to $\mathbf{A} \circledM \mathbf{V}$ by Theorem 4.2, the $\mathcal{C}$-variety of languages associated with $\mathbf{A} \circledM \mathbf{V}$ is contained in $\overline{\mathcal{V}}$. Let us now prove this theorem.

Theorem 4.8 *The $\mathcal{C}$-varieties of languages associated with $\mathbf{A} * \mathbf{V}$ and $\mathbf{V} *_r \mathbf{A}$ are both contained in $\overline{\mathcal{V}}$.*

Proof. Since the operators $\mathcal{V} \to \mathcal{V}^r$ and $\mathcal{V} \to \overline{\mathcal{V}}$ commute [14], the statement on $\mathbf{V} *_r \mathbf{A}$ follows immediately from the statement on $\mathbf{A} * \mathbf{V}$. Indeed, assuming the result on $\mathbf{A} * \mathbf{V}$, the languages corresponding to $\mathbf{V} *_r \mathbf{A}$ will be in $(\overline{\mathcal{V}^r})^r = \overline{\mathcal{V}}$.

The proof for $\mathbf{A} * \mathbf{V}$ relies on special case of Krohn-Rhodes theorem (see [3]): every aperiodic monoid divides an iterated wreath product of copies of U_2, where U_2 denotes the monoid $\{1, a, b\}$, defined by $xy = y$ for all x, y in $\{a, b\}$. Let us recall a well-known result (see for instance [3, p. 255]).

Lemma 4.9 *Let $L \subseteq A^*$ be a language recognized by an action (Q, A) whose transformation monoid divides U_2. Then L is a Boolean combination of languages of the form A^*aB^* where $B \subseteq A$ and $a \in A$.*

We now describe the languages recognized by a sequential product whose first operand divides U_2.

Proposition 4.10 *Let (Q, A) be an action and let $\mathcal{B}$ be the Boolean algebra generated by all languages recognized by (Q, A). Let $\overline{\mathcal{B}}$ be the smallest Boolean algebra of languages containing $\mathcal{B}$ and closed under marked product. Let $(P, Q \times A)$ be an action whose transformation monoid divides U_2. Then, any language recognized by $(P, Q \times A) \circ (Q, A)$ belongs to $\overline{\mathcal{B}}$.*

Proof. Let $L \subseteq A^*$ be a language recognized by $(P, Q \times A) \circ (Q, A)$. Then, by Theorem 3.2, L is a finite union of languages of the form $V \cap \sigma^{-1}(R)$ where V is recognized by (Q, A), σ is a sequential function associated with (Q, A), and R is recognized by $(P, Q \times A)$. Since V belongs to $\mathcal{B}$ by definition, it suffices to show that $\sigma^{-1}(R)$ belongs to $\overline{\mathcal{B}}$.

Since σ^{-1} commutes with Boolean operations, we may assume by Lemma 4.9 that $R = (Q \times A)^* cB^*$ where $c = (q, a)$ and $B \subseteq Q \times A$. Let $u = a_1 \cdots a_n$ be a word of A^*, where $a_1, \ldots, a_n$ are letters of A. If σ is the sequential function associated with (Q, A) and some element q_0 in Q, the word $\sigma(u)$ is in R if and only if the following holds:

(1) there exists i in $\{1, \ldots, n\}$ such that $(q_0 \cdot a_1 \cdots a_{i-1}, a_i) = (q, a)$,

(2) for any $j \geqslant i$, $(q_0 \cdot a_1 \cdots a_j, a_{j+1}) = (q \cdot aa_{i+1} \cdots a_j, a_{j+1})$ is in B.

Let $L_1 = \{v \in A^* \mid q_0 \cdot v = q\}$. The above two conditions amount to saying that $u = u_1 a u_2$ where u_1 is in L_1 and $u_2 = b_1 \cdots b_k$ satisfies

(2′) For $0 \leqslant j < k$, $(q \cdot ab_1 \cdots b_j, b_{j+1})$ is in B.

The negation of condition (2’) can be stated as

(3) There exists j in $\{0, \ldots, k - 1\}$ such that $(q \cdot ab_1 \cdots b_j, b_{j+1})$ is in $D = (Q \times A) \setminus B$.

Set, for each p in Q, $K_p = \{v \in A^* \mid q \cdot av = p\}$. Condition (2’) is then equivalent to saying that u_2 belongs to the language

$$L_2 = A^* \setminus \bigcup_{(p,d) \in D} K_p d A^*.$$

Altogether, we get $\sigma^{-1}(R) = L_1 a L_2$. Since L_1 and all languages K_p are recognized by (Q, A), the languages L_2 and $\sigma^{-1}(R)$ belong to $\overline{\mathcal{B}}$. $\square$

Let $\mathcal{U}$ be the $\mathcal{C}$-variety of languages associated with $\mathbf{A} * \mathbf{V}$.

Theorem 4.11 *Any language L in $\mathcal{U}(A^*)$ is recognized by a sequential product of the form*

$$(Q_k, Q_{k-1} \times Q_{k-2} \times \ldots \times Q_0 \times R \times A) \circ \cdots \\ \circ (Q_1, Q_0 \times R \times A) \circ (Q_0, R \times A) \circ (R, A),$$

where each of the actions $(Q_i, Q_{i-1} \times \cdots \times Q_0 \times R \times A)$ has a transformation monoid that divides U_2, and (R, A) is in $\mathbf{V}$.

Proof. Let L be a language recognized by an action (Q, A) in $\mathbf{A} * \mathbf{V}$. By Proposition 2.4, there exists a division from (Q, A) into an $(\mathbf{A}, \mathbf{V})$-sequential product $(P, R \times A) \circ (R, A)$. By Lemma 1.7, L is recognized by the action $(P, R \times A) \circ (R, A)$.

Let $(N, R \times A) = \mathrm{Act} \circ \mathrm{Stp}(P, R \times A)$. The monoid N is the transformation monoid of both actions $(P, R \times A)$ and $(N, R \times A)$. Thus, N is an aperiodic monoid. By Lemma 1.2, $(P, R \times A)$ divides the product $I_P(R \times A) \times (N, R \times A)$. Thus, by Proposition 2.3, $(P, R \times A) \circ (R, A)$ divides the action $D = (I_P(R \times A) \times (N, R \times A)) \circ (R, A)$. Further, it is easy to verify that D is isomorphic to the action $D' = I_P(A) \times ((N, R \times A) \circ (R, A))$. Therefore, L is recognized by D'. We need here the following lemma, whose proof is straightforward and omitted.

Lemma 4.12 *Any language recognized by an action of the form $I_{P_1}(A) \times (P_2, A)$ is recognized by (P_2, A).*

By Lemma 4.12, L is recognized by the sequential product $(N, R \times A) \circ (R, A)$. Let $\sigma : (R \times A)^* \to (N \times R \times A)^*$ be the *lp*-sequential function associated with the action $(N, R \times A)$ and the element 1_N, and let

$$\mathcal{T} = (N, R \times A, N \times R \times A, 1_N, \cdot, *_{\mathcal{T}})$$

be its minimal transducer. Its input action is $(N, R \times A)$ and its output function is defined by $n *_{\mathcal{T}} (r, a) = (n, r, a)$. The transformation monoid of $\mathcal{T}$ is N, which is aperiodic. By the Krohn-Rhodes theorem, N divides $U_2 \circ U_2 \circ \cdots \circ U_2$ (k terms), for some integer $k > 0$. Eilenberg showed in [3, Corollary 3.3, p. 167] that one can then write $\sigma = \sigma_k \circ \sigma_{k-1} \circ \cdots \circ \sigma_1$, where each $\sigma_i : A_{i-1}^* \to A_i^*$ is a sequential function realized by a transducer $\mathcal{T}_i = (Q_{i-1}, A_{i-1}, A_i, \cdot, *)$ whose input action (Q_{i-1}, A_{i-1}) has a transformation monoid that divides U_2. Further, a closer look at the proof of this result shows that, since σ is length-preserving, each σ_i is also length-preserving. Notice that $A_0 = R \times A$ and $A_k = N \times R \times A$. It follows that σ is also realized by a transducer $\mathcal{T}' = (Q_{k-1} \times \cdots \times Q_0, A_0, A_k, \cdot, *)$ whose input action S is defined by

$$(q_{k-1}, q_{k-2}, \ldots, q_0) \cdot (r, a) = \\ \Big(q_{k-1} \cdot \big(q_{k-2} * (\cdots (q_0 * (r, a)) \cdots)\big), \ldots, q_1 \cdot (q_0 * (r, a)), q_0 \cdot (r, a)\Big).$$

Since $\mathcal{T}$ is the minimal transducer of σ, there exists a unique morphism from $\mathcal{T}'$ onto $\mathcal{T}$. In particular, there is a surjective function $\varphi : Q_{k-1} \times \cdots \times Q_0 \to N$ such that, for each state $(q_{k-1}, \ldots, q_0)$,

$$\varphi((q_{k-1}, q_{k-2}, \ldots, q_0) \cdot (r, a)) = \varphi(q_{k-1}, q_{k-2}, \ldots, q_0) \cdot (r, a).$$

This morphism induces a division $(Id_{(R\times A)^*}, \varphi)$ from $(N, R\times A)$ into S. Moreover, the action S can be written as a sequential product

$$(Q_{k-1}, Q_{k-2}\times Q_{k-3}\times\cdots\times Q_0\times R\times A)\circ\cdots \\ \circ(Q_1, Q_0\times R\times A)\circ(Q_0, R\times A),$$

where the action $(Q_i, Q_{i-1}\times\cdots\times Q_0\times R\times A)$ is defined by

$$q_i\cdot(q_{i-1},\ldots,q_0,r,a) = q_i\cdot(q_{i-1}*(q_{i-2}*(\cdots(q_0*(r,a))\cdots))).$$

Since, for each $0\leqslant i<k$, the action $(Q_i, Q_{i-1}\times\cdots\times Q_0\times R\times A)$ *lp*-divides (Q_i, A_i), its transformation monoid divides U_2. By Proposition 2.3, $(N, R\times A)\circ(R,A)$ divides $S\circ(R,A)$. Finally, by Lemma 1.7, L is recognized by $S\circ(R,A)$. □

We now complete the proof of Theorem 4.8 by proving the inclusion $\mathcal{U}(A^*)\subseteq\overline{\mathcal{V}}(A^*)$.

Proof. Let L be a language in $\mathcal{U}(A^*)$. By Theorem 4.11, L is recognized by a sequential product

$$S = (Q_k, Q_{k-1}\times Q_{k-2}\times\cdots\times Q_1\times R\times A)\circ\cdots\circ(Q_1, R\times A)\circ(R,A),$$

where, for $i=1,\ldots,k$, the transformation monoid of the action $(Q_i, Q_{i-1}\times\cdots\times Q_1\times R\times A)$ divides U_2 and (R,A) is in $\mathbf{V}$. Set $S_0=(R,A)$ and for $n=1,\ldots,k$

$$S_n = (Q_n, Q_{n-1}\times Q_{n-2}\times\cdots\times Q_1\times R\times A)\circ\cdots\circ(Q_1, R\times A)\circ(R,A).$$

By associativity of the sequential product, $S_n=(Q_n, Q_{n-1}\times\cdots\times Q_1\times R\times A)\circ S_{n-1}$. We show by induction on n that any language recognized by S_n is in $\overline{\mathcal{V}}(A^*)$. The result holds for $n=0$, since (R,A) is in $\mathbf{V}$. Let K be a language recognized by S_{n+1}. By Proposition 4.10, K belongs to the smallest Boolean algebra containing all languages recognized by S_n (which, by the inductive hypothesis, belong to $\overline{\mathcal{V}}(A^*)$) and closed under marked product. This suffices to conclude that $K\in\overline{\mathcal{V}}(A^*)$. Finally, since L is recognized by $S=S_k$, L is in $\overline{\mathcal{V}}(A^*)$. □

Let us conclude this section with an example. If $\varphi: A^*\to M$ is a stamp, consider the set $\varphi(A)$ as an element of the monoid $\mathcal{P}(M)$ of the subsets of M. This element has a unique idempotent power, which is also a subsemigroup of M, called the *stable subsemigroup* of φ. Given a variety of finite semigroups $\mathbf{V}$, a stamp is said to be a *quasi-*$\mathbf{V}$ *stamp* if its stable subsemigroup belongs to $\mathbf{V}$. It is stated in [18] that the quasi-$\mathbf{V}$ stamps form an *lm*-variety (and therefore also an *lp*-variety), denoted by $\mathbf{QV}$.

We now recover a characterization of the languages corresponding to $\mathbf{QA}$, first given in [1].

Theorem 4.13 *A language $L\subseteq A^*$ is recognized by a stamp in* $\mathbf{QA}$ *if and only if it belongs to the smallest Boolean algebra closed under marked product containing the languages $(A^q)^*$ for $q>0$.*

Proof. It is proved in [11] that $\mathbf{QA} = \mathbf{A} \circledcirc{\!\!\!\scriptstyle M}\, \mathbf{MOD}$, where $\mathbf{MOD}$ is the class of all stamps φ from a free monoid A^* onto a finite cyclic group such that, for all $a, b \in A$, $\varphi(a) = \varphi(b)$. It is shown in [6, 11] that $\mathbf{MOD}$ is an lm-variety.

Furthermore, the languages of A^* corresponding to $\mathbf{MOD}$ are precisely the Boolean combinations of the languages $(A^q)^*A^r$ for $0 \leqslant r < q$. Thus the result follows from Theorem 4.1. $\square$

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