Semigroups with idempotent stabilizers and applications to automata theory.[1]

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Résumé

Nous prouvons que tout semigroupe fini est quotient d'un semigroupe fini dans lequel les stabilisateurs droits satisfont les identités $x = x^2$ et $xy = xyx$. Ce résultat a plusieurs conséquences. Tout d'abord, nous l'utilisons, en même temps qu'un résultat de I. Simon sur les congruences de chemins, pour obtenir une preuve purement algébrique d'un théorème profond de McNaughton sur les mots infinis. Puis, nous donnons une preuve algébrique d'un théorème de Brown sur des conditions de finitude pour les semigroupes.

Abstract

We show that every finite semigroup is a quotient of a finite semigroup in which every right stabilizer satisfies the identities $x = x^2$ and $xy = xyx$. This result has several consequences. We first use it together with a result of I. Simon on congruences on paths to obtain a purely algebraic proof of a deep theorem of McNaughton on infinite words. Next, we give an algebraic proof of a theorem of Brown on a finiteness condition for semigroups.

1 Introduction

Recall that the stabilizer of an element s of a semigroup S is the set of all t such that $st = s$. These stabilizers are themselves semigroups, and reflect rather well the structure of S: if S is a group, every stabilizer is trivial, but if S is arbitrary, certain stabilizers can be equal to S (especially if S has a zero !). The study of stabilizers occurs in many structure theorems for semigroups. This is the case for instance for the relation between Mal'cev product and wreath product, since stabilizers control the local structure of the derived category of a semigroup morphism [24]. Another typical example is the "Ideal Theorem" (as presented in [23]). This theorem states that if I is an ideal of a finite semigroup S, then

$$Sc \leq Ic + (S/I)c$$

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where Sc denotes the group complexity of S in the sense of Rhodes. This is relatively easy to prove if all the stabilizers of S are aperiodic, but, as mentioned above, this is far from the general case. In fact, the crucial point of the proof of the Ideal Theorem is the following result (see [23]): every semigroup is the quotient of a finite semigroup (of the same complexity) in which *the stabilizer of any element is aperiodic*. Actually, a slightly stronger result is proved in [22, 23]: every finite semigroup S is the quotient of a finite semigroup $\hat{S}$, called the Rhodes expansion, in which the stabilizer of any element is $\mathcal{R}$-trivial. Our main result gives a stronger version of this theorem:

Theorem 1.1 *Every finite semigroup S is a quotient of a finite semigroup $\hat{S}$ in which the stabilizer of any element is an idempotent and $\mathcal{R}$-trivial semigroup, that is, it satisfies the identities $x^2 = x$ and $xyx = xy$.*

In particular, this solves positively a conjecture of the first author [13]. Note that Theorem 1.1 is, in some sense, optimal. More precisely, for each identity $u = v$, one can ask whether it is possible to prove the following variant of Theorem 1.1: "every semigroup S is a quotient of a semigroup $\hat{S}$ in which the stabilizer of any element satisfies the identities $x^2 = x$ and $u = v$". We show that this is true if and only if $u = v$ is a consequence of the identities $x^2 = x$ and $xyx = xy$ (Proposition 4.5).

Returning to our comparison with Rhodes' result, it is fair to say that there is a price to pay to our strengthening of Rhodes theorem. Namely, in Rhodes' version, the natural morphism $\pi : \hat{S} \to S$ has the following property: for each idempotent $e \in S$, the subsemigroup $e\pi^{-1}$ of $\hat{S}$ is idempotent (in fact it satisfies the identity $xy = y$). In particular, it is aperiodic, and that is the reason why S and $\hat{S}$ have the same group complexity.

In our case, the morphism $\pi : \hat{S} \to S$ satisfies a weaker property: for each idempotent $e \in S$, the subsemigroup $e\pi^{-1}$ of $\hat{S}$ is the direct product of a commutative group by a rectangular band (a rectangular band is a simple type of idempotent semigroup, which satisfies $x^2 = x$ and $xyx = x$). In particular, $e\pi^{-1}$ is not aperiodic in general. By the way, one can show that it is impossible to require simultaneously that the stabilizers of $\hat{S}$ be idempotent and the semigroups $e\pi^{-1}$ be aperiodic (see Section 4).

Our result has several interesting applications. First it has a geometrical interpretation. A transformation semigroup is said to be fixpoint-free if every element which stabilizes a point is idempotent (for instance, translations of the plane form a fixpoint-free transformation semigroup). We show that every finite transformation semigroup is covered by a finite fixpoint-free transformation semigroup (the precise geometrical meaning of the word "cover" is explained in Section 4).

Another important application of our result (which was in fact the original motivation for this research) is a new proof of a fundamental theorem of McNaughton on infinite words. This theorem is one of the deepest results of the theory of finite automata. Roughly speaking, it states the equivalence of deterministic and nondeterministic automata to recognize sets of infinite words. One should emphasize that this result is much harder than the corresponding result for sets of finite words, and that all previous proofs of McNaughton's theorem involve complex constructions of automata. Our new proof relies strongly on the algebraic approach to automata theory, introduced by Schützenberger, Eilenberg and others, which consists essentially in converting combinatorial properties of automata into algebraic properties of semigroups. In particular, most of the technical aspects of our proof of McNaughton's theorem are crystallized in our Theorem 1.1. Interestingly, another algebraic ingredient of our proof is Simon's theorem on path congruences, which plays an important role in other branches of the theory of automata. Compared to others, our proof of McNaughton's theorem also requires a complex argument. But, assuming Theorem 1.1, it is a simpler proof. (In fact, a weaker version of Theorem 1.1 is sufficient for this construction, as we argue at the end of Section 6.) Moreover, the construction of a deterministic automaton starting from $\hat{S}$ is extremely simple.

Finally, we give a new proof of the following finiteness condition of Brown: let $\varphi\colon S \to T$ be a semigroup morphism, with T locally finite. If, for every idempotent e of T, the semigroup $e\varphi^{-1}$ is locally finite, then S itself is locally finite.

This paper is organized as follows. In Section 2 we review the basic definitions and properties of semigroups needed in this paper, including background on Rhodes expansion. Sections 3 and 4 contain the main results of the paper: Section 3 is devoted to the formal construction and Section 4 to its properties. This includes the proof of Theorem 1.1 and its geometrical interpretation. Two examples are presented in Section 5 and Section 6 is devoted to the new proof of McNaughton's theorem. For the convenience of the reader, this section also contains all the necessary definitions on automata. Section 7 contains our new proof of Brown's theorem, and our conclusions and suggestions for further investigations are presented in the last section.

2 Semigroup background

All semigroups considered in this paper are either finite or free, except in Section 7. In general, we use the postfixed notation for mappings and morphisms defined on semigroups. An exception to this rule is made for the $\mathcal{J}$-depth function defined below.

In this section, we will recall some basic definitions, in particular for what concerns Green's relations on a semigroup S, and the Rhodes expansion of S. For proofs and further details on Green's relations the reader is referred to [10, 11, 17]. Concerning the Rhodes expansion, see [22, 23, 18, 3].

If S is a semigroup, S^1 denotes the semigroup equal to S if S has an identity and to $S \cup \{1\}$ otherwise. In the latter case, the multiplication on S is extended by setting $s1 = 1s = s$ for every $s \in S^1$. An element s of S is said to be *idempotent* if $s^2 = s$. We denote by $E(S)$ the set of idempotents of S. The semigroup S itself is *idempotent* if each of its elements is idempotent. For instance, the semigroup $U_1 = \{0, 1\}$ under the usual multiplication of integers is an idempotent semigroup. Let s be an element of S. The *(right) stabilizer* of s in S is the subsemigroup $Stab(s)$ of S consisting of all the elements t of S such that $st = s$.

Green's relations on S are defined as follows. If s and t are elements of S, we say that

$$\begin{aligned}
s \leq_{\mathcal{L}} t \qquad & \text{if there exist } x \in S^1 \text{ such that } s = xt, \\
s \leq_{\mathcal{R}} t \qquad & \text{if there exist } y \in S^1 \text{ such that } s = ty \text{ and} \\
s \leq_{\mathcal{J}} t \qquad & \text{if there exist } x, y \in S^1 \text{ such that } s = xty.
\end{aligned}$$

These three relations are quasi-orders and the associated equivalence relations are denoted by $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$. For instance $\mathcal{L}$ is defined by $s \mathrel{\mathcal{L}} t$ if and only if $s = xt$ and $t = ys$ for some $x, y \in S^1$.

Example. A *rectangular band* is a semigroup of the form

$$B(n, m) = \{1, \ldots, n\} \times \{1, \ldots, m\}$$

$(n, m \geq 1)$, with $(i, j)(h, k) = (i, k)$. One can verify that $B(n, m)$ is an idempotent semigroup that has one $\mathcal{J}$-class, n $\mathcal{R}$-classes and m $\mathcal{L}$-classes.

We write $s <_{\mathcal{L}} t$ if $s \leq_{\mathcal{L}} t$ and s is not $\mathcal{L}$-equivalent to t. The relations $<_{\mathcal{R}}$ and $<_{\mathcal{J}}$ are defined in the same fashion. A finite sequence $(s_n, \ldots, s_1)$ of elements of S such that $s_{i+1} \leq_{\mathcal{L}} s_i$ for all $1 \leq i < n$ is called an *$\mathcal{L}$-chain.* We say that $(s_n, \ldots, s_1)$ is *strict* if $s_{i+1} <_{\mathcal{L}} s_i$ for all i. $\mathcal{J}$-chains and $\mathcal{R}$-chains are defined similarly. The following proposition summarizes the properties of Green's relations that will be used freely in the sequel.

Proposition 2.1 *Let S be a semigroup, and let s, t, $u \in S$.*

(1) *If s is idempotent, then $t \leq_{\mathcal{L}} s$ if and only if $ts = t$, and $t \leq_{\mathcal{R}} s$ if and only if $st = t$.*

(2) *If $s \mathrel{\mathcal{R}} t$ (resp. $s \leq_{\mathcal{R}} t$), then $us \mathrel{\mathcal{R}} ut$ (resp. $us \leq_{\mathcal{R}} ut$). Symmetrically, if $s \mathrel{\mathcal{L}} t$ (resp. $s \leq_{\mathcal{L}} t$), then $su \mathrel{\mathcal{L}} tu$ (resp. $su \leq_{\mathcal{L}} tu$).*

(3) *If $s \leq_{\mathcal{R}} t$ or $s \leq_{\mathcal{L}} t$, then $s \leq_{\mathcal{J}} t$. If $s \leq_{\mathcal{R}} t$ and $t \leq_{\mathcal{J}} s$ then $s \mathrel{\mathcal{R}} t$ and $s \mathrel{\mathcal{J}} t$. Symmetrically, if $s \leq_{\mathcal{L}} t$ and $t \leq_{\mathcal{J}} s$ then $s \mathrel{\mathcal{L}} t$ and $s \mathrel{\mathcal{J}} t$.*

(4) *If $s \mathrel{\mathcal{R}} t$ and $su = t$, then $r_u \colon x \mapsto xu$ defines a bijection from the $\mathcal{L}$-class of s onto the $\mathcal{L}$-class of t. If further $s \mathrel{\mathcal{L}} t$ and $s \neq t$, then r_u is a fixpoint-free permutation of the $\mathcal{L}$-class of S. In particular, if $st = s$ and $s \mathrel{\mathcal{L}} t$, then $t^2 = t$.*

If S is a finite semigroup and $s \in S$, the *$\mathcal{J}$-depth* of s is the maximal length $d(s)$ of a strict ascending $\mathcal{J}$-chain starting at s. Formally

$$d(s) = \max\{n \mid \exists\ s_1 <_{\mathcal{J}} s_2 <_{\mathcal{J}} \cdots <_{\mathcal{J}} s_n \text{ with } s = s_1\}.$$

We now turn to discuss the definition and certain properties of the Rhodes expansion of a semigroup S. The reduction ρ of $\mathcal{L}$-chains is defined inductively as follows:

$$\begin{aligned}
&(1) \qquad (s_1)\rho = (s_1) \\
&(2) \qquad (s_n, \ldots, s_1)\rho = \begin{cases} (s_n, s_{n-2}, \ldots, s_1)\rho & \text{if } s_n \mathrel{\mathcal{L}} s_{n-1} \\ (s_n, (s_{n-1}, \ldots, s_1)\rho) & \text{if } s_n <_{\mathcal{L}} s_{n-1}. \end{cases}
\end{aligned}$$

In other words, $(s_n,\ldots,s_1)\rho$ is the strict $\mathcal{L}$-chain obtained from $(s_n,\ldots,s_1)$ by removing all the terms s_i such that $s_{i+1}\ \mathcal{L}\ s_i$. The *Rhodes expansion of S* is the set $\widehat{S}$ of all strict $\mathcal{L}$-chains of S. One can verify that the following operation makes $\widehat{S}$ a semigroup:

$$(s_n,\ldots,s_1)(t_m,\ldots,t_1) = (s_nt_m, s_{n-1}t_m,\ldots,s_1t_m,t_m,t_{m-1},\ldots,t_1)\rho.$$

Furthermore, the projection π of an $\mathcal{L}$-chain onto its leftmost component is a morphism from $\widehat{S}$ onto S. Let now $\sigma\colon A^+\to S$ be an onto morphism, and let η be the morphism from A^+ into $\widehat{S}$ defined by $a\eta=(a\sigma)$. The image $A^+\eta$ is called the *Rhodes expansion of S cut-down to generators*, and it is written $\widehat{S}_A$. Note that σ factors through η, that is, $\sigma=\eta\pi$.

The following properties of $\widehat{S}$ will be used freely in the sequel.

Proposition 2.2 *Let S be a semigroup.*

(a) *The element $(s_n,\ldots,s_1)$ of $\widehat{S}$ is idempotent if and only if s_n is idempotent in S.*

(b) *If $\hat{s},\hat{t}\in\widehat{S}$ with $\hat{t}=(t_m,t_{m-1},\ldots,t_1)$, then $\hat{s}\leq_{\mathcal{L}}\hat{t}$ if and only if $\hat{s}$ is of the form $\hat{s}=(s_n,\ldots,s_m,t_{m-1},\ldots,t_1)$ with $n\geq m$ and $s_m\ \mathcal{L}\ t_m$.*

(c) *For each idempotent e of S, the subsemigroup $e\pi^{-1}$ of $\widehat{S}$ is idempotent and satisfies the identity $xy=y$.*

(d) *For each element s of $\widehat{S}$, the stabilizer $Stab(s)$ is an $\mathcal{R}$-trivial semigroup. In particular, $Stab(s)$ is aperiodic.*

3 The semigroup ${}^p\widehat{S_A}$

Let S be a finite semigroup, let A be a set of generators of S, let $\sigma\colon A^+\to S$ be an onto morphism, and let $p\geq 2$ be an integer. We denote by $\mathbb{Z}_p$ the ring of integers modulo p. We will construct a finite semigroup denoted ${}^p\widehat{S_A}$ that maps naturally onto S and whose properties will be discussed in the next section. The semigroup ${}^p\widehat{S_A}$ will actually arise as the transition semigroup of an automaton $\mathcal{A}(\mathbb{Z}_p,S,A)$, which we now proceed to construct.

Let $d\colon S^1\to\mathbb{N}\setminus\{0\}$ be the $\mathcal{J}$-depth function of S^1. In a first step we construct an automaton $(Q',A,\cdot)$ with initial state q_0, and then we restrict it to its accessible states to obtain $\mathcal{A}(\mathbb{Z}_p,S,A)$.

We set $Q'=\mathbb{Z}_p^{d(S)}\times(\widehat{S}_A)^1$. Thus the elements of Q' are pairs of the form $\big(f,(s_n,\ldots,s_1)\big)$ where $n\geq 0$, the s_i's are in S, $s_n<_{\mathcal{L}}\cdots<_{\mathcal{L}}s_1$ and f is a mapping $f\colon d(S)\to\mathbb{Z}_p$.

The alphabet A acts on Q' by

$$\big(f,(s_n,\ldots,s_1)\big)\cdot a=\big(g,(s_n,\ldots,s_1)(a\sigma)\big)=\big(g,(s_na\sigma,\ldots,s_1a\sigma,a\sigma)\rho\big)$$

where g is defined as follows. Set $s_0=1\in S^1$. For each i in $d(S)$, let j_i be maximal in $\{0,\ldots,n\}$ such that $d(s_{j_i})\leq i$. We let

$$g(i)=\begin{cases} f(i) & \text{if } d(s_{j_i}a\sigma)\leq i\\ f(i)+1 & \text{if } d(s_{j_i}a\sigma)>i.\end{cases}$$

Let $q_0 = (\bar{0}, 1)$, where $\bar{0}$ is the constant function equal to zero, let $Q = \{q_0 \cdot u \mid u \in A^*\}$, and let $\mathcal{A}(\mathbb{Z}_p, S, A) = (Q, A, \cdot)$. We will write ${}^p\widehat{S_A}$ the transition semigroup of $\mathcal{A}(\mathbb{Z}_p, S, A)$ and τ the projection of A^+ onto ${}^p\widehat{S_A}$.

For each word u of A^+, let $q_0 \cdot u = q_u = (f_u, u\eta)$. Note that if $i = \max d(S)$, then $f_u(i) = 0$ for all $u \in A^+$. Since each letter a of A acts on the right component of q_u as $a\eta$, the morphism η factors through τ, and hence so does σ.

$$\begin{tikzcd}
 & {}^p\widehat{S_A} \arrow[d, "\pi_1"] \\
A^+ \arrow[ur, "\tau"] \arrow[r, "\eta"] \arrow[dr, "\sigma"'] & \widehat{S_A} \arrow[d, "\pi_2"] \\
 & S
\end{tikzcd}$$

Figure 3.1: A commutative diagram.

The following lemma will be crucial for the proof of our main theorem.

Lemma 3.1 *Let $v \in A^+$ and let $q = (f, \hat{s})$, $q' = (f', \hat{s}') \in Q$ be such that $\hat{s}, \hat{s}' \leq_{\mathcal{L}} v\eta$. Finally let $q \cdot v = (g, \hat{s}v\eta)$ and $q' \cdot v = (g', \hat{s}'v\eta)$.*

(a) *For each $i < d(v\sigma)$, $g(i) - f(i) = g'(i) - f'(i)$ (in $\mathbb{Z}_p$), which only depends on $v\tau$ and i.*

(b) *If $v\sigma$ is idempotent and $i \geq d(v\sigma)$, then $g(i) = f(i)$ and $g'(i) = f'(i)$.*

Proof. Let $v = a_1 \cdots a_r$ and $v\eta = (v\sigma, s_{m-1}, \ldots, s_1)$. Then, by Proposition 2.2, $\hat{s}$ and $\hat{s}'$ are of the form $\hat{s} = (s_n, \ldots, s_m, s_{m-1}, \ldots, s_1)$ and $\hat{s}' = (s'_t, \ldots, s'_m, s_{m-1}, \ldots, s_1)$ with $n, t \geq m$ and $s'_m \mathrel{\mathcal{L}} s_m \mathrel{\mathcal{L}} v\sigma$. For each $1 \leq j \leq r$, let

$$q \cdot a_1 \cdots a_j = \big(f_j, \hat{s}(a_1 \cdots a_j)\eta\big) \text{ and } q' \cdot a_1 \cdots a_j = \big(f'_j, \hat{s}'(a_1 \cdots a_j)\eta\big).$$

We also write $f_0 = f$ and $f'_0 = f'$.

Let us first consider $i < d(v\sigma)$. In that case, the successive incrementations of $f_j(i)$ depend only on the action of $a_1 \cdots a_r$ on the $\mathcal{L}$-chain $(s_{m-1}, \ldots, s_1)$, which is common between $\hat{s}$ and $\hat{s}'$. More precisely, let $i_0 = 0$ and let us define $(i_1, \ldots, i_k)$ to be the longest sequence in $\{1, \ldots, r\}$ such that $i_{h+1} \geq i_h$ and i_{h+1} is minimal with $f_{i_{h+1}}(i) \neq f_{i_h}(i)$. In particular, for $1 \leq h \leq k$, $f_{i_h}(i) = f_{i_{h-1}}(i) + 1 = f(i) + h$ and $g(i) = f_r(i) = f(i) + k$. Since $d(s_m) = d(v\sigma) > i$, the sequence $(i_1, \ldots, i_k)$ is characterized by the following fact: if t_h is the leftmost of

$$s_{m-1}(a_1 \cdots a_{i_h})\sigma, \ldots, s_1(a_1 \cdots a_{i_h})\sigma, (a_1 \cdots a_{i_h})\sigma, \ldots, a_{i_h}\sigma$$

such that $d(t_h) \leq i$, then

$$d\big(t_h(a_{i_h+1} \cdots a_{i_{h+1}-1})\sigma\big) \leq i < d\big(t_h(a_{i_h+1} \cdots a_{i_{h+1}})\sigma\big)$$

and

$$d\big(t_k(a_{i_k+1}\cdots a_r)\sigma\big) \leq i.$$

Therefore, for each $0 \leq j \leq r$, if h is maximal with $i_h \leq j$, then $f'_j(i) = f'(i) + h$, so that $g'(i) = f'_r(i) = f'(i) + k$, thus proving (a).

We now prove the second statement: we assume $v\sigma = v^2\sigma$ and we consider $i \geq d(v\sigma)$. If j is maximal in $\{1, \ldots, n\}$ such that $d(s_j) \leq i$, then $j \geq m$, so that $s_j \leq_{\mathcal{L}} s_m \mathrel{\mathcal{L}} v\sigma$ and hence $s_jv\sigma = s_j$. Therefore

$$s_j \mathrel{\mathcal{R}} s_ja_1\sigma \mathrel{\mathcal{R}} \cdots \mathrel{\mathcal{R}} s_j(a_1\cdots a_{r-1})\sigma \mathrel{\mathcal{R}} s_j(a_1\cdots a_r)\sigma = s_j$$

and hence $f(i) = f_h(i) = g(i)$ for all $0 \leq h \leq r$. $\square$

4 Properties of ${}^p\widehat{S_A}$ and proof of the main theorem

Our main result, Theorem 1.1, will be proved if we show that for some p, ${}^p\widehat{S_A}$ is a semigroup in which the stabilizers satisfy the identities $x^2 = x$ and $xyx = xy$.

Let us assume that u and v are words of A^+ such that $(uv)\tau = u\tau$. It follows immediately:

(1) $(uv)\eta = u\eta$.

(2) $f_{uv} = f_u$.

In particular, $u\eta \leq_{\mathcal{L}} v\eta$ in $\widehat{S}_A$. So $u\eta$ and $v\eta$ are of the form $u\eta = (s_n, \ldots, s_1)$ and $v\eta = (v\sigma, s_{m-1}, \ldots, s_1)$, with $m \leq n$ and $s_m \mathrel{\mathcal{L}} v\sigma$. We finally set $v = a_1\cdots a_r$ ($a_i \in A$).

Proposition 4.1 *If $d(s_mv\sigma) = d(s_m)$, then $v\sigma$, $v\eta$ and $v\tau$ are idempotent.*

Proof. Since $s_mv\sigma \leq_{\mathcal{L}} v\sigma$, our hypothesis implies $s_mv\sigma \mathrel{\mathcal{L}} s_m \mathrel{\mathcal{L}} v\sigma$. Since further,

$$(uv)\eta = (s_nv\sigma, \ldots, s_mv\sigma, \ldots, s_1v\sigma, v\sigma, s_{m-1}, \ldots, s_1)\rho = u\eta,$$

and since $s_{m+1}v\sigma \leq_{\mathcal{R}} s_{m+1} <_{\mathcal{L}} s_m$, it follows $s_m = s_mv\sigma$, which implies that $v\sigma$ and $v\eta$ are idempotent.

Let us consider a state $q \in Q$. The states $q \cdot v$ and $q \cdot v^2$ are of the form

$$\begin{aligned} q \cdot v &= \big(f, (x_t, \ldots, x_m, s_{m-1}, \ldots, s_1)\big) \\ q \cdot v^2 &= \big(g, (x_tv\sigma, \ldots, x_mv\sigma, s_{m-1}v\sigma, \ldots, s_1v\sigma, v\sigma, s_{m-1}, \ldots, s_1)\rho\big) \end{aligned}$$

with $t \geq m$ and $x_m \mathrel{\mathcal{L}} s_m \mathrel{\mathcal{L}} v\sigma$.

If $i \geq d(s_m)$, then $g(i) = f(i)$ by Lemma 3.1 (b). Let us now consider $i < d(v\sigma)$. Using now Lemma 3.1 (a) we get $g(i) = f(i)$ since $f_{uv}(i) - f_u(i) = 0$. $\square$

Proposition 4.2 *If $p \geq \max d(S) - 1$, then $d(s_mv\sigma) = d(s_m)$.*

Proof. Let $f_j = f_{ua_1\cdots a_j}$. Suppose that $d(s_m v\sigma) > i = d(s_m)$. This implies in particular that $m < n$. Let $0 \le j < r$ be maximal such that $d(s_m(a_1 \cdots a_j)\sigma) = i$. Then $d(s_m(a_1 \cdots a_{j+1})\sigma) > i$ and we have

$$f_j(i) = f_u(i) \neq f_u(i) + 1 = f_{j+1}(i).$$

Therefore, as above, there exists a sequence $0 < i_1 < \cdots i_k \le r$ of length k, a non-zero multiple of p, and such that i_{h+1} be minimal with $f_{i_{h+1}}(i) \neq f_{i_h}(i)$ $(f_{i_1}(i) \neq f_u(i))$.

Here too, the sequence (i_h) is characterized by the fact that, if t_h is the leftmost of

$$s_m(a_1 \cdots a_{i_h})\sigma, s_{m-1}(a_1 \cdots a_{i_h})\sigma, \ldots, s_1(a_1 \cdots a_{i_h})\sigma, (a_1 \cdots a_{i_h})\sigma$$

such that $d(t_h) \le i$ (note that $d((a_1 \cdots a_{i_h})\sigma) \le d(v\sigma) = i$), then

$$d\big(t_h(a_{i_h+1} \cdots a_{i_{h+1}-1})\sigma\big) \le i < d\big(t_h(a_{i_h+1} \cdots a_{i_{h+1}})\sigma\big).$$

Since t_h is in the form $x(a_1 \cdots a_{i_h})\sigma$, this implies

$$(a_1 \cdots a_{i_{h+1}})\sigma <_{\mathcal{R}} (a_1 \cdots a_{i_h})\sigma.$$

Therefore the $\le_{\mathcal{R}}$-chain

$$\big((a_1 \cdots a_r)\sigma, (a_1 \cdots a_{r-1})\sigma, \ldots, a_1\sigma, 1\big)$$

contains at least p strict inequalities, and hence $d(v\sigma) = d\big((a_1 \cdots a_r)\sigma\big) \ge p + d(1) = p + 1$. Since $d(s_m v\sigma) > d(v\sigma)$, we have $p + 2 \le \max d(S)$, thus contradicting $p \ge \max d(S) - 1$. $\square$

We are now ready to prove

Theorem 4.3 *Let $p \ge \max d(S) - 1$. For each $u \in A^+$, the stabilizer $Stab(u\tau)$ satisfies the identities $x^2 = x$ and $xyx = xy$, that is, $Stab(u\tau)$ is an $\mathcal{R}$-trivial idempotent semigroup.*

Proof. Let $u\eta = (s_n, \ldots, s_1)$, and let $v, w \in A^+$ be such that $(uv)\tau = (uw)\tau = u\tau$. Then $v\eta = (v\sigma, s_{m-1}, \ldots, s_1)$ and $w\eta = (w\sigma, s_{\ell-1}, \ldots, s_1)$ for some $\ell, m \le n$ with $s_m \mathrel{\mathcal{L}} v\sigma$ and $s_\ell \mathrel{\mathcal{L}} w\sigma$. Let now $q \in Q$. Then $q \cdot v$ and $q \cdot vw$ are of the form

$$\begin{aligned} q \cdot v &= \big(f, (x_t, \ldots, x_m, s_{m-1}, \ldots, s_1)\big) \\ q \cdot vw &= \big(g, (x_t w\sigma, \ldots, x_m w\sigma, s_{m-1} w\sigma, \ldots, s_1 w\sigma, w\sigma, s_{\ell-1}, \ldots, s_1)\rho\big) \end{aligned}$$

with $t \ge m$ and $x_m \mathrel{\mathcal{L}} v\sigma$. By Propositions 4.1 and 4.2, we have $v\sigma = v^2\sigma$ and $w\sigma = w^2\sigma$, which implies $x_j v\sigma = x_j$ if $j \ge m$, $x_j w\sigma = x_j$ if $j \ge \ell$ and $s_j w\sigma = s_j$ if $j \ge \ell$.

Let us assume $\ell \le m$. Then

$$q \cdot vw = \big(g, (x_t, \ldots, x_m, s_{m-1}, \ldots, s_\ell, s_{\ell-1}, \ldots, s_1)\big).$$

We now show that $g = f$. Note that we have the inequalities $u\eta \le_{\mathcal{L}} w\eta$ and $(x_t, \ldots, x_m, s_{m-1}, \ldots, s_1) \le_{\mathcal{L}} w\eta$. If $i \ge d(w\sigma)$, then $g(i) = f(i)$ by Lemma 3.1 (b).

For $i < d(w\sigma)$, $g(i) = f(i)$ follows from Lemma 3.1 (a) and from the fact that $f_{uw}(i) - f_u(i) = 0$. This shows that $q \cdot vw = q \cdot v$ and hence $v\tau w\tau = v\tau$.

So we have $v\tau w\tau = v\tau$ if $\ell \leq m$, and $w\tau v\tau = w\tau$ otherwise. Therefore we find $(v\tau)^2 = v\tau$ (for $v = w$) and, in general, $v\tau w\tau v\tau = v\tau w\tau$. Thus $Stab(u\tau)$ satisfies the announced identities. $\square$

Remark. The proof of Theorem 4.3 above shows also that, if $p \geq \max d(S) - 1$, and if $v\tau, w\tau \in Stab(u\tau)$, then

$$d(v\sigma) \geq d(w\sigma) \Longleftrightarrow v\sigma \leq_{\mathcal{L}} w\sigma \Longleftrightarrow v\tau \leq_{\mathcal{L}} w\tau.$$

In particular, all the elements of $Stab(u\tau)$ are $\mathcal{L}$-comparable (in ${}^p\widehat{S_A}$ as well as in $Stab(u\tau)$).

Let π_1 and π_2 be the projections of ${}^p\widehat{S_A}$ onto $\widehat{S_A}$ and from $\widehat{S_A}$ onto S respectively, and let $\pi = \pi_1\pi_2$. We know (Proposition 2.2) that, if e is an idempotent of S, then $e\pi_2^{-1}$ is an idempotent semigroup which satisfies $xy = y$. It is interesting to note the following properties of π_1 and π.

Proposition 4.4 *Let $e \in E(S)$ and $\hat{e} \in e\pi_2^{-1}$.*

(a) *$\hat{e}\pi_1^{-1}$ satisfies the identities $xyz = xzy$ and $xy^p = x$, and $\hat{e}\pi_1^{-1}$ is the direct product of an idempotent $\mathcal{L}$-class with a subgroup of $\mathbb{Z}_p^{d(e)-1}$.*

(b) *$e\pi^{-1}$ satisfies the identities $xyzx = xzyx$, $xy^pz = xz$ and $x^{2p} = x^p$, and $e\pi^{-1}$ is the direct product of a rectangular band and of a subgroup of $\mathbb{Z}_p^{d(e)-1}$.*

Proof. Let us first prove (a). Let $u \in A^+$ be such that $u\sigma = e$ and $u\eta = \hat{e}$. If $q = (f, \hat{s}) \in Q$, then $q \cdot u = (g, \hat{s}\hat{e})$. If $\hat{s} \leq_{\mathcal{L}} \hat{e}$, then by Lemma 3.1 we have $g(i) = f(i)$ for $i \geq d(e)$ and $g(i) = f(i) + f_{u^2}(i) - f_u(i)$ for $i < d(e)$. It is now trivial to see that $\hat{e}\pi_1^{-1}$ satisfies the announced identities, and hence that $\hat{e}\pi_1^{-1}$ is an $\mathcal{L}$-class of abelian groups of exponent a divisor of p. To obtain the more precise result that $\hat{e}\pi_1^{-1}$ is the direct product of an idempotent $\mathcal{L}$-class with a subgroup of $\mathbb{Z}_p^{d(e)-1}$, it suffices to notice that if $u \in A^+$ is such that $u\eta = \hat{e}$, then $u\tau$ is entirely determined by its $\mathcal{R}$-class in $\hat{e}\pi_1^{-1}$ and by the $\big(d(e)-1\big)$-tuple $\big(f_{u^2}(i) - f_u(i)\big)_{1 \leq i < d(e)}$.

We now turn to proving (b). Let $u, v \in A^+$ such that $u\sigma = v\sigma = e$. Then $u\eta$ and $v\eta$ are of the form

$$u\eta = (e, s_{n-1}, \ldots, s_1) \qquad \text{and} \qquad v\eta = (e, t_{m-1}, \ldots, t_1),$$

so that it is easy to verify that $u\eta v\eta = v\eta$. Consequently, applying (a) to $v\eta\pi_1^{-1}$, we find that $v^p\tau$ is idempotent and that $u\tau v\tau \;\mathcal{L}\; v\tau$, and hence $v\tau \leq_{\mathcal{J}} u\tau$. By symmetry it follows that $e\pi^{-1}$ is a single $\mathcal{D}$-class where p-th powers are idempotent. Furthermore each $v\eta\pi_1^{-1}$ is an $\mathcal{L}$-class of $e\pi^{-1}$, so that each $\mathcal{H}$-class of $e\pi^{-1}$ is a subgroup of $\mathbb{Z}_p^{d(e)-1}$. To conclude our proof, we need to show that if $u\tau$ and $v\tau$ are idempotent, then $(uvu)\tau = u\tau$. Let $q \in Q$. Then $q \cdot u$ and $q \cdot uv$ are of the form $(f, \hat{s}u\eta)$ and $(g, \hat{s}u\eta v\eta) = (g, \hat{s}v\eta)$. According to Lemma 3.1, $g(i) = f(i)$ for all $i \geq d(e)$ and $g(i) = f(i) + f_{v^2}(i) - f_v(i)$ for $i < d(e)$. But $v\tau = v^2\tau$, so $f_v = f_{v^2}$ and hence $g = f$, that is, $q \cdot uv = (f, \hat{s}v\eta)$. Consequently $q \cdot uvu = (f, \hat{s}v\eta u\eta) = (f, \hat{s}u\eta) = q \cdot u$ and $u\tau = (uvu)\tau$. $\square$

Remark. Note that it is impossible to strengthen our construction in order to obtain simultaneously idempotent stabilizers and a projection morphism such that the inverse image of each idempotent be aperiodic. In fact, one can verify that an aperiodic semigroup with idempotent stabilizers is necessarily an idempotent semigroup. Indeed, the semigroup satisfies an identity of the form $x^r = x^{r+1}$, and hence each of its elements belongs to a stabilizer.

Theorem 1.1 shows that every finite semigroup is a quotient of a semigroup in which right stabilizers belong to the variety of idempotent semigroups $\mathbf{R_1}$ defined by the identities $x = x^2$ and $xy = xyx$. This variety is rather low in the lattice of varieties of idempotent semigroups, but it is natural to ask whether one can improve the result by choosing a lower variety. The answer to this question is negative. More precisely, we have the following result.

Proposition 4.5 *Let* $\mathbf{V}$ *be a variety of idempotent semigroups such that every finite semigroup is the quotient of a finite semigroup whose right stabilizers belong to* $\mathbf{V}$*. Then* $\mathbf{V}$ *contains* $\mathbf{R_1}$*.*

Proof. The structure of the lattice of all varieties of idempotent semigroups has been determined by Birjukov [2], Fennemore [8] and Gerhard [9]. The largest variety of idempotent semigroups not containing $\mathbf{R_1}$ is the variety $\mathbf{W}$ defined by the identities $x = x^2$ and $xyz = xzxyz$. Let $S = \{1, a, b\}$ be the semigroup in which 1 is an identity, $aa = ab = a$ and $ba = bb = b$. Let $\pi : T \to S$ be a surjective morphism. We claim that the right stabilizers in T cannot belong to $\mathbf{W}$. Let e, f, g be idempotents of T such that $e\pi = a$, $f\pi = b$ and $g\pi = 1$. Let n be an integer such that every element of the form t^n is idempotent in T and put $x = g$, $y = \big(e(fg)^n\big)^n$ and $z = (fg)^n$. Then $yx = y$, $yy = y$ and $yz = y$, and thus x, y and z belong to the right stabilizer R of y. If R belongs to $\mathbf{W}$, then in particular $xyz = xzxyz$, and thus $(xyz)\pi = (xzxyz)\pi$. But $x\pi = 1$, $y\pi = \big(e(fg)^n\big)^n\pi = (ab^n)^n = a$ and $z\pi = (fg)^n\pi = b$. Thus $(xyz)\pi = a \neq b = (xzxyz)\pi$. The proposition follows immediately. $\square$

In the sequel we will call *nice* any finite semigroup whose stabilizers are idempotent. Thus, Theorem 1.1 states that every finite semigroup is a quotient of a nice semigroup in which the stabilizers satisfy the identity $xyx = xy$. This terminology is only introduced for convenience in this paper, and should not survive it.

Another way of viewing Theorem 1.1 is to express it in geometric terms. We say that a transformation semigroup (P, S) is *fixpoint-free* if, for every $p \in P$ and every $s \in S$,

$$p \cdot s = p \quad \text{implies} \quad s = s^2.$$

A transformation semigroup (P, S) is *covered* by a transformation semigroup (Q, T) if there exists a surjective function $\varphi : Q \to P$ such that, for every $s \in S$, there exists $\hat{s} \in T$ satisfying, for every $q \in Q$,

$$(q\varphi) \cdot s = (q \cdot \hat{s})\varphi.$$

We recall the following well-known results.

Proposition 4.6 *Let (Q, S) be a transformation semigroup. Then (Q, S) is covered by the transformation semigroup $(Q, 1_Q) \times (S^1, S)$.*

Proof. Let $\varphi : Q \times S^1 \to Q$ be the function defined by

$$(q, s)\varphi = q \cdot s.$$

Set, for every $s \in S$, $\hat{s} = (1_Q, s)$. Then, for every $q \in Q$ and every $t \in S^1$, and for every $s \in S$,

$$\begin{aligned} (q, t)\varphi \cdot s &= (q \cdot t) \cdot s = q \cdot ts \quad \text{and} \\ ((q, t) \cdot \hat{s})\varphi &= ((q, t)(1_Q, s))\varphi = (q, ts)\varphi = q \cdot ts \end{aligned}$$

Thus $(q, t)\varphi \cdot s = ((q, t) \cdot \hat{s})\varphi$, and this proves the proposition. $\square$

Proposition 4.7 *Let $\varphi : S \to T$ be a surjective morphism. Then the transformation semigroup (S^1, S) covers (T^1, T).*

Proof. The function $\varphi : S \to T$ can be extended to a function $\varphi : S^1 \to T^1$ by setting $1\varphi = 1$. For every $t \in T$, choose an element $\hat{t} \in t\varphi^{-1}$. Then, for every $s \in S^1$,

$$\begin{aligned} &(s\varphi) \cdot t = (s\varphi)t \quad \text{and} \\ &(s \cdot \hat{t})\varphi = (s\hat{t})\varphi = (s\varphi)(\hat{t}\varphi) = (s\varphi)t. \end{aligned}$$

Thus (S^1, S) covers (T^1, T). $\square$

Proposition 4.8 *If S is nice, then (S^1, S) is fixpoint-free.*

Proof. If $st = s$ for some $s \in S^1$ and $t \in S$, then either $s = 1$ and then $t = 1$ (and $S = S^1$), or t is idempotent since S is nice. Thus (S^1, S) is fixpoint-free. $\square$

In this context, Theorem 1.1 has the following consequence.

Theorem 4.9 *Every finite transformation semigroup is covered by a finite fixpoint-free transformation semigroup.*

Proof. Let (P, S) be a transformation semigroup. By Proposition 4.6, (P, S) is covered by the transformation semigroup $(P, 1_P) \times (S^1, S)$. By Theorem 1.1, S is a quotient of a nice semigroup T, and by Proposition 4.7, (S^1, S) is covered by (T^1, T). It follows that (P, S) is covered by $X = (P, 1_P) \times (T^1, T)$. Now $(P, 1_P)$ is fixpoint-free and so is (T^1, T) by Proposition 4.8. Thus X is a fixpoint-free covering of (P, S). $\square$

5 Two examples

Our first example is rather simple and will be treated completely to illustrate our construction. The second example aims at showing that the full generality allowed by Proposition 4.4 may occur: the inverse image of an idempotent may be the direct product of a rectangular band which is neither $\mathcal{R}$- nor $\mathcal{L}$-trivial with several copies of $\mathbb{Z}_p$.

Let S be the set of integers modulo 4 under multiplication. This is a commutative semigroup with three $\mathcal{D}$-classes, represented in Figure 5.1 below

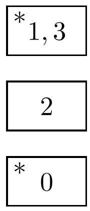

Figure 5.1: The semigroup S.

Then $S^1 = S$ and $\max d(S) - 1 = 2$. Let $A = \{a, b\}$. Then $a\sigma = 3$ and $b\sigma = 2$ defines an onto morphism from A^+ onto S. We will now construct the automaton $\mathcal{A}(\mathbb{Z}_2, S, A) = (Q, A, \cdot)$. If $u \in A^+$, then $q_u = (f_u, u\eta)$, f_u is a map from $\{1, 2, 3\}$ into $\mathbb{Z}_2 = \{0, 1\}$ and we know that $f_u(3) = 0$. So we can represent f_u by the two-letter binary word $f_u(2)f_u(1)$. $\mathcal{A}(\mathbb{Z}_2, S, A)$ is represented in Figure 5.2.

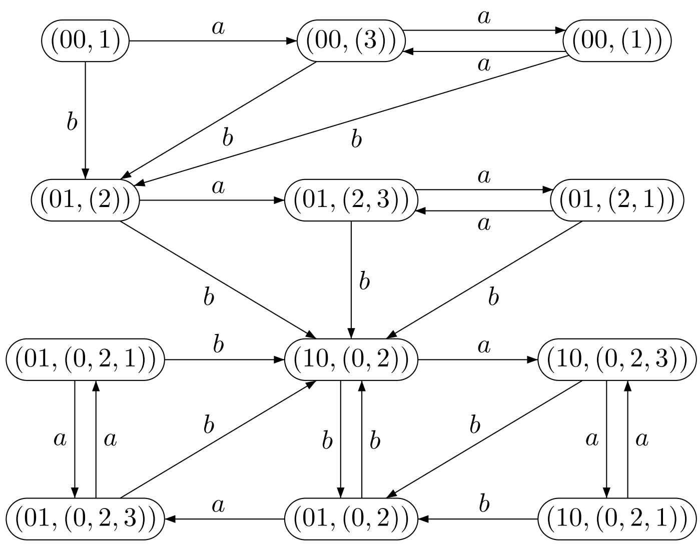

Figure 5.2:

One can then verify that ${}^2\widehat{S_A} = \langle a, b \mid ab = b, a^3 = a, b^4 = b^2 \rangle$ is as represented in Figure 5.3.

$^* \; a, a^2$

b

ba	ba^2

$^* \; b^2$ b^3	$^* \; b^2a$ b^3a	$^* \; b^2a^2$ b^3a^2

Figure 5.3: The semigroup ${}^2\widehat{S_A}$.

In particular $1\pi^{-1}$ is trivial and $0\pi^{-1}$ is isomorphic to $B(1,2) \times \mathbb{Z}_2$. The stabilizer of b is empty, the stabilizers of a, a^2, ba and ba^2 are trivial, and the stabilizers of the elements of the minimal ideal are isomorphic to U_1.

Remark. ${}^2\widehat{S_A}$ is not the smallest nice semigroup generated by A of which S is a quotient. Indeed, S is a quotient of $S' = \langle a, b \mid ab = ba, a^2 = 1, b^4 = b^2 \rangle$, represented in Figure 5.4.

$^* \; 1, a$

b, ab

$^* \; b^2, ab^2$ b^3, ab^3

Figure 5.4: The semigroup S'.

Let us now consider the semigroup $T = \langle a, b \mid a^2 = a, aba = ba, bab = b^2 = 0 \rangle$. T has five elements and is represented in Figure 5.5.

* a	b

ab

ba

* 0

Figure 5.5: The semigroup T.

Note that the maximal $\mathcal{L}$-chains in T are $(0, ba, a)$ and $(0, ab, b)$. Since T is not a monoid, $d(T) = \{2, 3, 4, 5\}$ so that $\max d(T) - 1 = 4$ but it so turns out that ${}^2\widehat{T_A}$ has idempotent and $\mathcal{R}$-trivial stabilizers. We leave it to the reader to verify that ${}^2\widehat{T_A}$ is defined by the relators

$$a^2 = a,\ b^4 = b^2,\ b^2aba = bab^2a,\ b^2ab^2 = bab^3,\ b^3ab = bab,\ babab^2 = b^2$$

and is represented as in Figure 5.6.

* a	b
ab	ba

aba

* b^2, bab^2 b^3, bab^3	* b^2a, bab^2a $b^3a, (ba)^2$	* $bab, (bab)^2$ $b^2ab, (ba)^2b$
* $ab^2, abab^2$ $ab^3, (ab^2)^2$	* $ab^2a, abab^2a$ $ab^3a, a(ba)^2$	* $(ab)^2, ab^2ab$ $(ab)^2b, (ab)^2bab$

Figure 5.6: The semigroup ${}^2\widehat{T_A}$.

Here $a\pi^{-1}$ is trivial and $0\pi^{-1} = B(2, 3) \times \mathbb{Z}_2^2$. Further the stabilizers of b, ab, ba and aba are empty, the stabilizer of a is trivial, the stabilizers of the elements $\mathcal{L}$-related to either b^2 or bab are isomorphic to $B(2, 1)$ while the stabilizers of the elements $\mathcal{L}$-related to b^2a are equal to $\{a, (ba)^2, a(ba)^2\}$ and hence isomorphic to the idempotent semigroup $\{e, f, ef\}$ where $fe = f$.

6 A new proof of Mc Naughton's theorem

Let A be a finite alphabet, and let A^+ (resp. $A^{\mathbb{N}}$) be the set of finite (resp. infinite) words on A. It is a well-known fact of the theory of finite automata that every set of finite words recognized by a non-deterministic automaton is also accepted by a deterministic one. When Büchi introduced the notion of an automaton accepting a set of infinite words, it became apparent that the situation was quite different for infinite words, since deterministic and non-deterministic automata are not equivalent. However, McNaughton showed that with a suitable change in the accepting conditions, deterministic automata are as powerful as non-deterministic ones. In order to state this result precisely, we need some definitions. For more details, see [6], [21] or [16].

An *automaton* is a triple $\mathcal{A} = (Q, A, E)$ where Q is a finite set, called the set of states, A is a finite alphabet, and E is a subset of $Q \times A \times Q$, called the set of *transitions* or *edges*. An automaton $\mathcal{A} = (Q, A, E)$ is said to be *deterministic* if, for every state q and every letter a, there exists at most one state q' such that (q, a, q') is a transition. In this case, the set of transitions is the graph of a partial function from $Q \times A$ into Q, called the *transition function*, usually denoted by $(q, a) \mapsto q \cdot a$. Thus, deterministic automata are often defined by a triple $(Q, A, \cdot)$.

Two transitions (p, a, q) and (p', a', q') are consecutive if $q = p'$. A finite (resp. infinite) path is a finite (resp. infinite) sequence of consecutive transitions:

$$p = (q_0, a_0, q_1)(q_1, a_1, q_2) \dots (q_{n-1}, a_{n-1}, q_n)$$

The word $a_0a_1a_2 \cdots a_n$ is the *label* of the path, q_0 its *origin* and q_n its end. A *loop* is a path whose origin is equal to its end. The states (resp. transitions) *visited* by p are q_0, q_1, ..., q_n (resp. (q_0, a_0, q_1), (q_1, a_1, q_2), ..., (q_{n-1}, a_{n-1}, q_n)). We denote by $Q_\infty(p)$ (resp. $T_\infty(p)$) the set of states (resp. transitions) that p visits infinitely often. It is also convenient to consider empty paths: namely we introduce an empty loop around each state, whose label is the empty word.

A *Büchi automaton* is a quintuple $\mathcal{A} = (Q, A, E, I, F)$ where

(1) (Q, A, E) is an automaton,

(2) I and F are subsets of Q, called respectively the set of *initial* and *final* states.

An infinite word u is *recognized* by $\mathcal{A}$ if there exists a path p with origin in I and label u such that p visits a final state infinitely often. The set of infinite words recognized by $\mathcal{A}$ is denoted by $L^{\mathbb{N}}(\mathcal{A})$. A set of infinite words is *recognizable* if it is recognized by some Büchi automaton.

A *Muller automaton* (or *state table automaton*) is a quintuple $\mathcal{A} = (Q, A, E, i, \mathcal{T})$ where

(1) (Q, A, E) is a deterministic automaton,

(2) i is an element of Q, called the *initial state*,

(3) $\mathcal{T}$ is a set of subsets of Q, called the *table of states*.

An infinite word u is *recognized* by $\mathcal{A}$ if there exists a path with origin i and label u such that $Q_\infty(p) \in \mathcal{T}$. We are now ready to state the theorem of McNaughton [14].

Theorem 6.1 *Every recognizable set of infinite words is recognized by some Muller automaton.*

One could also consider a slightly different class of table automata. A *transition table automaton* [12] is a quintuple $\mathcal{A} = (Q, A, E, i, \mathcal{T})$ where

(1) (Q, A, E) is a deterministic automaton,

(2) i is an element of Q, called the *initial state*,

(3) $\mathcal{T}$ is a set of subsets of E called the *table of transitions*.

An infinite word u is *recognized* by $\mathcal{A}$ if there exists a path p with origin i and label u such that $T_\infty(p) \in \mathcal{T}$. State table automata and transition table automata are in fact equivalent, but we need to prove the result in one direction only.

Proposition 6.2 *Every set of infinite words recognized by a transition table automaton is also recognized by some state table automaton.*

Proof. Let $\mathcal{A} = (Q, A, E, i, \mathcal{T})$ be a transition table automaton recognizing a set X. Let j be a new state and let $\mathcal{B} = (E \cup \{j\}, A, E', j, \mathcal{T})$ be the state table automaton defined by $E' = E_1 \cup E_2$ where

$$\begin{aligned}E_1 =& \big\{\big(j, a, (i, a, q)\big) \mid (i, a, q) \in E\big\}\\ E_2 =& \big\{\big((q, a, q'), b, (q', b, q'')\big) \mid (q, a, q') \in E \text{ and } (q', b, q'') \in E\big\}\end{aligned}$$

Then any infinite path with origin i

$$p = (i, a_0, q_1)(q_1, a_1, q_2) \ldots$$

in $\mathcal{A}$ corresponds to an infinite path with origin j

$$p' = \big(j, a_0, (i, a_0, q_1)\big)\big((i, a_0, q_1), a_1, (q_1, a_1, q_2)\big) \ldots$$

in $\mathcal{B}$ and conversely, every infinite path with origin j arises this way. Furthermore, p visits a transition (q, a, q') in $\mathcal{A}$ if and only if p' visits the state (q, a, q') in $\mathcal{B}$. Therefore $T_\infty(p) = Q_\infty(p')$ and thus $L^{\mathbb{N}}(\mathcal{A}) = L^{\mathbb{N}}(\mathcal{B})$. □

As is possible for sets of finite words, one can define recognizable sets of infinite words by using finite semigroups. Let $X \subseteq A^{\mathbb{N}}$ and let $\varphi : A^+ \to S$ be a semigroup morphism onto some finite semigroup S. The morphism φ is said to *recognize* X if, for every $u \in X$, and for every factorization

$$u = u_0 u_1 u_2 \cdots \qquad (u_i \in A^+)$$

of u, the set

$$(u_0\varphi\varphi^{-1})(u_1\varphi\varphi^{-1})(u_2\varphi\varphi^{-1}) \cdots$$

is contained in X. Equivalently, if

$$v = v_0 v_1 v_2 \cdots$$

with $v_n\varphi = u_n\varphi$ for every $n \geq 0$, then $v \in X$.

By extension, a finite semigroup recognizes a subset X of $A^{\mathbb{N}}$ if there is a morphism $\varphi : A^+ \to S$ that recognizes X. Then one can state the following result (see [5, 21, 16]).

Proposition 6.3 *A set of infinite words is recognizable if and only if it is recognized by some finite semigroup.*

In order to prove McNaughton's theorem, we need a more precise statement

Proposition 6.4 *A set of infinite words is recognizable if and only if it is recognized by some finite nice semigroup in which the stabilizers satisfy the identity $xyx = xy$.*

Proof. By Proposition 6.3, it suffices to show that if a subset X of $A^{\mathbb{N}}$ is recognized by a semigroup morphism $\varphi : A^+ \to S$ onto a finite semigroup, then X is recognized by a nice semigroup in which the stabilizers satisfy the identity $xyx = xy$. By Theorem 1.1, there exists such a nice semigroup N, and a surjective morphism $\pi : N \to S$. Therefore, by the universal property of the free semigroup, there exists a morphism $\psi : A^+ \to N$ such that $\psi\pi = \varphi$.

We claim that ψ recognizes X. Indeed, let $u \in X$ and let $u = u_0u_1u_2\cdots$ be a factorization of u. If $v = v_0v_1v_2\cdots$ is a factorization of v such that $u_n\psi = v_n\psi$ for every $n \geq 0$, then $u_n\psi\pi = v_n\psi\pi = u_n\varphi = v_n\varphi$ for every $n \geq 0$. It follows $v \in X$, since φ recognizes X. This proves the claim and the proposition. □

We also need two results of independent interest. The first one is a Ramsey type theorem (see [15]).

Proposition 6.5 *Let A be an alphabet, let S be a finite semigroup, and let $\varphi : A^+ \to S$ be a semigroup morphism. Let u be an infinite word of $A^{\mathbb{N}}$, and let $u = u_0u_1\ldots$ be a factorisation of u in words of A^+. Then there exists $s \in S$, $e \in E(S)$ and a strictly increasing sequence of integers $(k_n)_{n\geq 0}$ such that $\varphi(u_0u_1\cdots u_{k_0-1}) = s$ and $\varphi(u_{k_n}u_{k_n+1}\cdots u_{k_{n+1}-1}) = e$ for every $n \geq 0$.*

The second one is a result of I. Simon on path congruences. A proof of this lemma can be found in [7]. Given an automaton $\mathcal{A}$, a path congruence is an equivalence relation on the set of finite paths of $\mathcal{A}$ satisfying the following conditions:

(1) any two equivalent paths are coterminal (that is, they have the same origin and the same end),

(2) if p and q are equivalent paths, and if r, p and s are consecutive paths, then rps is equivalent to rqs.

Proposition 6.6 *Let $\sim$ be a path congruence such that, for every pair of loops p, q around the same state, $p^2 \sim p$ and $pq \sim qp$. Then two coterminal paths visiting the same sets of transitions are equivalent.*

We can now give our proof of McNaughton's theorem. Let X be a recognizable subset of $A^{\mathbb{N}}$. By Proposition 6.4, there exists a morphism $\varphi : A^+ \to S$ onto a finite nice semigroup S which recognizes X and in which the stabilizers satisfy the identity $xyx = xy$. One naturally associates a deterministic automaton $(S^1, A, \cdot)$ to this morphism by setting, for every $s \in S^1$ and every $a \in A$

$$s \cdot a = s(a\varphi).$$

Let s be a fixed state of S^1. Then every word u is the label of exactly one path with origin s, called the path with origin s defined by u.

Let $\mathcal{A} = (S^1, A, \cdot, 1, \mathcal{T})$ be the transition table automaton with 1 as initial state and such that

$$\mathcal{T} = \{T_\infty(u) \mid u \in X\}.$$

We claim that $\mathcal{A}$ recognizes X. First, if $u \in X$, then $T_\infty(u) \in \mathcal{T}$ by definition, and thus u is recognized by $\mathcal{A}$. Conversely, let u be an infinite word recognized by $\mathcal{A}$. Then

$$T_\infty(u) = T_\infty(v) = T \quad \text{for some } v \in X.$$

Thus, both paths u and v visit only finitely many times transitions out of T. Therefore, after a certain point, every transition of u (resp. v) belongs to T, and every transition of T is visited infinitely often. Consequently, one can find two factorizations $u = u_0u_1u_2\cdots$ and $v = v_0v_1v_2\cdots$ and a state $s \in S$ such that

(1) u_0 and v_0 define paths from 1 to s,

(2) for every $n > 0$, u_n and v_n define loops around s that visit at least once every transition in T and visit no other transition.

The situation is summarized in Figure 6.1 below

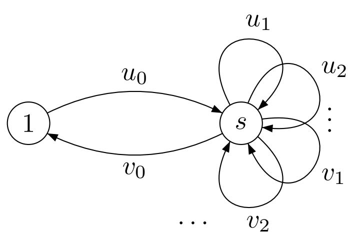

Figure 6.1:

Furthermore, Proposition 6.5 shows that, by grouping the u_i's (resp. v_i's) together, we may assume that

$$u_1\varphi = u_2\varphi = u_3\varphi = \ldots \quad \text{and} \quad v_1\varphi = v_2\varphi = v_3\varphi = \ldots$$

It follows in particular

$$u_0v_1^\omega \in X \tag{1}$$

since $u_0\varphi = v_0\varphi = s$, $v_1\varphi = v_2\varphi = \ldots$ and $v_0v_1v_2\cdots \in X$. Furthermore,

$$u \in X \quad \text{if and only if} \quad u_0u_1^\omega \in X \tag{2}$$

To decongest notations, we shall denote by the same letter a path and its label. We define a path equivalence $\sim$ as follows. Two paths p and q are equivalent if p and q are coterminal, and if, for every non-empty path x from 1 to the origin of p, and for every path r from the end of p to its origin, $x(pr)^\omega \in X$ if and only if $x(qr)^\omega \in X$.

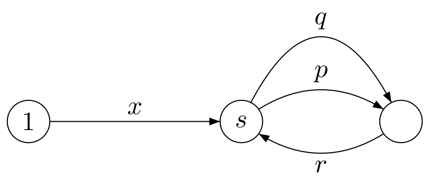

Figure 6.2:

Lemma 6.7 *The equivalence $\sim$ is a path congruence such that, for every pair of loops p, q around the same state, $p^2 \sim p$ and $pq \sim qp$.*

Proof. We first verify that $\sim$ is a congruence. Suppose that $p \sim q$ and let u and v be paths such that u, p and v are consecutive. Since $p \sim q$, p and q are coterminal, and thus upv and uqv are also coterminal. Furthermore, if x is a non-empty path from 1 to the origin of upv, and if r is a path from the end of upv to its origin such that $x(upvr)^\omega \in X$, then $(xu)(p(vru))^\omega \in X$, whence $(xu)(q(vru))^\omega \in X$ since $p \sim q$, and thus $x(uqvr)^\omega \in X$. Symmetrically, $x(uqvr)^\omega \in X$ implies $x(upvr)^\omega \in X$, showing that $upv \sim uqv$.

Next we show that if p is a loop around $s \in S$, then $p^2 \sim p$. Let x be a non-empty path from 1 to the origin of p, and let r be a path from the end of p to its origin. Then, since p is a loop, $(x\varphi)(p\varphi) = x\varphi$. Now since S is a nice semigroup, $p\varphi = p^2\varphi$ and thus $x(pr)^\omega \in X$ if and only if $x(p^2r)^\omega \in X$ since φ recognizes X. Note that this is the only place in the proof where we use the fact that S is nice.

Finally, we show that if p and q are loops around the same state s, then $pq \sim qp$. Let, as before, x be a non-empty path from 1 to the origin of p, and let r be a path from the end of p to its origin. Then r is a loop around s. We first observe that

$$x(pq)^\omega \in X \iff x(qp)^\omega \in X \tag{3}$$

Indeed $x(pq)^\omega = xp(qp)^\omega$, and since p is a loop, $(x\varphi)(p\varphi) = x\varphi$. Thus $xp(qp)^\omega \in X$ if and only if $x(qp)^\omega \in X$, then proving (3). Now, we have the following sequence of equivalences

$$\begin{aligned} x(pqr)^\omega \in X &\iff x(pqrq)^\omega \in X \\ &\iff x(rqpq)^\omega \in X \\ &\iff x(rqp)^\omega \in X \\ &\iff x(qpr)^\omega \in X, \end{aligned}$$

where the second and fourth equivalences follow from (3) and the first and third from the identity $xyx = xy$ satisfied by $Stab(x\varphi)$. $\square$

We can now conclude the proof of theorem 6.1. By assumption, the two loops around s defined by u_1 and v_1 visit exactly the same sets of transitions (namely T). Thus, by Lemma 6.7 and by Proposition 6.6, these two paths are equivalent. In particular, since $u_0v_1^\omega \in X$ by (1), we have $u_0u_1^\omega \in X$, and thus $u \in X$ by (2). Therefore $\mathcal{A}$ recognizes X. $\square$

Remark. Lemma 6.7 does not actually require the full strength of our hypothesis on S. In fact, the only place where we use the fact that the stabilizers in S satisfy the identity $xyx = xy$ is in the last lines of the proof of Lemma 6.7. If we assume only that S is nice (without any further assumption on its stabilizers), then we can still derive (3). Next we can show

$$x(pqr)^\omega \in X \iff x(qpqr)^\omega \in X \tag{4}$$

Indeed, we have the following sequence of equivalences

$$\begin{aligned}
x(pqr)^\omega \in X &\iff x(pqrqr)^\omega \in X && \text{since } qr \sim (qr)^2\\
&\iff x(pqrqpqrqr)^\omega \in X && \text{since } pqrq \sim (pqrq)^2\\
&\iff x(pqrqpqr)^\omega \in X && \text{since } qr \sim (qr)^2\\
&\iff x(qpqrpqr)^\omega \in X && \text{by (3) applied to } pqr \text{ and } qpqr\\
&\iff x(qpqr)^\omega \in X && \text{since } pqr \sim (pqr)^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
x(pqr)^\omega \in X &\iff x(qpqr)^\omega \in X && \text{by (4)}\\
&\iff x(qrqp)^\omega \in X && \text{by (3) applied to } qr \text{ and } qp\\
&\iff x(rqp)^\omega \in X && \text{by (4)}\\
&\iff x(qpr)^\omega \in X && \text{by (3) applied to } qp \text{ and } r
\end{aligned}$$

and hence $pq \sim qp$. $\square$

Remark. We can weaken further the conditions on S under which Lemma 6.7 still holds, and hence under which the transition table automaton $\mathcal{A} = (S^1, A, \cdot, 1, \mathcal{T})$ defined above recognizes X. If $\varphi\colon A^+ \to S$ recognizes X, it is enough to assume that there exists a morphism $\pi\colon S \to V$ such that $\psi = \varphi\pi\colon A^+ \to V$ recognizes X and such that $Stab(s)\pi \subseteq E(V)$ for each $s \in S$. That is, we do not assume that the stabilizers in S are idempotent, but we assume that their images in a semigroup V recognizing X are idempotent.

The only parts of the proof of Lemma 6.7 that need modifying are the following. Let us consider a path x from 1 to $s \in S^1$ and loops p, q and r around s. Then $x\varphi(p\varphi) = x\varphi$, so that $p\varphi\pi = p\psi$ is idempotent, that is $p\psi = p^2\psi$. Therefore $x(pr)^\omega \in X$ is equivalent to $x(p^2r)^\omega \in X$, and hence $p \sim p^2$. Similarly, to prove (3) we verify as above that $x(pq)^\omega = xp(qp)^\omega$ and that $(xp)\psi = x\psi$ so that $x(qp)^\omega \in X$ if and only if $xp(qp)^\omega = x(pq)^\omega \in X$. We then apply these facts as in the previous remark to prove (4) and then $qp \sim pq$, thus completing the verification of Lemma 6.7. $\square$

7 A new proof of a theorem on locally finite semigroups

We say that a semigroup S is *locally finite* if each of its finitely generated subsemigroups is finite.

Theorem 7.1 (Brown) *Let $\varphi : S \to T$ be a semigroup morphism such that T is locally finite and such that $e\varphi^{-1}$ is locally finite for each idempotent e of T. Then S is locally finite.*

The original proof of this theorem [4] relies on the study of combinatorial properties of infinite strings. We use Theorem 1.1 above to avoid the study of such strings and to give a completely algebraic proof for this result. Simon [19, 20] gave a different algebraic proof of the same result.

We will use Lemmas 7.2 and 7.3 below. The first one is due to Tilson [24] and we will not prove it here. Let A be an alphabet, let $\eta : A^+ \to V$ be an onto morphism, and let $\tau : V \to W$ be a semigroup morphism. For convenience, η is extended to a morphism from A^* onto V^1 by letting $1\eta = 1$. The derived category D_τ is defined as follows. Its set of objects is $Ob(D_\tau) = W^1$. For each pair $(w, u) \in W^1 \times A^*$ we let $[w, u]$ be the right translation of $w\tau^{-1}$ by $u\eta$. The arrow set $D_\tau(w_1, w_2)$ is then defined as the set of all $[w_1, u]$ such that $w_1(u\eta\tau) = w_2$. Multiplication of consecutive arrows is the composition of functions, that is, if $w_1(u\eta\tau) = w_2$, then $[w_1, u][w_2, u'] = [w_1, uu']$. Then, the following holds.

Lemma 7.2 *Let $\eta : A^+ \to V$, $\tau : V \to W$ and D_τ be defined as above. If W and D_τ are finite, then so is V.*

Remark. Tilson's results in [24] are actually much more precise. He shows that V divides the wreath products $X \circ W$ for all monoids X divided by D_τ. In particular, V divides $X \circ W$ where X is the union of all the arrow sets of D_τ and of a zero and a unit, with $x \cdot y = xy$ if x and y can be multiplied in D_τ, and $x \cdot y = 0$ otherwise.

Our second lemma deals with locally finite categories. A category is said to be *locally finite* if any subcategory generated by a finite set of arrows is finite.

Lemma 7.3 *Let C be a category such that each of its base monoids $C(c, c)$ ($c \in Ob(C)$) is locally finite. Then C is locally finite.*

Proof. Let A be a finite set of arrows of C and let $\pi : A^+ \to C$ be the partial function which maps a word u on the product of its letters if u represents a path in C. We need to show that $A^+\pi$ is finite. We use a variant of the famous algorithm of McNaughton and Yamada which computes the rational expression associated with a finite automaton (see [1]).

Let us enumerate in an arbitrary fashion the objects of C, say, $Ob(C) = \{c_1, \ldots, c_n\}$. For each $1 \le i, j \le n$ and $1 \le k \le n+1$, let $A^k_{i,j}$ denote the set of all words in A^+ which represent paths in C from i to j, all of whose interior nodes are in $\{c_h \mid h < k\}$. Then

$$A^+\pi = \bigcup_{1 \le i,j \le n} A^{n+1}_{i,j}\pi.$$

We will show by induction on k that each $A^k_{i,j}\pi$ is finite. First let us note that $A^1_{i,j}$ is contained in A, so that it is finite. Let us now assume that for some $k \geq 1$, all $A^k_{i,j}\pi$ $(1 \leq i, j \leq n)$ are finite. Note that

$$A^{k+1}_{i,j} = A^k_{i,j} \cup A^k_{i,k}(A^k_{k,k})^* A^k_{k,j}.$$

Since $A^k_{k,k}\pi$ is finite and is contained in $C(c_k, c_k)$, the submonoid it generates, that is, $(A^k_{k,k})^*\pi$ is finite. Therefore, $A^{k+1}_{i,j}\pi = A^k_{i,j}\pi \cup A^k_{i,k}\pi(A^k_{k,k})^*\pi A^k_{k,j}\pi$ is finite by induction. $\square$

We can now start the proof of Brown's result (Theorem 7.1 above). As a first step, we show that we can assume T to be finite and to have all its stabilizers in $\mathbf{R}_1$. Let us indeed consider a finite alphabet A and a morphism $\sigma : A^+ \to S$. Since T is locally finite, the semigroup $T' = A^+\sigma\varphi$ is finite. Furthermore, for each idempotent e of T', $e\varphi^{-1} \cap A^+\sigma$ is locally finite. So we may assume that T is finite. By Theorem 1.1 above, there exists a finite semigroup $\hat{T}$ whose right stabilizers are in $\mathbf{R}_1$, and a surjective morphism π from $\hat{T}$ onto T. Let $R = \{(s,t) \in S \times \hat{T} \mid s\varphi = t\pi\}$ be the pull-back of φ and π, with α and β the projections of R onto its first and second component.

$$\begin{tikzcd}
 & R \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \widehat{T} \arrow[d, "\pi"] \\
A^+ \arrow[r, "\sigma"'] & S \arrow[r, "\varphi"'] & T
\end{tikzcd}$$

For each idempotent e of $\hat{T}$, $e\beta^{-1} = \{(s,e) \mid s \in e\pi\varphi^{-1}\}$ is isomorphic to $e\pi\varphi^{-1}$ and hence is locally finite. Finally, we can lift σ to a morphism $\rho : A^+ \to R$ such that $\rho\alpha = \sigma$. We then need to show that $A^+\rho$ is finite.

Therefore we may assume that the stabilizers of T are in $\mathbf{R}_1$ and that the morphism $\sigma : A^+ \to S$ is surjective, and we want to show that S is finite. According to Lemma 7.2 we need to show that the category D_φ is finite. But D_φ is finitely generated (by the $[t,a]$, $t \in T^1$, $a \in A$), so by Lemma 7.3 we need to show that the base monoids of D_φ are locally finite. Now for each $t \in Ob(D_\varphi) = T^1$, $D_\varphi(t,t)$ is the set of the right translations of $t\varphi^{-1}$ by an element of $Stab(t)\varphi^{-1}$, and hence is a quotient of $Stab(t)\varphi^{-1}$.

So we have finally reduced the problem to proving the theorem in the case where T is in $\mathbf{R}_1$ and each $t\varphi^{-1}$ $(t \in T)$ is locally finite. Let $\tau = \sigma\varphi : A^+ \to T$. We know that we may assume σ, φ and τ to be onto. We now proceed by induction on $\mathrm{Card}(T)$.

If $T = \{e\}$, then $S = e\varphi^{-1}$ is locally finite by assumption. Let us now assume that $\mathrm{Card}(T) \geq 2$ and let us consider an element t in a $\leq_{\mathcal{J}}$-maximal $\mathcal{J}$-class of T. Then, for each $a \in A$, $u \in A^*$, if $t = (au)\tau$, then $t = a\tau$. Indeed, $t \leq_{\mathcal{R}} a\tau$, which implies by maximality that $t \mathrel{\mathcal{R}} a\tau$ and hence $t = a\tau$. Let $B = A \cap t\tau^{-1}$ and let $C = A \setminus B$. According to our assumption, $B^+\tau \neq T$ and $C \neq \emptyset$ (otherwise, $T = a^+\tau = \{a\tau\}$). Furthermore $t \notin (CA^*)\tau$. Let us now notice that

$$A^+ = B^+ \cup (CB^*)^+ \cup B^+(CB^*)^+.$$

Since $B^+\tau$ and $(CB^*)^+\tau$ are strictly contained in T, the semigroups $S_B = B^+\sigma$ and $S_C = (CB^*)^+\sigma$ are locally finite by induction. But S_B is finitely generated (by $B\sigma$), so it is finite. Also, S_C is generated by the finite set $C\sigma S_B$ and hence is finite as well. Therefore $A^+\sigma = S_B \cup S_C \cup S_B S_C$ is finite. $\square$

8 Conclusion

Given a finite semigroup S, we have constructed effectively a finite covering $\widehat{S}$ of S, the stabilizers of which are idempotent and $\mathcal{R}$-trivial. This result is used in particular to obtain a new proof of McNaughton's theorem. As we noticed, a weaker version of our theorem is sufficient for this proof. Since our construction is rather involved, a simpler construction yielding only idempotent stabilizers would be of great interest. This would lead to an improved proof of McNaughton's theorem.

Another remarkable fact is the duality between the properties of the projection morphism $\pi\colon \widehat{S} \to S$ and the properties of the stabilizers, in Rhodes' expansion and in our construction. Intuitively, it seems that the more is required on the stabilizers, the less can be imposed on the projection. It would be interesting to know whether there are other results in this direction.

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