A conjecture on the concatenation product

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Abstract

In a previous paper, the authors studied the polynomial closure of a variety of languages and gave an algebraic counterpart, in terms of Mal'cev products, of this operation. They also formulated a conjecture about the algebraic counterpart of the boolean closure of the polynomial closure — this operation corresponds to passing to the upper level in any concatenation hierarchy. Although this conjecture is probably true in some particular cases, we give a counterexample in the general case. Another counterexample, of a different nature, was independently given recently by Steinberg. Taking these two counterexamples into account, we propose a modified version of our conjecture and some supporting evidence for that new formulation. We show in particular that a solution to our new conjecture would give a solution of the decidability of the levels 2 of the Straubing-Thérien hierarchy and of the dot-depth hierarchy. Consequences for the other levels are also discussed.

All semigroups and monoids considered in this paper are either finite or free.

1 Introduction.

The study of the concatenation product goes back to the early years of automata theory. The first major result in this direction was the characterization of star-free languages obtained by Schützenberger in 1965 [38]. A few years later, Cohen and Brzozowski [11] defined the dot-depth of a star-free language and subdivided the class of star-free languages into Boolean algebras, according to their dot-depth. In the original definition, the languages of dot-depth 0 were the finite or cofinite languages and the hierarchy was built by alternating concatenation product and boolean operations.

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It took a few years [8] to show that the dot-depth hierarchy is infinite and a few more years to find an algorithm to decide whether a given star-free language is of dot-depth one [21, 22]. Since then, the dot-depth problem — to find an algorithm to compute the dot-depth of a given star-free language — is wide open, and is, together with the star-height problem, one of the most important open questions of the theory of finite automata.

It does not mean no significant progress was made on the dot-depth problem during the recent years. Elaborating on the precursor work of Büchi [9] and McNaughton [23], Thomas [49] discovered a remarkable connection with logic. It made the dot-depth problem appealing to researchers in model theory, and the articles of Thomas [50, 51] and Selivanov [39] are a good illustration of the contribution of these techniques. Another key fact was the arrival of non-commutative algebra on the scene. Indeed, both the characterizations of star-free languages and of dot-depth one languages were given in terms of an algebraic property of their syntactic semigroups, leading to the hope that every level of the dot-depth hierarchy would have a similar characterization. This hope became fact with Eilenberg's variety theory [13] (see also its extension by the first author [29]).

Roughly speaking, the variety theorem gives a one-to-one correspondence between certain classes of recognizable languages, the varieties of languages, and certain classes of algebraic structures, the varieties of semigroups (resp. monoids, ordered semigroups, ordered monoids). For instance, each level of the dot-depth hierarchy forms a variety of languages and corresponds to a variety of (ordered) semigroups. Furthermore, again by general results, every variety of semigroups can be defined by a set of identities. The problem is that the theory usually does not tell how to find these identities, and for instance, identities are known only for the lower levels of the dot-depth hierarchy.

In view of the variety theorem, one may expect some relationship between the operators on languages (of combinatorial nature) and the operators on semigroups (of algebraic nature). The following table, extracted from [30], summarizes some of the results of this type related to the concatenation product. We shall not attempt to explain in detail the meaning of the symbols in the right column except for one: the symbol Ⓜ corresponds to the Mal'cev product, whose definition is given in Section 2.2.

Closure under the operations ...	$\mathbf{V}$
Product and union	$[\![x^\omega y x^\omega \leq x^\omega]\!] \circledM \mathbf{V}$
Unambiguous product and union	$[\![x^\omega y x^\omega = x^\omega]\!] \circledM \mathbf{V}$
Left deterministic product and union	$[\![x^\omega y = x^\omega]\!] \circledM \mathbf{V}$
Right deterministic product and union	$[\![y x^\omega = x^\omega]\!] \circledM \mathbf{V}$
Product, boolean operations	$\mathbf{A} \circledM \mathbf{V}$
Product with counters, boolean operations	$\mathbf{LGsol} \circledM \mathbf{V}$
Product, product with counters, bool. op.	$\mathbf{L\overline{Gsol}} \circledM \mathbf{V}$
Product followed by boolean closure	?

The operator on the last line is precisely the one used to pass from dot-depth n to dot-depth $n+1$. In view of this table, it is tempting to guess that the question mark should also be some Mal'cev product. If this is the case, a short argument shows it has to be $\mathbf{B}_1 \circledM \mathbf{V}$, where $\mathbf{B}_1$ is the variety corresponding to dot-depth one. This is exactly the conjecture that was formulated in [35].

We prove in this paper that this conjecture is false. Actually, another counterexample was proposed recently by Steinberg [41]. Taking these two counterexamples into account, we propose a new formulation of the conjecture, still deeply related to Mal'cev products, we give supporting evidence for this new conjecture, and we discuss some of its consequences.

2 Semigroups and varieties

2.1 Varieties

A *variety of semigroups* is a class of semigroups closed under taking subsemigroups, quotients and finite direct products [13]. *Varieties of ordered semigroups* are defined analogously [29]. If $\mathbf{V}$ is a variety of monoids, we denote by $\mathbf{V_S}$ the smallest variety of semigroups containing the monoids of $\mathbf{V}$. We also denote by $\mathbf{LV}$ the variety of all semigroups S, such that, for each idempotent $e \in S$, the monoid eSe belongs to $\mathbf{V}$. In particular, $\mathbf{LI}$ is the variety of *locally trivial* semigroups. Finally, if $\mathbf{V}$ is a variety of semigroups, we denote by $\mathbf{V_M}$ the variety of monoids consisting of all monoids in $\mathbf{V}$.

Varieties of ordered semigroups are conveniently defined by identities [34]. Precise definitions can be found in Almeida's book [1], or in the first sections of the survey paper [30]. See also [29, 35] for more specific information. For instance, the identity $x \leq 1$ defines the variety of ordered monoids M such that, for all $x \in M$, $x \leq 1$. This variety is denoted $[\![x \leq 1]\!]$. Following Almeida [1], we use the symbols e and f to represent idempotents in identities. For instance, the variety $[\![exe \leq e]\!]$ is the variety of ordered semigroups $(S, \leq)$ such that, for each idempotent $e \in S$ and for each $x \in S$, $exe \leq e$. The following varieties will be used in this paper.

$$\begin{aligned} \mathbf{I} &= \{1\} & \mathbf{LI} &= [\![exe = e]\!] \\ \mathbf{G} &= [\![e = 1]\!] & \mathbf{LG} &= [\![(exe)^\omega = e]\!] \\ \mathbf{J}^+ &= [\![x \leq 1]\!] & \mathbf{LJ}^+ &= [\![exe \leq e]\!] \\ \mathbf{J}^- &= [\![x \geq 1]\!] & \mathbf{LJ}^- &= [\![exe \geq e]\!] \end{aligned}$$

The varieties on the left are varieties of monoids and those one the right are varieties of semigroups. The variety $\mathbf{G}$ is the variety of all groups (considered as monoids). Note that $\mathbf{LI} = \mathbf{LJ}^+ \cap \mathbf{LJ}^-$.

The join of two varieties $\mathbf{V}_1$ and $\mathbf{V}_2$, that is, the smallest variety containing $\mathbf{V}_1$ and $\mathbf{V}_2$, is denoted by $\mathbf{V}_1 \vee \mathbf{V}_2$. For instance, it is shown in [37] that the join of $\mathbf{J}^+$ and $\mathbf{J}^-$ is the variety of $\mathcal{J}$-trivial monoids

$$\mathbf{J} = [\![(xy)^\omega x = (xy)^\omega = y(xy)^\omega]\!].$$

Similarly, it is shown in [37] that the join of $\mathbf{LJ}^+$ and $\mathbf{LJ}^-$ is the variety

$$\mathbf{B}_1 = [\![(esfte)^\omega (eufve)^\omega = (esfte)^\omega esfve(eufve)^\omega]\!].$$

This identity was discovered by Knast [22] in his study of dot-depth one languages (see Section 7). Another example will be considered in this paper. Let $\mathbf{H}$ be a variety of groups. The semigroups in $\mathbf{LI} \vee \mathbf{H_S}$ are easy to describe: their unique regular $\mathcal{D}$-class is their minimal ideal and is isomorphic to the product of a rectangular band by a group in $\mathbf{H}$. The next proposition is just another way of stating the same result.

Proposition 2.1 *Let $\mathbf{H}$ be a variety of groups. Then a semigroup S belongs to $\mathbf{LI} \vee \mathbf{H_S}$ if and only if $S \in \mathbf{LH}$ and $efe = e$ for every pair (e, f) of idempotents of S.*

Let S and T be semigroups. We write the product in S additively to provide a more transparent notation, but it is not meant to suggest that S is commutative. A *left action* of T on S is a map $(t, s) \mapsto t{\cdot}s$ from $T^1 \times S$ into S such that, for all $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$,

(1) $(t_1t_2){\cdot}s = t_1(t_2{\cdot}s)$

(2) $t{\cdot}(s_1 + s_2) = t{\cdot}s_1 + t{\cdot}s_2$

(3) $1{\cdot}s = s$

If S is a monoid with identity 0, the action is *unitary* if it satisfies, for all $t \in T$,

(4) $t{\cdot}0 = 0$

The *semidirect product* of S and T (with respect to the given action) is the semigroup $S * T$ defined on $S \times T$ by the multiplication

$$(s,t)(s',t') = (s + t{\cdot}s', tt')$$

Given two varieties of ordered semigroups $\mathbf{V}$ and $\mathbf{W}$, their *semidirect product* $\mathbf{V} * \mathbf{W}$ is the variety generated by all semidirect products of the form $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$. If $\mathbf{V}$ is a monoid variety, we always assume that the action of T on S is unitary.

We illustrate these notions by an elementary result, which will be used in Section 6.

Proposition 2.2 *For every variety of groups* $\mathbf{H}$, $\mathbf{H}*\mathbf{LI} = \mathbf{LH}$ *and* $\mathbf{LI}*\mathbf{H} = \mathbf{LI} \vee \mathbf{H_S}$.

Proof. The equality $\mathbf{H} * \mathbf{LI} = \mathbf{LH}$ is proved for instance in [13]. The inclusion $\mathbf{LI} \vee \mathbf{H_S} \subseteq \mathbf{LI} * \mathbf{H}$ is clear. To establish the opposite inclusion, consider a semidirect product $S*H$, with $S \in \mathbf{LI}$ and $H \in \mathbf{H}$. Since $\mathbf{LI}*\mathbf{H}$ is contained in $\mathbf{LH}$ [13], Proposition 2.1 can be applied if we prove the identity on idempotents. Let $(e,1)$ and $(f,1)$ be idempotents in $S * H$. Writing the product in S additively, we have

$$(e,1)(f,1)(e,1) = (e + 1f, 1)(e,1) = (e + 1f + 1e, 1) = (e,1)$$

and thus $(e,1)(f,1)(e,1) = (e,1)$ as required. $\square$

A much more difficult result is the equality

$$\mathbf{B}_1 = \mathbf{J}_1 * \mathbf{LI}$$

established by Knast [21, 22].

2.2 Relational morphisms

If $(S, \leq)$ and $(T, \leq)$ are ordered semigroups, a *relational morphism* from S to T is a relation $\tau : (S, \leq) \to (T, \leq)$, i.e. a mapping from S into $\mathcal{P}(T)$ such that:

(1) $\tau(s)\tau(t) \subseteq \tau(st)$ for all $s, t \in S$,

(2) $\tau(s)$ is non-empty for all $s \in S$,

For a relational morphism between two ordered monoids $(S, \leq)$ and $(T, \leq)$, a third condition is required

(3) $1 \in \tau(1)$

Equivalently, τ is a relation whose graph

$$\mathrm{graph}(\tau) = \{\ (s,t) \in S \times T \mid t \in \tau(s)\ \}$$

is an ordered subsemigroup (resp. submonoid if S and T are monoids) of $S \times T$, with first-coordinate projection surjective onto S.

Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be varieties of ordered semigroups. A relational morphism $\tau : S \to T$ is a $(\mathbf{V}_1, \mathbf{V}_2)$ *relational morphism* if, for every ordered subsemigroup R of T in $\mathbf{V}_2$, the ordered semigroup $\tau^{-1}(R)$ belongs to $\mathbf{V}_1$. A $(\mathbf{V}, \mathbf{V})$ *relational morphism* is simply called a $\mathbf{V}$-relational morphism.

Let $\mathbf{W}$ be a variety of ordered semigroups (resp. monoids). The class of all ordered semigroups (resp. monoids) S such that there exists a $(\mathbf{V}_1, \mathbf{V}_2)$ relational morphism $\tau : S \to T$, with $T \in \mathbf{W}$, is a variety of ordered semigroups (resp. monoids), denoted by $(\mathbf{V}_1, \mathbf{V}_2) \circledM \mathbf{W}$. If $\mathbf{V}_1 = \mathbf{V}$ and if $\mathbf{V}_2$ is the trivial variety $\mathbf{I_S}$, the notation simplifies to $\mathbf{V} \circledM \mathbf{W}$ (this is the *Mal'cev product* of $\mathbf{V}$ and $\mathbf{W}$). Note that even though the inclusion

$$\mathbf{V}_1 \circledM (\mathbf{V}_2 \circledM \mathbf{V}_3) \subseteq (\mathbf{V}_1 \circledM \mathbf{V}_2) \circledM \mathbf{V}_3 \tag{1}$$

always holds [54], the Mal'cev product is not associative.

In [35], the authors gave a description of a set of identities defining $\mathbf{V} \circledM \mathbf{W}$, given a set of identities describing $\mathbf{V}$ and $\mathbf{W}$.

Let us recall a useful characterization of $\mathbf{LI}$- (resp. $\mathbf{LG}$-) relational morphisms, given in [27, Chapter 3].

Proposition 2.3 *A relational morphism is a* $\mathbf{LI}$*-relational morphism if and only if it is a* $(\mathbf{LI}, \mathbf{I_S})$ *relational morphism. A relational morphism is a* $\mathbf{LG}$*-relational morphism if and only if it is a* $(\mathbf{LG}, \mathbf{I_S})$ *relational morphism.*

Corollary 2.4 $\mathbf{LI} \circledM \mathbf{LI} = \mathbf{LI}$ *and* $\mathbf{LG} \circledM \mathbf{LG} = \mathbf{LG}$.

We now state a few elementary results on relational morphisms.

Lemma 2.5 *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be varieties of ordered semigroups and let* $\tau : S \to T$ *be a* $(\mathbf{V}, \mathbf{W})$ *relational morphism. Let* S' *and* T' *be subsemigroups of* S *and* T*, respectively, and let* $\tau' : S' \to T'$ *be a relational morphism such that, for each* $s \in S'$, $\tau'(s) \subseteq \tau(s)$*. Then* τ' *is a* $(\mathbf{V}, \mathbf{W})$ *relational morphism.*

Proof. We note that if W is a subsemigroup of T', then ${\tau'}^{-1}(W)$ is a subsemigroup of $\tau^{-1}(W)$. Now if $W \in \mathbf{W}$, then $\tau^{-1}(W) \in \mathbf{V}$ and hence also ${\tau'}^{-1}(W) \in \mathbf{W}$. $\square$

Proposition 2.6 *Let* $\mathbf{V}_1$*,* $\mathbf{V}_2$*,* $\mathbf{V}_3$ *and* $\mathbf{V}$ *be varieties of ordered semigroups. Any* $(\mathbf{V}_1, \mathbf{V}_2)$ *relational morphism is a* $((\mathbf{V}_1, \mathbf{V}_3) \circledM \mathbf{V}, (\mathbf{V}_2, \mathbf{V}_3) \circledM \mathbf{V})$ *relational morphism.*

Proof. Let $\tau : S \to T$ be a $(\mathbf{V}_1, \mathbf{V}_2)$ relational morphism between ordered semigroups. Let T' be a subsemigroup of T such that $T' \in (\mathbf{V}_2, \mathbf{V}_3) \circledM \mathbf{V}$. Then there exists a $(\mathbf{V}_2, \mathbf{V}_3)$ relational morphism $\tau' : T' \to W$ with $W \in \mathbf{V}$.

Now let $S' = \tau^{-1}(T')$ and $\rho : S' \to T'$ be the relational morphism given by $\rho(s) = \tau(s) \cap T'$ for each $s \in S'$. By Lemma 2.5, ρ is a $(\mathbf{V}_1, \mathbf{V}_2)$ relational morphism.

We claim that the composite $\tau' \circ \rho : S' \to W$ is a $(\mathbf{V}_1, \mathbf{V}_3)$ relational morphism. Indeed, if U is a subsemigroup of W, then $(\tau' \circ \rho)^{-1}(U) = \rho^{-1}(\tau'^{-1}(U))$. If $U \in \mathbf{V}_3$, then $\tau'^{-1}(U) \in \mathbf{V}_2$ since τ' is a $(\mathbf{V}_2, \mathbf{V}_3)$ relational morphism and $\rho^{-1}(\tau'^{-1}(U)) \in \mathbf{V}_1$ since ρ is a $(\mathbf{V}_1, \mathbf{V}_2)$ relational morphism. This proves the claim. It follows that $S' \in (\mathbf{V}_1, \mathbf{V}_3) \circledM \mathbf{V}$, which concludes the proof. $\square$

Specializing Proposition 2.6 with $\mathbf{V}_3 = \mathbf{I_S}$ shows that any $(\mathbf{V}_1, \mathbf{V}_2)$ relational morphism is a $(\mathbf{V}_1 \circledM \mathbf{V}, \mathbf{V}_2 \circledM \mathbf{V})$ relational morphism. This yields the following corollary.

Corollary 2.7 *Let* $\mathbf{W}$ *be a variety of semigroups (resp. monoids) and let* $\mathbf{V}_1$*,* $\mathbf{V}_2$ *and* $\mathbf{V}$ *be varieties of semigroups. Then the following inclusion holds*

$$(\mathbf{V}_1, \mathbf{V}_2) \circledM \mathbf{W} \subseteq (\mathbf{V}_1 \circledM \mathbf{V}, \mathbf{V}_2 \circledM \mathbf{V}) \circledM \mathbf{W}$$

As an illustration of these definitions, let us compute a few Mal'cev products.

Proposition 2.8 *Let* $\mathbf{H}$ *be a variety of groups. Then*

$$\begin{array}{ll} \mathbf{LI} * \mathbf{H} = \mathbf{LI} \circledM \mathbf{H_S} = \mathbf{LI} \vee \mathbf{H_S} & \mathbf{H} * \mathbf{LI} = \mathbf{H_S} \circledM \mathbf{LI} = \mathbf{LH} \\ \mathbf{LI} * \mathbf{LH} = \mathbf{LI} \circledM \mathbf{LH} = \mathbf{LH} & \mathbf{LI} * (\mathbf{LI} * \mathbf{H}) = \mathbf{LI} \circledM (\mathbf{LI} * \mathbf{H}) = \mathbf{LI} * \mathbf{H} \\ \mathbf{LH} * \mathbf{LI} = \mathbf{LH} \circledM \mathbf{LI} = \mathbf{LH} & (\mathbf{LI} * \mathbf{H}) * \mathbf{LI} = (\mathbf{LI} * \mathbf{H}) \circledM \mathbf{LI} = \mathbf{LH} \end{array}$$

Proof. The equalities $\mathbf{H_S} \circledM \mathbf{LI} = \mathbf{LH}$ and $\mathbf{LI} \circledM \mathbf{H_S} = \mathbf{LI} \vee \mathbf{H_S}$ are proved in [19]. It follows by Proposition 2.2 that $\mathbf{H} * \mathbf{LI} = \mathbf{LH}$ and $\mathbf{LI} * \mathbf{H} = \mathbf{LI} \vee \mathbf{H_S}$. This settles the first line of the proposition. Next, we have $\mathbf{LI} * (\mathbf{LI} * \mathbf{H}) = (\mathbf{LI} * \mathbf{LI}) * \mathbf{H} = \mathbf{LI} * \mathbf{H}$ and since $\mathbf{LI} * \mathbf{H} \subseteq \mathbf{H} * \mathbf{LI}$, $(\mathbf{LI} * \mathbf{H}) * \mathbf{LI} = \mathbf{H} * \mathbf{LI} = \mathbf{LH}$. Furthermore,

$$\begin{aligned} \mathbf{LI} * \mathbf{H} \subseteq \mathbf{LI} \circledM (\mathbf{LI} * \mathbf{H}) &= \mathbf{LI} \circledM (\mathbf{LI} \circledM \mathbf{H_S}) \subseteq (\mathbf{LI} \circledM \mathbf{LI}) \circledM \mathbf{H_S} \\ &= \mathbf{LI} \circledM \mathbf{H_S} = \mathbf{LI} * \mathbf{H} \end{aligned}$$

Thus, $\mathbf{LI} \circledM (\mathbf{LI} * \mathbf{H}) = \mathbf{LI} * \mathbf{H}$. Now, we have

$$\begin{aligned} \mathbf{LH} \subseteq \mathbf{LI} \circledM \mathbf{LH} &= \mathbf{LI} \circledM (\mathbf{H_S} \circledM \mathbf{LI}) \subseteq (\mathbf{LI} \circledM \mathbf{H_S}) \circledM \mathbf{LI} \\ &= (\mathbf{LI} * \mathbf{H}) \circledM \mathbf{LI} \subseteq \mathbf{LH} \circledM \mathbf{LI} \end{aligned}$$

Finally, let $S \in \mathbf{LH} \circledM \mathbf{LI}$. By definition, there exists a semigroup $T \in \mathbf{LI}$ and a $\mathbf{LH}$-relational morphism $\tau : S \to T$. Let e be an idempotent of S, and let f be an idempotent of $\tau(e)$. Then $e \in \tau^{-1}(f)$. Now if $s \in S$ and $t \in \tau(s)$,

$f = ftf \in \tau(e)\tau(s)\tau(e) \subseteq \tau(ese)$. It follows that eSe is a subsemigroup of $\tau^{-1}(f)$. Since $\tau^{-1}(f) \in \mathbf{LH}$, $eSe \in \mathbf{LH}$. Therefore, $e(eSe)e = eSe \in \mathbf{H}$ and $S \in \mathbf{LH}$. This proves that $\mathbf{LH} \,ⓜ\, \mathbf{LI} \subseteq \mathbf{LH}$, completing the proof of the proposition. □

3 Languages

A $+$-*class* (resp. $*$-*class*) of recognizable languages is a correspondence $\mathcal{C}$ which associates with each alphabet A a set $\mathcal{C}(A^+)$ (resp. $\mathcal{C}(A^*)$) of recognizable languages of A^+ (resp. A^*).

3.1 Varieties of languages

A *positive $+$-variety of languages* is a class of recognizable languages $\mathcal{V}$ such that

(1) for every alphabet A, $\mathcal{V}(A^+)$ contains $\emptyset$ and A^+ and is closed under finite intersection and finite union,

(2) if $\varphi : A^+ \to B^+$ is a morphism of semigroups, $L \in \mathcal{V}(B^+)$ implies $\varphi^{-1}(L) \in \mathcal{V}(A^+)$,

(3) if $L \in \mathcal{V}(A^+)$ and if $a \in A$, then $a^{-1}L$ and La^{-1} are in $\mathcal{V}(A^+)$.

Positive $*$-varieties are defined analogously by replacing each instance of $+$ by $*$ and morphisms of semigroups by morphisms of monoids. A *variety* is a positive variety closed under complement.

Eilenberg's theorem [13] provides a bijective correspondence between varieties of languages and varieties of semigroups. There is an analogous correspondence [29] between positive varieties of languages and varieties of ordered semigroups. In the sequel, we shall use freely the term "corresponding variety" to refer to these correspondences in both directions. In particular, if $\mathbf{V}$ is a variety of ordered semigroups, a $\mathbf{V}$-*language* is a language recognized by an ordered semigroup of $\mathbf{V}$, that is, a language of the positive variety of languages corresponding to $\mathbf{V}$. The next proposition will serve as an illustration of this definition.

Proposition 3.1 *Let $\mathbf{V} = \mathbf{LI} \vee \mathbf{H_S}$, where $\mathbf{H}$ is a variety of groups, and let $\mathcal{V}$ be the corresponding variety of languages. For each alphabet A, $\mathcal{V}(A^+)$ consists of the finite unions of languages of the form w, where $w \in A^+$, or uLv, where $uv \in A^+$ and L is an $\mathbf{H}$-language of A^*.*

Proof. Let $\mathcal{C}$ be the class of languages described in the second part of the statement. First, it is well known [13] that the $\mathbf{LI}$-languages are finite unions of languages of the form w, where $w \in A^+$, or uA^*v, where $uv \in A^+$. Thus $\mathcal{C}$ contains the $\mathbf{LI}$-languages, since A^* is an $\mathbf{H}$-language of A^* for any $\mathbf{H}$.

Next, an $\mathbf{H_S}$-language is a language of the form $L \cap A^+$, where L is an $\mathbf{H}$-language. Now, $L \cap A^+ = \bigcup_{a \in A} a(a^{-1}L)$, and since, for each letter a, $a^{-1}L$ is an $\mathbf{H}$-language, $L \cap A^+$ is in $\mathcal{C}(A^+)$. This proves that $\mathcal{C}$ contains all the $\mathbf{V}$-languages.

To establish the opposite inclusion, it suffices to consider the languages of the form $\{w\}$ or uLv described above. It was already mentioned that the singletons are $\mathbf{LI}$-languages, and hence $\mathbf{V}$-languages. Next, let L be an $\mathbf{H}$-language and let $\eta : A^* \to H$ be its syntactic morphism. Since $\mathbf{H}$ is a variety of groups, H is a group. Let u, v be words of A^* such that $uv \neq 1$ and let $\bar{u}$ and $\bar{v}$ be words such that $\eta(\bar{u}) = \eta(u)^{-1}$ and $\eta(\bar{v}) = \eta(v)^{-1}$. We claim that

$$uLv = uA^*v \cap \bar{u}^{-1}L\bar{v}^{-1}$$

First observe that, for every $x \in A^*$, $\eta(\bar{u}uxv\bar{v}) = \eta(x)$. Thus $x \in L$ if and only if $\bar{u}uxv\bar{v} \in L$. It follows that if $x \in L$, then $uxv \in uA^*v \cap \bar{u}^{-1}L\bar{v}^{-1}$. Conversely, if $uxv \in uA^*v \cap \bar{u}^{-1}L\bar{v}^{-1}$, then $\bar{u}uxv\bar{v} \in L$, and hence $x \in L$, which proves the claim. Now uA^*v is a $\mathbf{LI}$-language and $A^+ \cap \bar{u}^{-1}L\bar{v}^{-1}$ is an $\mathbf{H_S}$-language. This proves that uLv is a $\mathbf{LI} \vee \mathbf{H_S}$-language. □

3.2 Polynomial closure

The *polynomial closure* of a set of languages $\mathcal{L}$ of A^* is the set of languages of A^* that are finite unions of languages of the form

$$L_0a_1L_1 \cdots a_nL_n$$

where $n \geq 0$, the a_i's are letters and the L_i's are elements of $\mathcal{L}$.

A product $L = L_0a_1L_1 \cdots a_nL_n$ is *unambiguous* if every word u of L admits a unique factorization of the form $u_0a_1u_1 \cdots a_nu_n$ with $u_0 \in L_0$, $u_1 \in L_1$, ..., $u_n \in L_n$. The *unambiguous polynomial closure* of a set of languages $\mathcal{L}$ of A^* is the set of languages that are finite disjoint unions of unambiguous products of the form

$$L_0a_1L_1 \cdots a_nL_n$$

where $n \geq 0$, the a_i's are letters and the L_i's are elements of $\mathcal{L}$.

The *polynomial closure* of a set of languages $\mathcal{L}$ of A^+ is the set of languages of A^+ that are finite unions of languages of the form

$$u_0L_1u_1 \cdots L_nu_n$$

where $n \geq 0$, the u_i's are words of A^* and the L_i's are elements of $\mathcal{L}$. If $n = 0$, one requires of course that u_0 is not the empty word.

A product $L = u_0L_1u_1 \cdots L_nu_n$ of n languages L_1, ..., L_n of A^+ is *unambiguous* if every word u of L admits a unique factorization of the form

$u_0v_1u_1\cdots v_nu_n$ with $v_1 \in L_1, \ldots, v_n \in L_n$. The *unambiguous polynomial closure* of a set of languages $\mathcal{L}$ of A^+ is the set of languages that are finite disjoint unions of unambiguous products of the form

$$u_0L_1u_1\cdots L_nu_n$$

where $n \geq 0$, the u_i's are words of A^* and the L_i's are elements of $\mathcal{L}$.

Finally, the *boolean closure* of a set of languages $\mathcal{L}$ of A^* (resp. A^+) is the smallest set of languages containing $\mathcal{L}$ and closed under finite boolean operations (finite union and complement).

By extension, if $\mathcal{C}$ is a $*$-class (resp. $+$-class), we denote by Pol $\mathcal{C}$ its *polynomial closure*, that is, the class of languages such that, for every alphabet A, Pol $\mathcal{C}(A^*)$ (resp. Pol $\mathcal{C}(A^+)$) is the polynomial closure of $\mathcal{C}(A^*)$ (resp. $\mathcal{C}(A^+)$). The *unambiguous polynomial closure* UPol $\mathcal{C}$ and the *boolean closure* B$\mathcal{C}$ of a class of languages $\mathcal{C}$ is defined analogously. One can show that if $\mathcal{C}$ is a variety of languages, then Pol $\mathcal{C}$ is a positive variety of languages, while UPol $\mathcal{C}$ and BPol $\mathcal{C}$ are varieties of languages.

The following theorem summarizes the results of [36].

Theorem 3.2 *Let* $\mathbf{V}$ *be a variety of monoids (resp. semigroups) and let* $\mathcal{V}$ *be the corresponding variety of languages.*

(1) *The variety of ordered monoids (resp. semigroups) corresponding to Pol* $\mathcal{V}$ *is the variety* $\mathbf{LJ}^+ \circledM \mathbf{V}$.

(2) *The variety of monoids (resp. semigroups) corresponding to UPol* $\mathcal{V}$ *is the variety* $\mathbf{LI} \circledM \mathbf{V}$.

We shall denote by $\widehat{\mathbf{V}}$ the variety of semigroups (resp. monoids) corresponding to BPol $\mathcal{V}$. It follows from the results of [35, 36] that

$$\mathbf{LI} \circledM \mathbf{V} = (\mathbf{LJ}^+ \circledM \mathbf{V}) \cap (\mathbf{LJ}^- \circledM \mathbf{V}) \text{ and} \tag{2}$$

$$\widehat{\mathbf{V}} = (\mathbf{LJ}^+ \circledM \mathbf{V}) \vee (\mathbf{LJ}^- \circledM \mathbf{V}) \tag{3}$$

In view of Theorem 3.2, it is tempting to guess that $\widehat{\mathbf{V}}$ is also of the form $\mathbf{V} \to \mathbf{W} \circledM \mathbf{V}$ for some variety $\mathbf{W}$. The inclusion $\widehat{\mathbf{V}} \subseteq \mathbf{B}_1 \circledM \mathbf{V}$ follows from [28, Theorems 3.1 and 3.2]. In [35], it was conjectured that $\widehat{\mathbf{V}} = \mathbf{B}_1 \circledM \mathbf{V}$, but a counterexample will be given in Section 4. An improved version of the conjecture will be stated in Section 6.

For any class of languages $\mathcal{C}$, Pol $\mathcal{C}$ = Pol (UPol $\mathcal{C}$) and thus BPol $\mathcal{C}$ = BPol (UPol $\mathcal{C}$). In view of Theorem 3.2, the latter equality can be translated in terms of varieties as follows.

Corollary 3.3 *For any variety of monoids (resp. semigroups)* $\mathbf{V}$, *the equality* $\widehat{\mathbf{LI} \circledM \mathbf{V}} = \widehat{\mathbf{V}}$ *holds.*

The variety $\widehat{\mathbf{V}}$ can also be described in terms of Schützenberger products. We refer the reader to [32] for precise definitions. Given a variety of semigroups (resp. monoids) $\mathbf{V}$, we denote by $\Diamond\mathbf{V}$ the variety generated by all Schützenberger products of semigroups in $\mathbf{V}$.

It was shown in [26] that if $\mathbf{V}$ is a variety of monoids, then $\widehat{\mathbf{V}} = \Diamond\mathbf{V}$. It follows from [32, Theorem 5.3] that the same result holds for varieties of semigroups, provided that the variety of languages corresponding to $\mathbf{V}$ is closed under the operations $L \to uL$ and $L \to Lu$ (where u is a word).

4 A counterexample

In this section, we give an aperiodic counterexample to the conjecture $\widehat{\mathbf{V}} = \mathbf{B}_1 \circledM \mathbf{V}$. It is interesting to note that Steinberg [41] gave another counterexample, in a quite different realm: he showed that if $\mathbf{H}$ is a variety of groups, $\widehat{\mathbf{H}} = \mathbf{J} * \mathbf{H}$, $\mathbf{B}_1 \circledM \mathbf{H} = \mathbf{J} \circledM \mathbf{H}$, yet $\mathbf{J} * \mathbf{H} \neq \mathbf{J} \circledM \mathbf{H}$ as soon as $\mathbf{H}$ consists only of commutative groups, or of groups satisfying a fixed identity of the form $x^n = 1$.

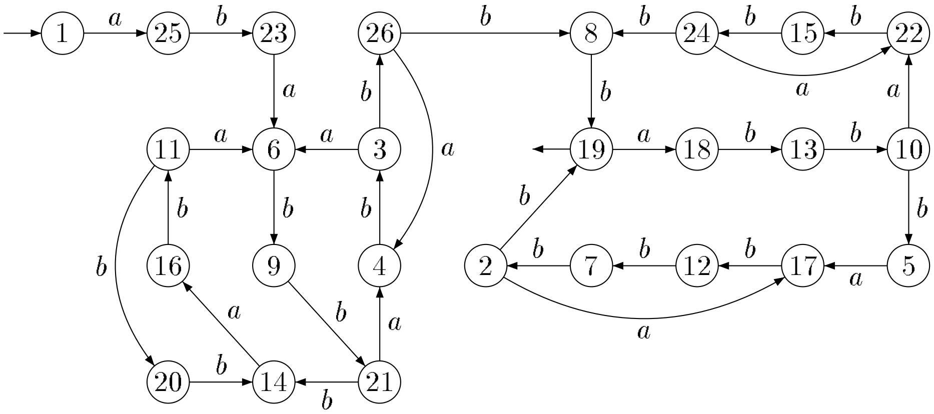
Our starting point is a semigroup in $\mathbf{LJ}$ which is not in $\mathbf{B}_1$, discovered by Knast [21] as a byproduct of his characterization of the languages of dot-depth one. This semigroup is the syntactic semigroup of the following language:

$$L_0 = (ab^+ \cup ac^+)^* ab^+ d(c^+ d \cup b^+ d)^*.$$

We study in some detail the syntactic monoid of the image L of L_0 under the morphism which maps a, b, c, d respectively to ab, ab^2, ab^3, ab^4:

$$L = (ab(ab^2)^+ \cup ab(ab^3)^+)^* ab(ab^2)^+ ab^4((ab^3)^+ ab^4 \cup (ab^2)^+ ab^4)^*.$$

The minimal automaton $\mathcal{A}$ of L has 26 states, and is depicted below[1]. The initial state is 1 and the unique final state is 19.



[1]This computation was performed using Champarnaud's AUTOMATE software [10].

Let S be the syntactic monoid of L and let $\sigma : A^+ \to S$ be its syntactic morphism. It is aperiodic and it has 445 elements, of which are 85 idempotents. Its $\mathcal{D}$-class structure is as follows[2]. The maximal $\mathcal{D}$-class is the singleton $\{1\}$. Next there are 19 non-regular singleton $\mathcal{D}$-classes (with elements respectively the images by σ of a, b, ab, ba, b^2, ab^2, bab, b^2a, bab^2, b^2ab, b^2ab^2 and b^3, ab^3, b^3a, bab^3, b^3ab, b^2ab^3, b^3ab^2, b^3ab^3). Next there are two incomparable regular $\mathcal{D}$-classes. The first one has 3 $\mathcal{R}$-classes, 3 $\mathcal{L}$-classes and 3 idempotents $\sigma(b^2ab^2a)$, $\sigma(bab^2ab)$, $\sigma(ab^2ab^2)$. The second one has 4 $\mathcal{R}$-classes, 4 $\mathcal{L}$-classes and 4 idempotents $\sigma(b^3ab^3a)$, $\sigma(b^2ab^3ab)$, $\sigma(bab^3ab^2)$, $\sigma(ab^3ab^3)$. (The first group of non-regular elements listed above is $\mathcal{J}$-above both these $\mathcal{D}$-classes, and the second group, the images of words having b^3 as a factor, sits above the second $\mathcal{D}$-class only.) All the other elements have rank at most 1. Consisting of such elements, there are 2 incomparable regular $\mathcal{D}$-classes, one with 10 $\mathcal{R}$-classes, 10 $\mathcal{L}$-classes and 32 idempotents, and the other with 13 $\mathcal{R}$-classes, 13 $\mathcal{L}$-classes and 44 idempotents. Below both these classes, there is a non-regular $\mathcal{D}$-class with 10 $\mathcal{R}$-classes and 13 $\mathcal{L}$-classes; and finally the trivial $\mathcal{D}$-class 0.

Using this monoid, we show that $\widehat{\mathbf{R}} \neq \mathbf{B}_1 \circledM \mathbf{R}$. More precisely, we use the known fact that $\widehat{\mathbf{J}_1} = \widehat{\mathbf{R}}$, where $\mathbf{R}$ is the variety of $\mathcal{R}$-trivial monoids [33]. We verify that the monoid S lies in $\mathbf{B}_1 \circledM \mathbf{R}$ and not in $\mathbf{B}_1 \circledM \mathbf{J}_1$. Thus $\mathbf{B}_1 \circledM \mathbf{J}_1$ is strictly contained in $\mathbf{B}_1 \circledM \mathbf{R}$. Since $\widehat{\mathbf{V}} \subseteq \mathbf{B}_1 \circledM \mathbf{V}$ for every variety $\mathbf{V}$, it follows that

$$\widehat{\mathbf{R}} = \widehat{\mathbf{J}_1} \subseteq \mathbf{B}_1 \circledM \mathbf{J}_1 \subsetneq \mathbf{B}_1 \circledM \mathbf{R}$$

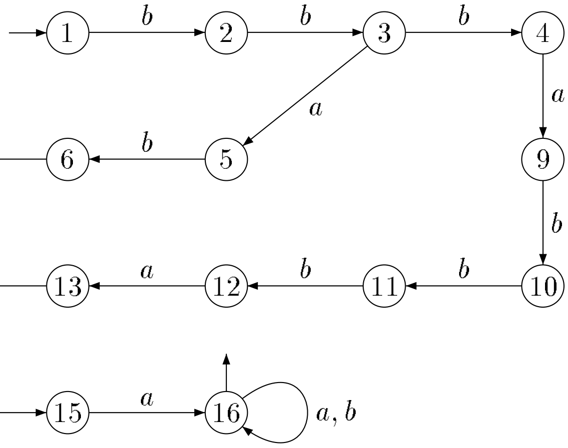
and thus $\widehat{\mathbf{R}} \neq \mathbf{B}_1 \circledM \mathbf{R}$.

In order to verify that $S \notin \mathbf{B}_1 \circledM \mathbf{J}_1$, it suffices to find words e, f, s, t, u, v in $\{a, b\}^*$ with the same alphabetic content, and such that the transitions labeled by e, f, $esfte$, $eufve$ (that is, their images by σ) are idempotent, and the transitions labeled $(esfte)(eufve)$ and $(esfte)esfve(eufve)$ are different (see [54, Proposition 1.3] or [35, Theorem 5.1]).

Let $e = ab^2ab^2$, $f = ab^3ab^3$, $s = t = ab$ and $u = v = ab^4$. It was remarked above that $\sigma(e)$ and $\sigma(f)$ are idempotent. Now $\sigma(esfte)$ is the rank 1 transition with domain $\{3, 6, 9, 12, 13\}$ and range $\{13\}$. Also $\sigma(eufve)$ is the rank 1 transition with domain $\{6, 13, 18, 91, 23, 26\}$ and range $\{26\}$. And $\sigma(esfve)$ is the empty transition. Thus $\sigma((esfte)(eufve)) \neq 0$ while $\sigma((esfte)esfve(eufve)) = 0$, and $S \notin \mathbf{B}_1 \circledM \mathbf{J}_1$.

We now prove that $S \in \mathbf{B}_1 \circledM \mathbf{R}$. Let $\rho \colon A^* \to R$ be the syntactic morphism of the language $(b^2a)^3A^* \cup (b^3a)^3A^*$, whose minimal automaton is the following:

[2] This computation was performed using Pin's SEMIGROUPE software [14].



It is immediately verified that $R \in \mathbf{R}$ and we consider the relational morphism $\tau = \rho \circ \sigma^{-1} \colon S \to R$. To show that $S \in \mathbf{B}_1 \textcircled{\scriptsize M} \mathbf{R}$, it suffices to show that if e, f, s, t, u, v are elements of A^* whose ρ-images are equal and idempotent in R and such that $\sigma(e)$ and $\sigma(f)$ are idempotent, then in S we have

$$\sigma((esfte)^\omega(eufve)^\omega) = \sigma((esfte)^\omega esfve(eufve)^\omega)$$

If $\sigma(esfte)^\omega = 0$ or $\sigma(eufve)^\omega = 0$, the above equality holds trivially, with both terms equal to 0. Let us now assume that $\sigma(esfte)^\omega \neq 0$ and $\sigma(eufve)^\omega \neq 0$.

If $\sigma(e)$ has rank 1, then $\sigma(e)$, $\sigma(esf)$, $\sigma(euf)$, $\sigma(fte)$, $\sigma(fve)$ are all $\mathcal{D}$-equivalent. Indeed, if it were not the case, we would have $\sigma(esfte) <_{\mathcal{J}} \sigma(e)$ or $\sigma(eufve) <_{\mathcal{J}} \sigma(e)$, and in particular, $\sigma(esfte)^\omega = 0$ or $\sigma(eufve)^\omega = 0$, a contradiction. It follows that $\sigma(esfte)$, $\sigma(eufve)$, $\sigma(esfve)$ and $\sigma(e)$ are $\mathcal{H}$-equivalent, and hence equal. The equality above is now immediate.

If $\sigma(e) = 1$, then $e = 1 \in A^*$, so $\rho(e) = 1$. But $\rho^{-1}(1) = \{1\}$, so we have $f = s = t = u = v = e = 1$, and the expected equality holds again in this case.

The remaining cases are those where

$$e \in F = \{b^2ab^2a, bab^2ab, ab^2ab^2, b^3ab^3a, b^2ab^3ab, bab^3ab^2, ab^3ab^3\}.$$

Note that $\sigma^{-1}(\sigma(b^2ab^2a)) = (b^2ab^2a)^+$, and $\tau(\sigma(b^2ab^2a)) = \rho((b^2ab^2a)^+)$ contains a single idempotent, equal to $\rho(b^2ab^2a)^2$. The same is true of the other idempotents of rank 2 in S: if $x \in F$, then $\tau(\sigma(x))$ contains a single idempotent, namely $\rho(x^2)$. Finally one can verify that these seven idempotents of R are pairwise distinct, for instance by checking that the corresponding transitions have distinct domains.

Thus, if $\sigma(e) \in F$, $\sigma(f)$ is idempotent and $\rho(e) = \rho(f)$, then we must have $\sigma(e) = \sigma(f)$. Now it suffices to check that each of the seven local submonoids of S of the form gSg ($g \in F$) are $\mathcal{J}$-trivial (since a monoid is in $\mathbf{B}_1$ if and only if it is $\mathcal{J}$-trivial), which is readily verified.

Thus we have proved that $S \in \mathbf{B}_1 \textcircled{m} \mathbf{R}$ and $S \notin \mathbf{B}_1 \textcircled{m} \mathbf{J}_1$. $\square$

The reader may wonder how an appropriate $\mathcal{R}$-trivial monoid was found, to show that a specific 445-element monoid sits in $\mathbf{B}_1 \textcircled{m} \mathbf{R}$. In fact the authors first used general results on free profinite objects on Mal'cev products [35] and structure theorems on free pro-$\mathcal{R}$ monoids [2, 52] to ascertain that $S \in \mathbf{B}_1 \textcircled{m} \mathbf{R}$, and then to exhibit a specific $\mathbf{B}_1$-relational morphism into an $\mathcal{R}$-trivial monoid.

5 Properties of the concatenation product

For $1 \leq i \leq n$, let L_i be a recognizable language of A^+, S_i be its syntactic semigroup and $\eta_i : A^+ \to S_i$ be its syntactic morphism. We also denote by S_i^+ its ordered syntactic semigroup. Let

$$\eta : A^+ \to S_1^+ \times S_2^+ \times \cdots \times S_n^+$$

be the morphism of ordered semigroups defined by

$$\eta(u) = (\eta_1(u), \eta_2(u), \ldots, \eta_n(u))$$

Let u_0, u_1, ..., u_n be words of A^*, let $L = u_0 L_1 u_1 \cdots L_n u_n$. Let S (resp. S^+) be the (resp. ordered) syntactic semigroup of L and let $\mu : A^+ \to S^+$ be its syntactic morphism. The following property of the relational morphism

$$\tau_+ = \eta \circ \mu^{-1} : S^+ \to S_1^+ \times S_2^+ \times \cdots \times S_n^+$$

was established in [32], as an improvement over similar former results [44, 28, 36].

Proposition 5.1 *The relational morphism $\tau_+ : S^+ \to S_1^+ \times S_2^+ \times \cdots \times S_n^+$ is a $\mathbf{LJ}^+$-relational morphism.*

In order to turn this result into a property of the relational morphism

$$\tau = \eta \circ \mu^{-1} : S \to S_1 \times S_2 \times \cdots \times S_n$$

we need an auxiliary result, which is interesting in its own right.

Lemma 5.2 *Let $\tau_1 : S_1 \to T_1$ and $\tau_2 : S_2 \to T_2$ be relational morphisms between ordered semigroups, and let $\tau : S_1 \times S_2 \to T_1 \times T_2$ be the relational morphism given by $\tau(s_1, s_2) = \tau_1(s_1) \times \tau_2(s_2)$. Let $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{W}_1$, $\mathbf{W}_2$ be varieties of ordered semigroups. If τ_1 is a $(\mathbf{V}_1, \mathbf{W}_1)$ relational morphism and τ_2 is a $(\mathbf{V}_2, \mathbf{W}_2)$ relational morphism, then τ is a $(\mathbf{V}_1 \vee \mathbf{V}_2, \mathbf{W}_1 \cap \mathbf{W}_2)$ relational morphism.*

Proof. Let $\pi_i : T_1 \times T_2 \to T_i$ be the projection $(i = 1, 2)$, and let W be a subsemigroup of $T_1 \times T_2$. Then $\tau^{-1}(W)$ is a subsemigroup of $\tau_1^{-1}(\pi_1(W)) \times \tau_2^{-1}(\pi_2(W))$. Now suppose that $W \in \mathbf{W}_1 \cap \mathbf{W}_2$. In particular, $W \in \mathbf{W}_1$, so $\pi_1(W) \in \mathbf{W}_1$ and $\tau_1^{-1}(\pi_1(W)) \in \mathbf{V}_1$. Similarly, $\tau_2^{-1}(\pi_2(W)) \in \mathbf{V}_2$, so that $\tau^{-1}(W) \in \mathbf{V}_1 \vee \mathbf{V}_2$. □

Proposition 5.3 *For each variety of semigroups* $\mathbf{V}$*, the relational morphism* $\tau : S \to S_1 \times S_2 \times \cdots \times S_n$ *is a* $(\widehat{\mathbf{V}}, \mathbf{LI} \circledM \mathbf{V})$ *relational morphism.*

Proof. Let $\tau_+ : S^+ \to S_1^+ \times \cdots \times S_n^+$ be the canonical relational morphism. By Proposition 5.1, τ^+ is a $\mathbf{LJ}^+$-relational morphism, and hence by Proposition 2.6, it is also a $(\mathbf{LJ}^+ \circledM \mathbf{V})$-relational morphism.

Let $\tau_- : S^- \to S_1^- \times \cdots \times S_n^-$ be the relational morphism given by $\tau_-(s) = \tau_+(s)$ for each $s \in S$. It is an elementary verification that τ_- is a $\mathbf{LJ}^-$-relational morphism, and hence τ_- is also a $(\mathbf{LJ}^- \circledM \mathbf{V})$-relational morphism.

Now, Formula 3 and Lemma 5.2 show that the product relational morphism

$$\tau_+ \times \tau_- : S^+ \times S^- \longrightarrow S_1^+ \times \cdots S_n^+ \times S_1^- \times \cdots \times S_n^-$$

is a $(\widehat{\mathbf{V}}, \mathbf{LI} \circledM \mathbf{V})$ relational morphism. Let us identify S (resp. $S_1 \times \cdots \times S_n$) with the diagonal of $S^+ \times S^-$ (resp. $(S_1^+ \times \cdots \times S_n^+) \times (S_1^- \times \cdots \times S_n^-)$. Up to this identification, $\tau(s) \subseteq \tau_+ \times \tau_-(s)$ for every $s \in S$. It follows by Lemma 2.5 that τ is a $(\widehat{\mathbf{V}}, \mathbf{LI} \circledM \mathbf{V})$ relational morphism. □

There is a similar result for ordered syntactic monoids. Let, for $0 \leq i \leq n$, L_i be recognizable languages of A^*, M_i^+ be its ordered syntactic monoid and $\eta_i : A^* \to M_i^+$ be its syntactic morphism. Let

$$\eta : A^* \to M_0^+ \times M_1^+ \times \cdots \times M_n^+$$

be the morphism defined by

$$\eta(u) = (\eta_0(u), \eta_1(u), \ldots, \eta_n(u))$$

Let $a_1, a_2, \ldots, a_n$ be letters of A, let $L = L_0 a_1 L_1 \cdots a_n L_n$ and let $\mu : A^* \to M^+$ be the syntactic morphism of L. Finally, consider the relational morphism

$$\tau_+ = \mu^{-1}\eta : M^+ \to M_0^+ \times M_1^+ \times \cdots \times M_n^+$$

Proposition 5.4 *The relational morphism* $\tau : M^+ \to M_1^+ \times M_2^+ \times \cdots \times M_n^+$ *is a* $\mathbf{LJ}^+$*-relational morphism. For each variety of semigroups* $\mathbf{V}$*, the relational morphism* $\tau : M \to M_1 \times M_2 \times \cdots \times M_n$ *is a* $(\widehat{\mathbf{V}}, \mathbf{LI} \circledM \mathbf{V})$ *relational morphism.*

6 A new conjecture and its consequences

In Section 4, we discussed counterexamples to the conjecture $\bigcap_{\mathbf{V}} = \mathbf{B}_1 Ⓜ \mathbf{V}$. Taking these counterexamples into account, we formulate a new conjecture and discuss its consequences.

Conjecture 6.1 For any variety of semigroups (resp. monoids) $\mathbf{W}$,

$$\widehat{\mathbf{W}} = \bigcap_{\mathbf{H}} (\mathbf{J} * \mathbf{LI} * \mathbf{H}, \mathbf{LI} * \mathbf{H}) Ⓜ \mathbf{W}$$

where the intersection is taken over all varieties of groups.

If all semigroups (resp. monoids) in $\mathbf{W}$ are aperiodic, Conjecture 6.1 simplifies to:

Conjecture 6.2 For any variety of aperiodic semigroups (resp. monoids) $\mathbf{W}$,

$$\widehat{\mathbf{W}} = (\mathbf{J} * \mathbf{LI}, \mathbf{LI}) Ⓜ \mathbf{W}.$$

Let us prove formally the logical connection between the two conjectures.

Proposition 6.1 *For any variety of aperiodic semigroups (resp. monoids)* $\mathbf{W}$*,*

$$\bigcap_{\mathbf{H}} (\mathbf{J} * \mathbf{LI} * \mathbf{H}, \mathbf{LI} * \mathbf{H}) Ⓜ \mathbf{W} = (\mathbf{J} * \mathbf{LI}, \mathbf{LI}) Ⓜ \mathbf{W}$$

Proof. The proof relies on the fact that an aperiodic semigroup in $\mathbf{LI} * \mathbf{H}$ is actually in $\mathbf{LI}$. Let us call $\mathbf{V}_1$ (resp. $\mathbf{V}_2$) the variety on the left (resp. right) of the formula. The inclusion $\mathbf{V}_1 \subseteq \mathbf{V}_2$ is clear. If $S \in \mathbf{V}_2$, there exists a $(\mathbf{J} * \mathbf{LI}, \mathbf{LI})$ relational morphism τ from S into some $W \in \mathbf{W}$. Let T be a subsemigroup of W and let $\mathbf{H}$ be a variety of groups. If $T \in \mathbf{LI} * \mathbf{H}$, then $T \in \mathbf{LI}$, since W is aperiodic. Therefore $\tau^{-1}(T) \in \mathbf{J} * \mathbf{LI}$ and hence $\tau^{-1}(T) \in \mathbf{J} * \mathbf{LI} * \mathbf{H}$. Thus τ is actually a $(\mathbf{J} * \mathbf{LI} * \mathbf{H}, \mathbf{LI} * \mathbf{H})$ relational morphism, and consequently $S \in \mathbf{V}_1$. Thus $\mathbf{V}_1 = \mathbf{V}_2$. □

Corollary 6.2 *Conjecture 6.1 implies Conjecture 6.2.*

With the same proof, one can show that for each variety $\mathbf{W}$,

$$\bigcap_{\mathbf{H}} (\mathbf{J} * \mathbf{LI} * \mathbf{H}, \mathbf{LI} * \mathbf{H}) Ⓜ \mathbf{W} = \bigcap_{\mathbf{H} \subseteq \mathbf{G} \cap \mathbf{W}} (\mathbf{J} * \mathbf{LI} * \mathbf{H}, \mathbf{LI} * \mathbf{H}) Ⓜ \mathbf{W}.$$

Our conjecture originates in the following general result:

Theorem 6.3 *For any variety of monoids (resp. semigroups)* $\mathbf{W}$*,*

$$\widehat{\mathbf{W}} = \bigcap\, (\hat{\mathbf{V}}, \mathbf{LI} \circledM \mathbf{V}) \circledM \mathbf{W} \tag{4}$$

where the intersection is taken over all varieties of semigroups $\mathbf{V}$*.*

Proof. Let us prove the result in the monoid case (the semigroup case is analogous). We consider a generator of $\widehat{\mathbf{W}}$, in the form of the syntactic monoid M of a language $L = L_0 a_1 L_1 \cdots a_n L_n$, where for each i, a_i is a letter and L_i is a language whose syntactic monoid M_i lies in $\mathbf{W}$. By Proposition 5.3, the relational morphism $\tau : M \to M_1 \times M_2 \times \cdots \times M_n$ is a $(\hat{\mathbf{V}}, \mathbf{LI} \circledM \mathbf{V})$ relational morphism for each variety of semigroups $\mathbf{V}$. This proves the inclusion

$$\widehat{\mathbf{W}} \subseteq \bigcap\, (\hat{\mathbf{V}}, \mathbf{LI} \circledM \mathbf{V}) \circledM \mathbf{W}$$

To establish the opposite inclusion, we observe that if $\mathbf{W}$ is a variety of semigroups, then $(\widehat{\mathbf{W}}, \mathbf{LI} \circledM \mathbf{W}) \circledM \mathbf{W}$ is contained in $\widehat{\mathbf{W}}$. If $\mathbf{W}$ is a variety of monoids, let us take $\mathbf{V} = \mathbf{W} * \mathbf{LI}$. Then, since $\mathbf{W_S} \subseteq \mathbf{LI} \circledM \mathbf{V}$, we have

$$(\hat{\mathbf{V}}, \mathbf{LI} \circledM \mathbf{V}) \circledM \mathbf{W} \subseteq \hat{\mathbf{V}}_{\mathbf{M}}.$$

Furthermore, by the results of [45, 37, 32], $\hat{\mathbf{V}} = \widehat{\mathbf{W}} * \mathbf{LI}$. Now, it follows from [13, Proposition 5.6, page 131, and page 155] that $\widehat{\mathbf{W}} * \mathbf{LI} \subseteq \mathbf{L}\widehat{\mathbf{W}}$ and thus $(\widehat{\mathbf{W}} * \mathbf{LI})_{\mathbf{M}} = \widehat{\mathbf{W}}$ and $(\hat{\mathbf{V}}, \mathbf{LI} \circledM \mathbf{V}) \circledM \mathbf{W} \subseteq \widehat{\mathbf{W}}$. □

A closer study of the varieties of the form $\mathbf{LI} \circledM \mathbf{V}$ and $\hat{\mathbf{V}}$ is in order to establish the links between Formula (4) and our conjectures. We shall consider successively varieties contained in $\mathbf{LI}$ and varieties of the form $\mathbf{LI} * \mathbf{H}$, $\mathbf{H}$ or $\mathbf{H_S}$, where $\mathbf{H}$ is a variety of groups.

Proposition 6.4 *Let* $\mathbf{V}$ *be a variety of semigroups. If* $\mathbf{V}$ *is contained in* $\mathbf{LI}$*, then* $\mathbf{LI} \circledM \mathbf{V} = \mathbf{LI}$ *and* $\hat{\mathbf{V}} = \mathbf{B}_1$*.*

Proof. Let $\mathcal{V}$ and $\mathcal{LI}$ be the varieties of languages corresponding to $\mathbf{V}$ and $\mathbf{LI}$, respectively. If $\mathbf{V} \subseteq \mathbf{LI}$, then $\mathbf{LI} \subseteq \mathbf{LI} \circledM \mathbf{V} \subseteq \mathbf{LI} \circledM \mathbf{LI} = \mathbf{LI}$. It follows, by Theorem 3.2, that UPol $\mathcal{V} = \mathcal{LI}$, and thus

$$\text{Pol } \mathcal{V} = \text{Pol } (\text{UPol } \mathcal{V}) = \text{Pol } \mathcal{LI}$$

and therefore BPol $\mathcal{V}$ = BPol $\mathcal{LI}$. This proves the equality of the corresponding varieties of semigroups, namely $\hat{\mathbf{V}}$ for BPol $\mathcal{V}$ and $\mathbf{B}_1$ for BPol $\mathcal{LI}$. □

The next result, which follows immediately from Theorem 6.3 and Proposition 6.4, can be considered as the easy part of Conjecture 6.2.

Corollary 6.5 *For any variety of aperiodic semigroups (resp. monoids)* $\mathbf{W}$*,*

$$\widehat{\mathbf{W}} \subseteq (\mathbf{J} * \mathbf{LI}, \mathbf{LI}) \circledM \mathbf{W}.$$

Corollary 6.6 *Conjecture 6.2 is true for any variety contained in* $\mathbf{LI}$*.*

Proof. Let $\mathbf{V}$ be a variety contained in $\mathbf{LI}$. Proposition 6.4 shows that $\widehat{\mathbf{V}} = \mathbf{B}_1$ and by Corollary 6.5, $\mathbf{B}_1 \subseteq (\mathbf{J} * \mathbf{LI}, \mathbf{LI}) \circledM \mathbf{V}$. Finally $(\mathbf{J} * \mathbf{LI}, \mathbf{LI}) \circledM \mathbf{V} \subseteq (\mathbf{J} * \mathbf{LI}, \mathbf{LI}) \circledM \mathbf{LI} \subseteq \mathbf{J} * \mathbf{LI} = \mathbf{B}_1$. Thus $\widehat{\mathbf{V}} = (\mathbf{J} * \mathbf{LI}, \mathbf{LI}) \circledM \mathbf{V}$. $\square$

Next we consider varieties of the form $\mathbf{V} * \mathbf{H}$, where $\mathbf{V}$ is a subvariety of $\mathbf{LI}$ and $\mathbf{H}$ is a variety of groups.

Proposition 6.7 *Let* $\mathbf{V}$ *be a variety such that* $(\mathbf{H})_{\mathbf{S}} \subseteq \mathbf{V} \subseteq \mathbf{LI} * \mathbf{H}$*. Then* $\widehat{\mathbf{W}} = \mathbf{J} * \mathbf{LI} * \mathbf{H}$*.*

Proof. By Corollary 3.3, $\widehat{\mathbf{LI} \circledM \mathbf{V}} = \widehat{\mathbf{V}}$. Now, since $(\mathbf{H})_{\mathbf{S}} \subseteq \mathbf{V} * \mathbf{H} \subseteq \mathbf{LI} * \mathbf{H}$, Proposition 2.8 gives $\mathbf{LI} \circledM \mathbf{H}_{\mathbf{S}} = \mathbf{LI} * \mathbf{H} = \mathbf{LI} \circledM (\mathbf{LI} * \mathbf{H})$ and thus $\mathbf{LI} \circledM \mathbf{V} = \mathbf{LI} * \mathbf{H}$. Therefore we may assume without loss of generality that $\mathbf{V} = \mathbf{LI} \circledM \mathbf{V}$.

Let $\mathcal{V}$ be the variety of languages corresponding to $\mathbf{V}$. Then the variety of languages corresponding to $\widehat{\mathbf{V}}$ is BPol $\mathcal{V}$ by definition. Proposition 3.1 shows that $\mathcal{V}$ is closed under the operations $L \to uL$ and $L \to Lu$ (where u is a word). Therefore by [32, Theorem 5.3], $\widehat{\mathbf{V}} = \Diamond\mathbf{V}$. It follows by [32, Theorem 4.8] that $\Diamond\mathbf{V} = (\Diamond\mathbf{LI}) * \mathbf{H}$ and since $\Diamond\mathbf{LI} = \mathbf{B}_1 = \mathbf{J} * \mathbf{LI}$, we finally obtain $\widehat{\mathbf{V}} = \mathbf{J} * \mathbf{LI} * \mathbf{H}$. $\square$

The "easy part" of Conjecture 6.1 now follows from Theorem 6.3 and Proposition 6.7.

Corollary 6.8 *For any variety of semigroups (resp. monoids)* $\mathbf{W}$*,*

$$\widehat{\mathbf{W}} \subseteq \bigcap_{\mathbf{H}} (\mathbf{J} * \mathbf{LI} * \mathbf{H}, \mathbf{LI} * \mathbf{H}) \circledM \mathbf{W}$$

where the intersection is taken over all varieties of groups.

Corollary 6.9 *Conjecture 6.1 is true for any variety of semigroups* $\mathbf{V}$ *such that* $(\mathbf{H})_{\mathbf{S}} \subseteq \mathbf{V} \subseteq \mathbf{LI} * \mathbf{H}$*, where* $\mathbf{H}$ *is a variety of groups.*

Proof. Proposition 6.7 shows that $\widehat{\mathbf{V}} = \mathbf{J} * \mathbf{LI} * \mathbf{H}$. Furthermore

$$(\mathbf{J} * \mathbf{LI} * \mathbf{H}, \mathbf{LI} * \mathbf{H}) \circledM \mathbf{V} \subseteq \mathbf{J} * \mathbf{LI} * \mathbf{H} = \widehat{\mathbf{V}}$$

The conjecture now follows from Corollary 6.8. $\square$

Next, let us consider varieties of groups.

Proposition 6.10 *Conjecture 6.1 is true for any variety of groups.*

Proof. If $\mathbf{H}$ is a variety of groups, then $\widehat{\mathbf{H}} = \mathbf{J} * \mathbf{H}$ [41, 32]. Furthermore

$$(\mathbf{J} * \mathbf{LI} * \mathbf{H}, \mathbf{LI} * \mathbf{H}) \ⓜ \mathbf{H} \subseteq (\mathbf{J} * \mathbf{LI} * \mathbf{H})_{\mathbf{M}} \subseteq (\mathbf{J} * \mathbf{H} * \mathbf{LI})_{\mathbf{M}}$$

Now, for any variety of monoids $\mathbf{V}$, $\mathbf{V} * \mathbf{LI} \subseteq \mathbf{LV}$ and thus $(\mathbf{V} * \mathbf{LI})_{\mathbf{M}} = \mathbf{V}$. It follows in particular that $(\mathbf{J} * \mathbf{LI} * \mathbf{H}, \mathbf{LI} * \mathbf{H}) ⓜ \mathbf{H} \subseteq \mathbf{J} * \mathbf{H}$. The conjecture now follows from Corollary 6.8. $\square$

Finally, let us consider the variety $\mathbf{LG} = \mathbf{G} * \mathbf{LI}$.

Proposition 6.11 *Conjecture 6.1 is true for the variety* $\mathbf{LG}$.

Proof. The reader is referred to [20] for the definition of a kernel of a monoid, and its connection with the Mal'cev product. It is shown in [25] that $\widehat{\mathbf{LG}} = \mathbf{LBG}$, where $\mathbf{J} * \mathbf{G} = \mathbf{BG}$ is the variety of block groups. We claim that

$$(\mathbf{J} * \mathbf{LI} * \mathbf{G}, \mathbf{LI} * \mathbf{G}) ⓜ \mathbf{LG} \subseteq \mathbf{LBG}$$

Let $S \in (\mathbf{J} * \mathbf{LI} * \mathbf{G}, \mathbf{LI} * \mathbf{G}) ⓜ \mathbf{LG}$. Then there exists a semigroup $T \in \mathbf{LG}$ and a $(\mathbf{J} * \mathbf{LI} * \mathbf{G}, \mathbf{LI} * \mathbf{G})$ relational morphism $\tau : S \to T$. Let e be an idempotent of S, f an idempotent of the semigroup $\tau(e)$ and let G be the maximal subgroup of T containing f. If $s \in S$ and $t \in \tau(s)$, then $ftf \in G$ since $T \in \mathbf{LG}$. Thus $ftf \in \tau(e)\tau(s)\tau(e) \subseteq \tau(ese)$. It follows that $G \cap \tau(ese) \neq \emptyset$, and thus $eSe \subseteq \tau^{-1}(G)$. Now since τ is $(\mathbf{J} * \mathbf{LI} * \mathbf{G}, \mathbf{LI} * \mathbf{G})$, it follows that $\tau^{-1}(G) \in \mathbf{J} * \mathbf{LI} * \mathbf{G}$, and thus $eSe \in \mathbf{J} * \mathbf{LI} * \mathbf{G}$. Now by [20, Theorem 3.1], $\mathbf{J} * \mathbf{LI} * \mathbf{G} \subseteq (\mathbf{J} * \mathbf{LI}) ⓜ \mathbf{G} \subseteq \mathbf{LJ} ⓜ \mathbf{G}$. It follows, by [20, Theorem 3.4], that $K(eSe) \in \mathbf{LJ}$. But eSe is a monoid and so is $K(eSe)$. Thus $K(eSe) \in \mathbf{J}$, and by [20, Theorem 3.4] again, $eSe \in \mathbf{J} * \mathbf{G} = \mathbf{BG}$. Thus $S \in \mathbf{LBG}$, proving the claim. The conjecture now follows from Corollary 6.8. $\square$

Let us give a last result supporting Conjecture 6.1. Corollary 3.3 shows that, for any variety of monoids (resp. semigroups) $\mathbf{V}$, $\widehat{\mathbf{LI} ⓜ \mathbf{V}} = \widehat{\mathbf{V}}$. The following proposition shows that if Conjecture 6.1 is true for $\mathbf{V}$, then it is also true for $\mathbf{LI} ⓜ \mathbf{V}$.

Proposition 6.12 *For any variety of groups* $\mathbf{H}$ *and any variety of semigroups (resp. monoids)* $\mathbf{V}$, $(\mathbf{J} * \mathbf{LI} * \mathbf{H}, \mathbf{LI} * \mathbf{H}) ⓜ \mathbf{V} = (\mathbf{J} * \mathbf{LI} * \mathbf{H}, \mathbf{LI} * \mathbf{H}) ⓜ (\mathbf{LI} ⓜ \mathbf{V})$.

Proof. The proof is given in the case where $\mathbf{V}$ is a variety of monoids, but the semigroup case is entirely similar. Let M be a monoid in $(\mathbf{J} * \mathbf{LI} * \mathbf{H}, \mathbf{LI} * \mathbf{H}) ⓜ (\mathbf{LI} ⓜ \mathbf{V})$. By definition, there exists a $(\mathbf{J} * \mathbf{LI} * \mathbf{H}, \mathbf{LI} * \mathbf{H})$ relational morphism $\tau : M \to N$, where $N \in \mathbf{LI} ⓜ \mathbf{V}$ and a $\mathbf{LI}$-relational morphism $\sigma : N \to V$ with $V \in \mathbf{V}$.

Let S be a subsemigroup of V in $\mathbf{LI}*\mathbf{H}$. Then $\sigma^{-1}(V)$ is in $\mathbf{LI} \textcircled{m} (\mathbf{LI}*\mathbf{H})$, and by Proposition 2.8, this variety is equal to $\mathbf{LI}*\mathbf{H}$. Therefore $(\sigma\circ\tau)^{-1}(V)$ is in $\mathbf{J}*\mathbf{LI}*\mathbf{H}$, showing that $\sigma\circ\tau : M \to V$ is a $(\mathbf{J}*\mathbf{LI}*\mathbf{H}, \mathbf{LI}*\mathbf{H})$ relational morphism. Thus $(\mathbf{J}*\mathbf{LI}*\mathbf{H}, \mathbf{LI}*\mathbf{H}) \textcircled{m} (\mathbf{LI} \textcircled{m} \mathbf{V}) \subseteq (\mathbf{J}*\mathbf{LI}*\mathbf{H}, \mathbf{LI}*\mathbf{H}) \textcircled{m} \mathbf{V}$. The opposite inclusion is obvious. $\square$

In particular, $(\mathbf{B}_1, \mathbf{LI}) \textcircled{m} \mathbf{J}_1 = (\mathbf{B}_1, \mathbf{LI}) \textcircled{m} (\mathbf{LI} \textcircled{m} \mathbf{J}_1) = (\mathbf{B}_1, \mathbf{LI}) \textcircled{m} \mathbf{DA}$ and thus

$$\mathbf{B}_1 \textcircled{m} \mathbf{J}_1 = (\mathbf{B}_1, \mathbf{LI}) \textcircled{m} \mathbf{J}_1 = (\mathbf{B}_1, \mathbf{LI}) \textcircled{m} \mathbf{R} = (\mathbf{B}_1, \mathbf{LI}) \textcircled{m} \mathbf{DA}$$

The counterexample given in Section 4 shows that, although $\widehat{\mathbf{J}_1} = \widehat{\mathbf{R}}$, the inclusion $\mathbf{B}_1 \textcircled{m} \mathbf{J}_1 \subseteq \mathbf{B}_1 \textcircled{m} \mathbf{R}$ is strict, ruining the conjecture $\widehat{\mathbf{V}} = \mathbf{B}_1 \textcircled{m} \mathbf{V}$. It is no longer a counterexample to Conjecture 6.2, since $(\mathbf{B}_1, \mathbf{LI}) \textcircled{m} \mathbf{J}_1 = (\mathbf{B}_1, \mathbf{LI}) \textcircled{m} \mathbf{R} = (\mathbf{B}_1, \mathbf{LI}) \textcircled{m} \mathbf{DA}$ (since $\mathbf{DA} = \mathbf{LI} \textcircled{m} \mathbf{J}_1$).

Conjecture 6.1 relies on the hope that the inclusion proved in Corollary 6.8 is actually an equality. Besides, Formula (4) shows that if the conjecture is true, then the inclusion

$$\bigcap_{\mathbf{H}} (\mathbf{J}*\mathbf{LI}*\mathbf{H}, \mathbf{LI}*\mathbf{H}) \textcircled{m} \mathbf{W} \subseteq (\widehat{\mathbf{V}}, \mathbf{LI} \textcircled{m} \mathbf{V}) \textcircled{m} \mathbf{W} \qquad (5)$$

should hold for each variety $\mathbf{V}$ and $\mathbf{W}$. It would be interesting to prove or disprove this inclusion for particular instances of $\mathbf{V}$ and $\mathbf{W}$, for instance for $\mathbf{V} = \mathbf{LH}$. Disproving one of these inclusions would of course disprove Conjecture 6.1.

7 Concatenation hierarchies

By alternating the use of the polynomial closure and of the boolean closure one can obtain hierarchies of recognizable languages. Let $\mathcal{W}$ be a variety of languages. The concatenation hierarchy of basis $\mathcal{W}$ is the hierarchy of classes of languages defined as follows.

(1) $\mathcal{W}_0 = \mathcal{W}$

(2) for every integer $n \geq 0$, $\mathcal{W}_{n+1/2} = \text{Pol } \mathcal{W}_n$,

(3) for every integer $n \geq 0$, $\mathcal{W}_{n+1} = \text{BPol } \mathcal{W}_n$.

Levels of the form $\mathcal{W}_n$ (resp. $\mathcal{W}_{n+1/2}$) for some integer n are called *full* (resp. *half*) levels. It is known that every full level is a variety of languages and every half level is a positive variety of languages.

Three concatenation hierarchies have been considered so far in the literature. The first one, introduced by Cohen and Brzozowski [11, 7] and called the *dot-depth hierarchy*, is the hierarchy of positive $+$-varieties whose basis is the trivial variety. The second one, first considered implicitly in [48] and explicitly in [43, 45] is called the *Straubing-Thérien hierarchy*: it is the hierarchy of positive $*$-varieties whose basis is the trivial variety. The third one, introduced in [24], is the hierarchy of positive $*$-varieties whose basis is the variety of group-languages. It is called the *group hierarchy.* The hierarchies of basis $\mathbf{H}$ and $\mathbf{LH}$, where $\mathbf{H}$ is a variety of groups, are also worth studying.

The variety of monoids (or ordered monoids for the half levels) corresponding to the n-th level of the Straubing-Thérien hierarchy (resp. of the group hierarchy) is denoted by $\mathbf{V}_n$ (resp. $\mathbf{G}_n$). The variety of (ordered) semigroups corresponding to the n-th level of the dot-depth hierarchy is denoted by $\mathbf{B}_n$. These varieties are related by the following formulas [45, 37, 31, 32], which hold for each level (full or half):

$$\mathbf{B}_n = \mathbf{V}_n * \mathbf{LI} \qquad \mathbf{G}_n = \mathbf{V}_n * \mathbf{G}$$

It follows, but this is a non trivial result [45, 37], that $\mathbf{B}_n$ is decidable if and only if $\mathbf{V}_n$ is decidable (for each integer or half integer n). In particular, the identities defining the variety $\mathbf{B}_1$ were given in Section 2.1. It is not yet known whether a similar result holds for $\mathbf{G}_n$, but nevertheless the formula $\mathbf{G}_n = \mathbf{V}_n * \mathbf{G}$ gives evidence that the Straubing-Thérien hierarchy should be simpler to study.

The first level has been actually thoroughly studied by Simon [40] long before the hierarchy was formally introduced. Simon's result states that $\mathbf{V}_1$ is the variety $\mathbf{J}$ of $\mathcal{J}$-trivial monoids. It follows that $\mathcal{V}_1$ is decidable. The decidability of level 1/2 is quite easy, and was formally established in [29]. The decidability of level 3/2 was first established by Arfi [3, 4] and a better algorithm was given by the authors in [36]. A promising approach based on forbidden patterns, was also considered by Glaßer and Schmitz [15, 16, 17].

The decidability problem for level 2 is still open, although much progress has been made in recent years [5, 6, 12, 33, 46, 47, 53, 54]. The more important of these partial results is Straubing's proof [46] of the decidability of the level 2 for languages on a two-letter alphabet. This result was recently extended to the level 5/2 by Glaßer and Schmitz [18]. One of the main results about $\mathbf{V}_2$, proved in [33], is that $\mathbf{V}_2 = \widehat{\mathbf{J}}_1$, where $\mathbf{J}_1$ denotes the variety of idempotent and commutative monoids. Conjecture 6.2, applied to $\mathbf{W} = \mathbf{J}_1$ states that

$$\mathbf{V}_2 = (\mathbf{J} * \mathbf{LI}, \mathbf{LI}) \circledM \mathbf{J}_1.$$

Now, since $\mathbf{LI} \cap \mathbf{J}_1 = \mathbf{I_S}$, the following equality holds

$$(\mathbf{J} * \mathbf{LI}, \mathbf{LI}) \circledM \mathbf{J}_1 = (\mathbf{J} * \mathbf{LI}, \mathbf{I_S}) \circledM \mathbf{J}_1 = \mathbf{B}_1 \circledM \mathbf{J}_1$$

This means that, for the special case $\mathbf{V} = \mathbf{J}_1$, there is no difference between Conjecture 6.2 and the conjecture made in [36]. In particular, our guess of the identities of $\mathbf{V}_2$, which would prove the decidability of $\mathbf{V}_2$, is unchanged.

What would be the consequences of our conjecture for the upper levels of the hierarchy? First, since for every integer n, $\widehat{\mathbf{V}}_n = \mathbf{V}_{n+1}$, Conjecture 6.2 would imply that

$$\mathbf{V}_{n+1} = (\mathbf{J} * \mathbf{LI}, \mathbf{LI}) \circledM \mathbf{V}_n.$$

To obtain the identities of $\mathbf{V}_{n+1}$, one could then use the following result, whose proof will be published elsewhere.

Proposition 7.1 *Let* $\mathbf{V}$ *be a variety of semigroups (resp. monoids). Then* $(\mathbf{J} * \mathbf{LI}, \mathbf{LI}) \,Ⓜ\, \mathbf{V}$ *is defined by the identities of the form*

$$(esfte)^\omega esfve(eufve)^\omega = (esfte)^\omega (eufve)^\omega$$

for all e, f, s, t, u, v *such that* $\mathbf{V}$ *satisfies*

$$\begin{array}{lll} e = e^\omega, & efe = e, & s = u = ef, \\ f = f^\omega, & fef = f, & t = v = fe \end{array}$$

However, the decidability of each variety $\mathbf{V}_n$ would not follow immediately from such a description.

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