Profinite semigroups, Mal'cev products and identities[1]

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Résumé: Nous calculons un ensemble d'identités définissant le produit de Mal'cev de pseudovariétés de semigroupes finis ou de semigroupes finis ordonnés. Nous caractérisons aussi les parties ponctuelles d'un semigroupe fini à l'aide d'un morphisme relationnel dans un semigroupe profini. Enfin, nous appliquons nos résultats à la preuve de la décidabilité des produits de Mal'cev d'une pseudovariété décidable avec la pseudovariété des semigroupes nilpotents et avec celle des semigroupes $\mathcal{J}$-triviaux.

Abstract: We compute a set of identities defining the Mal'cev product of pseudovarieties of finite semigroups or finite ordered semigroups. We also characterize the pointlike subsets of a finite semigroup by means of a relational morphism into a profinite semigroup. Finally, we apply our results to the proof of the decidability of the Mal'cev products of a decidable pseudovariety with the pseudovarieties of nilpotent and of $\mathcal{J}$-trivial semigroups.

The aim of this paper is to study the Mal'cev product of pseudovarieties of semigroups and monoids. The Mal'cev product is a very important operation in the lattice of pseudovarieties, with applications in group theory, in semigroup theory and in language theory.

It was established by Reiterman [20] that each pseudovariety is defined by a set of identities in some profinite structure. However, the identity problem, i.e. the problem of finding a defining set of identities for a pseudovariety, is known to be very difficult, even for the simple operation of join of two pseudovarieties [1]. Recent work of Almeida and the second author addressed the problem for the semidirect product of pseudovarieties [5, 6]. In this paper, we solve the analogous problem for the Mal'cev product, that is, we describe a defining set of identities for a Mal'cev product. Our methods also apply to the case of pseudovarieties of finite ordered semigroups. Such pseudovarieties can also be defined by pseudoidentities in a profinite structure as was established by the authors [18].

Let us say immediately that our solution is not effective: even if the pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ have decidable membership problems (we say that such pseudovarieties are decidable), the defining set of identities which we compute for the Mal'cev product $\mathbf{W} ⓜ \mathbf{V}$ does not permit in general to prove the decidability of $\mathbf{W} ⓜ \mathbf{V}$. But we could hardly hope

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for such a strong result in view of Albert, Baldinger and Rhodes's result according to which there exist decidable pseudovarieties — and even pseudovarieties defined by a finite number of word identities — whose join is not decidable [1].

The question of the decidability of a Mal'cev product is very difficult in general. In order to address this question, Henckell and Rhodes introduced the pointlike subsets of a finite semigroup [9]. Along with our result on the identities defining a Mal'cev product, we give a general theorem describing the pointlike subsets of a finite semigroup. Our results also lead to specific decidability results: for instance, if $\mathbf{V}$ is a decidable pseudovariety of semigroups, then $\mathbf{V} \circledM \mathbf{J}_1$, $\mathbf{V} \circledM \mathbf{Nil}$, $\mathbf{V} \circledM \mathbf{J}$, $\mathbf{V} \circledM \mathbf{G}$, $\mathbf{V} \circledM \mathbf{G}_p$ and $\mathbf{V} \circledM \mathbf{G}_{nil}$ are decidable as well. Here $\mathbf{J}_1$, $\mathbf{Nil}$, $\mathbf{J}$, $\mathbf{G}$, $\mathbf{G}_p$ and $\mathbf{G}_{nil}$ denote respectively the pseudovarieties of idempotent and commutative semigroups, nilpotent semigroups, $\mathcal{J}$-trivial semigroups, groups, p-groups (p prime) and nilpotent groups. Our results also have consequences in the study of the decidability of the dot-depth hierarchy, a long-standing open problem in the theory of formal languages with connections in logic, and more generally, in the study of the polynomial closure of varieties of recognizable languages. These latter sets of consequences are explored in another paper by the authors [19].

The two main ingredients in our proofs are the following: the theory of relatively free profinite semigroups, as it was developed by Almeida [2] and by Almeida and the second author [4] in order to build upon Reiterman's theorem; and a lemma by Hunter on the properties of certain congruences in compact semigroups [13].

The paper is organized as follows. The first section contains the necessary definitions and results on free profinite semigroups and on identities, as well as Hunter's lemma. Section 2 introduces Mal'cev products, and our central result on the characterization of a Mal'cev product by means of a relational morphism into a certain profinite semigroup is stated and proved in Section 3 (Theorem 3.1). In the same section, we give a description of the pointlike subsets of a finite semigroup. In Section 4, we derive from our central result the computation of a defining set of identities for a Mal'cev product. Finally, in Section 5, we give several applications of our results, and in particular we show that if $\mathbf{V}$ is decidable, then so are $\mathbf{V} \circledM \mathbf{J}_1$, $\mathbf{V} \circledM \mathbf{Nil}$ and $\mathbf{V} \circledM \mathbf{J}$.

1 Preliminaries

We first review the basic definitions and results concerning relatively free profinite semigroups and ordered semigroups and some important properties of compact semigroups. For more details on free profinite semigroups, the reader is referred to Almeida's book [2] and to the survey [4]. For the ordered case, see [18].

1.1 Relatively free profinite semigroups

A class $\mathbf{V}$ of finite semigroups is called a *pseudovariety* if it is closed under taking subsemigroups, homomorphic images and finite direct products. Pseudovarieties of finite monoids are defined in the same fashion. We will also consider topological semigroups and monoids. Recall that, by definition, the product is a continuous operation in these objects.

We say that a semigroup (resp. monoid) is *profinite* if it is a projective limit of finite semigroups (resp. monoids). More generally, if $\mathbf{V}$ is a pseudovariety of semigroups (resp. monoids), we say that a semigroup (resp. monoid) is *pro*-$\mathbf{V}$ if it is a projective limit of elements of $\mathbf{V}$. It is known that a semigroup (resp. monoid) is profinite if and only if it is compact and 0-dimensional [15]. If $\mathbf{V}$ is a pseudovariety of semigroups (resp. monoid), then a profinite semigroup (resp. monoid) is pro-$\mathbf{V}$ if and only if all its finite continuous homomorphic images are in $\mathbf{V}$. Of course, all elements of $\mathbf{V}$ are pro-$\mathbf{V}$, and if $\mathbf{W}$ is a pseudovariety contained in $\mathbf{V}$, then all pro-$\mathbf{W}$ semigroups (resp. monoids) are pro-$\mathbf{V}$.

A class of profinite semigroups is called a *pro-variety of semigroups* if it is closed under taking closed subsemigroups, continuous homomorphic images and arbitrary direct products. If $\mathbf{V}$ is a pseudovariety of semigroups, then the class $\widehat{\mathbf{V}}$ of all pro-$\mathbf{V}$ semigroups is a pro-variety. Conversely, if $\mathcal{V}$ is a pro-variety of semigroups and $\mathbf{V}$ is the class of finite elements of $\mathcal{V}$, then $\mathbf{V}$ is a pseudovariety and $\mathcal{V} = \widehat{\mathbf{V}}$. So $\mathbf{V} \mapsto \widehat{\mathbf{V}}$ establishes a lattice isomorphism between pseudovarieties of semigroups and pro-varieties of semigroups. *Pro-varieties of monoids* are defined similarly, and the analogous connection with pseudovarieties of monoids holds.

We say that a topological space A is a *profinite set* if it is a projective limit of finite sets, or equivalently, if it is compact and totally disconnected [15]. A profinite semigroup (resp. monoid) S is said to be generated by a profinite set A, *A-generated* for short, if there exists a continuous mapping $\sigma\colon A \to S$ such that the subsemigroup (resp. submonoid) generated by $A\sigma$ is dense in S.

Let $\mathbf{V}$ be a pseudovariety and let A be a profinite set. We denote by $\hat{F}_A(\mathbf{V})$ the projective limit of the A-generated elements of $\mathbf{V}$. The topological semigroups (resp. monoids) of the form $\hat{F}_A(\mathbf{V})$ have been widely studied, see Almeida [2], Almeida and Weil [4] or Zeitoun [25]. The semigroup (resp. monoid) $\hat{F}_A(\mathbf{V})$ can be viewed as the completion of a certain uniform structure on the free semigroup A^+ (resp. free monoid A^*), or as the semigroup (resp. monoid) of A-ary implicit operations on $\mathbf{V}$ (when A is finite).

If $\mathbf{V}$ is the pseudovariety of all finite semigroups (resp. monoids), we write $\widehat{A^+}$ (resp. $\widehat{A^*}$) for $\hat{F}_A(\mathbf{V})$. We summarize below the main properties of the $\hat{F}_A(\mathbf{V})$ which will be used freely in the sequel.

Theorem 1.1 *Let A be a profinite set and let $\mathbf{V}$ be a pseudovariety of semigroups (resp. monoids).*

1. *$\hat{F}_A(\mathbf{V})$ is A-generated and pro-$\mathbf{V}$: in particular there exists a continuous mapping $\iota\colon A \to \hat{F}_A(\mathbf{V})$, which is one-to-one if $\mathbf{V}$ is non-trivial, such that $A\iota$ generates a dense subsemigroup (resp. submonoid) of $\hat{F}_A(\mathbf{V})$.*

2. *$\hat{F}_A(\mathbf{V})$ is the free pro-$\mathbf{V}$ semigroup (resp. monoid) over A: If S is pro-$\mathbf{V}$ and if $\sigma\colon A \to S$ is a continuous mapping, then σ induces a unique continuous morphism $\hat{\sigma}\colon \hat{F}_A(\mathbf{V}) \to S$ such that $\iota\hat{\sigma} = \sigma$.*

3. *A finite semigroup (resp. monoid) is in $\mathbf{V}$ if and only if it is a continuous homomorphic image of $\hat{F}_A(\mathbf{V})$ for some finite set A.*

The case of a profinite set of generators A is considered here for the sake of completeness. In many applications, it suffices to consider the case where A is finite [2, 4]. However, the profinite case proved to be helpful in the study of the semidirect product of pseudovarieties, see Almeida and Weil [5]. Since this extension to profinite generating sets does not introduce specific difficulties in the proofs, the reader of this paper may be tempted to always assume that the generating sets are finite.

Let us observe the following immediate but useful corollary of Theorem 1.1.

Corollary 1.2 *Let $\mathbf{V}$ be a pseudovariety of semigroups (resp. monoids), let S and T be pro-$\mathbf{V}$ semigroups (resp. monoids) and let A be a profinite set. If $\sigma\colon \hat{F}_A(\mathbf{V}) \to S$ and*

$\varphi\colon T \to S$ *are continuous morphisms with* φ *onto, then there exists a continuous morphism* $\tau\colon \hat{F}_A(\mathbf{V}) \to T$ *such that* $\tau\varphi = \sigma$.

$$\begin{tikzcd} & \widehat{B^+} \arrow[d, "\varphi"] \\ \hat{F}_A(\mathbf{V}) \arrow[ur, "\tau"] \arrow[r, "\sigma"] & S \end{tikzcd}$$

Another important application of Theorem 1.1 is the following: If $\mathbf{W}$ is a sub-pseudovariety of $\mathbf{V}$, then the identity of A induces a continuous onto morphism $\pi\colon \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{W})$, which is called the *natural projection of* $\hat{F}_A(\mathbf{V})$ *onto* $\hat{F}_A(\mathbf{W})$.

Since $\hat{F}_A(\mathbf{V})$ is A-generated, its elements can be seen as limits of sequences of words on the alphabet A. An important such limit is the ω-power, which traditionally denotes the idempotent power of an element of a finite semigroup [8, 16].

Proposition 1.3 *Let* A *be a profinite set and let* $\mathbf{V}$ *be a pseudovariety of semigroups. Let* $x \in \hat{F}_A(\mathbf{V})$. *The sequence* $x^{n!}$ *converges in* $\hat{F}_A(\mathbf{V})$. *Its limit, written* x^ω, *is idempotent and, if* $\sigma\colon \hat{F}_A(\mathbf{V}) \to S$ *is a continuous morphism into a finite semigroup, then* $x^\omega\sigma$ *is the unique idempotent power of* $x\sigma$.

We will also consider here classes of finite ordered semigroups (resp. monoids). Such a class is called a *pseudovariety of ordered semigroups* (resp. *monoids*) if it closed under taking ordered subsemigroups (that is, subsemigroups equipped with the restriction of the order), images under order-preserving homomorphisms and finite direct products (equipped with the product order). A general discussion of pseudovarieties of first-order structures, including the case of ordered semigroups and monoids, can be found in [18]. We say that a topological semigroup (resp. monoid) S equipped with an order relation is a *topological ordered semigroup* (resp. *monoid*) if the order relation is a closed subset of $S \times S$.

When dealing with ordered semigroups and monoids, we will always assume that the morphisms under consideration are order-preserving. In particular, the projective limit of a directed system of ordered semigroups or monoids is itself ordered. For any pseudovariety $\mathbf{V}$ of ordered semigroups (resp. monoids) and any profinite set A, we can define as above the profinite ordered semigroup (resp. monoid) $\hat{F}_A(\mathbf{V})$. Then the analogues of Theorem 1.1 and Corollary 1.2 hold, where all morphisms are assumed to be order-preserving [18].

Observe that for the pseudovariety $\mathbf{V}$ of all finite ordered semigroups (resp. monoids), $\hat{F}_A(\mathbf{V})$ coincides with $\widehat{A^+}$ (resp. $\widehat{A^*}$), equipped with the trivial order where $x \leq y$ if and only if $x = y$.

Pro-varieties of ordered semigroups (resp. *monoids*) are also defined in the obvious way: they are the classes of profinite ordered semigroups (resp. monoids) closed under taking closed ordered semigroups (resp. submonoids), order-preserving continuous homomorphic images and arbitrary direct products. As in the unordered case, there is a lattice isomorphism between pro-varieties and pseudovarieties of ordered semigroups (resp. monoids).

Unless mentioned otherwise, the results reported in this paper hold for pseudovarieties of finite semigroups, monoids, ordered semigroups and ordered monoids. In order to avoid cumbersome statements, the word pseudovariety will now denote any one of these four kinds of pseudovarieties. Most proofs will silently assume that we are dealing with pseudovarieties of semigroups, and unless otherwise indicated, the proofs concerning the other kinds of pseudovarieties can be handled similarly.

1.2 Identities

Theorem 1.1 leads to the statement of Reiterman's theorem on the definition of pseudovarieties by identities (see Reiterman [20], and Pin and Weil for the ordered case [18]). Let A be a profinite set and let $x, y \in \widehat{A^+}$. We say that a profinite semigroup S *satisfies the identity* $x = y$ if $x\sigma = y\sigma$ for each continuous morphism $\sigma\colon \widehat{A^+} \to S$. In the ordered case, we also say that S *satisfies the identity* $x \leq y$ if $x\sigma \leq y\sigma$ for each continuous morphism $\sigma\colon \widehat{A^+} \to S$. (Any such morphism is order-preserving, since $\widehat{A^+}$ is equipped with the trivial order.) The identities of the form $x = y$ or $x \leq y$, with $x, y \in \widehat{A^*}$, satisfied by a profinite monoid or ordered monoid are defined in the same fashion. In all cases, we say that a class $\mathcal{V}$ of profinite semigroups (resp. monoids, ordered semigroups, ordered monoids) satisfies a given identity if each element of $\mathcal{V}$ satisfies it. The class $\mathcal{V}$ satisfies a set Σ of identities if each element of $\mathcal{V}$ satisfies each identity of Σ. The class of all finite (resp. profinite) semigroups (monoids, ordered semigroups, ordered monoids) which satisfy Σ is said to be *defined by* Σ and is denoted $[\![\Sigma]\!]$ (resp. $\widehat{[\![\Sigma]\!]}$). It is important to remark the following [18, Proposition 3.2].

Proposition 1.4 *Let* $\mathbf{V}$ *be a pseudovariety and let* A *be a profinite set. Let* $\pi\colon \widehat{A^+} \to \hat{F}_A(\mathbf{V})$ *(resp.* $\pi\colon \widehat{A^*} \to \hat{F}_A(\mathbf{V})$*) be the natural projection and let* $x, y \in \widehat{A^+}$ *(resp.* $\widehat{A^*}$*). Then* $\mathbf{V}$ *satisfies* $x = y$ *if and only if* $x\pi = y\pi$*. Similarly, in the ordered case,* $\mathbf{V}$ *satisfies* $x \leq y$ *if and only if* $x\pi \leq y\pi$*.*

Usually (for instance in [2, 4, 18]) it is assumed that the identities we consider are in finitely many variables, that is, each is of the form $x = y$ (or $x \leq y$ in the ordered case) with $x, y \in \widehat{A^+}$ (resp. $x, y \in \widehat{A^*}$) for some finite set A. Indeed, it suffices to consider such identities to describe all pseudovarieties, as expressed by the following theorem, originally proved by Reiterman in the unordered case [20] and extended by the authors in the ordered case [18].

Theorem 1.5 *Let* $\mathbf{V}$ *be a class of finite semigroups (resp. monoids, ordered semigroups, ordered monoids). Then* $\mathbf{V}$ *is a pseudovariety if and only if there exists a set* Σ *of identities such that* $\mathbf{V} = [\![\Sigma]\!]$*. Moreover, the set* Σ *can be chosen in such a way that all its elements are in finitely many variables.*

Since the elements of $\widehat{A^+}$ and $\widehat{A^*}$ are limits of sequences of words, an identity (in the unordered case) can be seen as the limit of a sequence of word identities. In this sense, Reiterman's theorem above is a topological version of Eilenberg and Schützenberger's theorem which states that pseudovarieties of finite semigroups (resp. monoids) are ultimately defined by sequences of word identities [8, 16].

Example. The following pseudovarieties will play a role in Section 5:

- the class $\mathbf{Com}$ of all finite commutative semigroups, $\mathbf{Com} = [\![xy = yx]\!]$;
- the class $\mathbf{J}_1$ of all finite idempotent and commutative semigroups, $\mathbf{J}_1 = [\![xy = yx,\ x^2 = x]\!]$;

- the class $\mathbf{G}$ of all finite groups (a pseudovariety of monoids), $\mathbf{G} = [\![x^\omega = 1]\!]$.

Finally, let us note the following consequence of the above results.

Corollary 1.6 *Let $\mathbf{V}$ be a pseudovariety, and let Σ be a set of identities. Then $\mathbf{V} = [\![\Sigma]\!]$ if and only if $\widehat{\mathbf{V}} = \widehat{[\![\Sigma]\!]}$.*

Proof. Since $\mathbf{V}$ is the class of finite elements of $\widehat{\mathbf{V}}$, it is clear from the definitions that $\widehat{\mathbf{V}} = \widehat{[\![\Sigma]\!]}$ implies $\mathbf{V} = [\![\Sigma]\!]$. Let us now assume that $\mathbf{V} = [\![\Sigma]\!]$, and let S be a profinite semigroup in $\widehat{\mathbf{V}}$. Then S is the projective limit of a directed set $(S_i)_{i\in I}$ of elements of $\mathbf{V}$. Let $\pi_i \colon S \to S_i$ $(i \in I)$ be the canonical projections. If a morphism $\sigma\colon \widehat{A^+} \to S$ is continuous, then so are the $\sigma\pi_i$, since the projections π_i are continuous. Let $x = y$ be an identity of Σ. Since each S_i satisfies $x = y$, we have $x\sigma\pi_i = y\sigma\pi_i$ for each $i \in I$, and hence $x\sigma = y\sigma$. Thus S satisfies $x = y$. So $\widehat{\mathbf{V}} \subseteq \widehat{[\![\Sigma]\!]}$

Conversely, let us assume that S satisfies Σ. Then the continuous homomorphic images of S also satisfy Σ. Let indeed $\varphi\colon S \to T$ be an onto continuous morphism, let $x, y \in \widehat{A^+}$ such that $(x = y) \in \Sigma$, and let $\sigma\colon \widehat{A^+} \to T$ be a continuous morphism. By Corollary 1.2, there exists a continuous morphism $\tau\colon \widehat{A^+} \to S$ such that $\sigma = \tau\varphi$. But $x\tau = y\tau$ by hypothesis, so $x\sigma = y\sigma$. That is, T satisfies Σ. In particular, the finite continuous homomorphic images of S satisfy Σ, and hence are in $\mathbf{V}$. Since S is profinite, this implies that $S \in \widehat{\mathbf{V}}$, thus concluding the proof. □

1.3 Some properties of compact semigroups

Compact semigroups share a number of important properties with finite semigroups. In particular, we will use freely the fact that a non empty compact semigroup has at least one idempotent [12].

We will also use the following results on congruences in compact semigroups.

Lemma 1.7 *Let S be a compact semigroup and let $\sim$ be an open congruence on S, that is, a congruence whose graph is an open subset of $S \times S$. Then $\sim$ is a clopen congruence, $\sim$ has finite index and the $\sim$-classes are clopen.*

Proof. Let R be the graph of $\sim$. By hypothesis, R is an open subset of $S \times S$. Let $s \in S$. Then $(s, s) \in R$ and hence there exists an open neighborhood U of s in S such that $U \times U \subseteq R$. In particular, U lies in the $\sim$-class $[s]_\sim$ of s, and hence $[s]_\sim$ is open. So the $\sim$-classes form an open partition of S. It follows by compactness that $\sim$ has finite index, and hence each $\sim$-class is clopen. Finally, R is a finite union of products of the form $[s]_\sim \times [s]_\sim$, so R is clopen. □

If S is a semigroup and X is a subset of S, the *syntactic congruence* of X is defined, for all $s, s' \in S$, by

$$s \sim s' \qquad \text{if and only if} \qquad \text{for all } u, v \in S \cup \{1\} \;\; usv \in X \Leftrightarrow us'v \in X.$$

It is the coarsest congruence of S which saturates X. Hunter proved the following result on the syntactic congruence of a clopen set [13, Lemma 4].

Proposition 1.8 *If S is a compact semigroup and X is a clopen subset of S, then the syntactic congruence of X in S is clopen.*

2 Relational morphisms and Mal'cev products

Relational morphisms were first defined by Tilson [23]. If S and T are semigroups, a *relational morphism* $\varphi\colon S \to T$ is a relation from S into T, i.e. a mapping from S into the set of subsets of T, such that

1. $s\varphi \neq \emptyset$ for all $s \in S$;
2. $(s\varphi)(t\varphi) \subseteq (st)\varphi$.

For instance, morphisms and inverses of morphisms are relational morphisms. Also, the composite of two relational morphisms is a relational morphism.

Let $\varphi\colon S \to T$ be a relation and let R be its graph:

$$R = \{(s,t) \in S \times T \mid t \in s\varphi\}.$$

Let also $\alpha\colon R \to S$ and $\beta\colon R \to T$ be the projections into the first and second coordinates. We observe that φ is a relational morphism if and only if R is a subsemigroup of $S \times T$ and α is onto. In particular $\varphi = \alpha^{-1}\beta$. Thus, any relational morphism can be written as the composite of the inverse of an onto morphism with a morphism. The factorization $\varphi = \alpha^{-1}\beta$ is called the *canonical factorization* of φ.

When dealing with monoids, we include in the definition of relational morphisms the fact that $1 \in 1\varphi$. Thus the relational morphisms from S to T are exactly the relations whose graph is a submonoid of $S \times T$ with first-coordinate projection onto S. It is not necessary however to introduce a special notion of relational morphisms for ordered semigroups or monoids. Indeed, if S and T are ordered, then the graph R of φ is naturally ordered, as a sub-algebra of the direct product $S \times T$, and the projection morphisms α and β are order-preserving.

Let $\mathcal{C}$ be a class of semigroups (resp. ordered semigroups). We say that the relational morphism $\varphi\colon S \to T$ is a *$\mathcal{C}$-relational morphism* if, for each idempotent e of T, the subsemigroup $e\varphi^{-1} = \{s \in S \mid e \in s\varphi\}$ of S lies in $\mathcal{C}$. Let $\mathbf{V}$ be a pseudovariety of semigroups (resp. monoids). If $\mathbf{W}$ is a pseudovariety of semigroups, we define the *Mal'cev product* $\mathbf{W} \textcircled{\scriptsize M} \mathbf{V}$ to be the class of all finite semigroups (resp. monoids) S such that there exists a $\mathbf{W}$-relational morphism from S into an element of $\mathbf{V}$. If $\mathbf{W}$ is a pseudovariety of ordered semigroups, we define similarly $\mathbf{W} \textcircled{\scriptsize M} \mathbf{V}$ to be the class of all finite ordered semigroups (resp. monoids) S such that there exists a $\mathbf{W}$-relational morphism from S into an element of $\mathbf{V}$. Using the notion of canonical factorization, it is not difficult to verify that, in all cases, $\mathbf{W} \textcircled{\scriptsize M} \mathbf{V}$ is again a pseudovariety.

Note. Another definition is sometimes adopted, where $\mathbf{W} \textcircled{\scriptsize M} \mathbf{V}$ is defined to be the class of all finite S such that there exists a $\mathbf{W}$- (functional) morphism from S into an element of $\mathbf{V}$. The two definitions yield different classes. However, the pseudovariety generated by the latter class coincides with our definition.

Also, observe that we will not use — and we have not defined — products of the form $\mathbf{W} \textcircled{\scriptsize M} \mathbf{V}$ where $\mathbf{V}$ is a pseudovariety of ordered semigroups or ordered monoids.

If S and T are topological semigroups (resp. monoids), we will be interested only in *closed* relational morphisms from S to T, that is, relational morphisms whose graph R is a closed subsemigroup (resp. submonoid) of $S \times T$. The cases most important to us are those where S and T are either finite, in which case we consider them endowed with the discrete topology, or compact. We will use the following observation, which follows immediately from the fact that the continuous image of a compact set is compact.

Lemma 2.1 *Let S be a finite semigroup, let R be a compact semigroup and let $\sigma\colon R \to S$ and $\pi\colon R \to T$ be continuous morphisms with σ onto. Then the relational morphism $\varphi = \sigma^{-1}\pi\colon S \to T$ is closed.*

Let $\psi\colon S \to T$ be a closed relational morphism with S and T compact. For each closed subset X of S, its image $X\psi$ is closed.

If $\mathbf{V}$ is a pseudovariety of semigroups (resp. monoids) and $\mathbf{W}$ is a pseudovariety of semigroups, we define $\widehat{\mathbf{W}} \textcircled{\scriptsize M} \hat{\mathbf{V}}$ to be the class of all profinite semigroups (resp. monoids) S such that there exists a closed $\widehat{\mathbf{W}}$-relational morphism $\varphi\colon S \to T$ with $T \in \hat{\mathbf{V}}$. If $\mathbf{W}$ is a pseudovariety of ordered semigroups, the class $\widehat{\mathbf{W}} \textcircled{\scriptsize M} \hat{\mathbf{V}}$ of profinite ordered semigroups (resp. monoids) is defined similarly. The connection between $\widehat{\mathbf{W}} \textcircled{\scriptsize M} \hat{\mathbf{V}}$ and the pro-($\mathbf{W} \textcircled{\scriptsize M} \mathbf{V}$) semigroups (resp. monoids) is described in Proposition 4.4 below.

Finally, we will need the following technical factorization lemma.

Proposition 2.2 *Let $\mathbf{V}$ be a pseudovariety of semigroups (resp. ordered semigroups), let A be a profinite set, let S and T be profinite semigroups (resp. ordered semigroup) with S A-generated and T pro-$\mathbf{V}$, and let $\varphi\colon S \to T$ be a closed relational morphism. Then φ factors through $\hat{F}_A(\mathbf{V})$. More precisely, if $\sigma\colon \widehat{A^+} \to S$ is an onto continuous morphism and $\pi\colon \widehat{A^+} \to \hat{F}_A(\mathbf{V})$ is the natural projection, then there exists a continuous morphism $\tau\colon \hat{F}_A(\mathbf{V}) \to T$ such that $\varphi = \sigma^{-1}\pi\tau$.*

The same result holds if we replace everywhere semigroups by monoids and $\widehat{A^+}$ by $\widehat{A^*}$.

Proof. Let R be the graph of φ and let $\varphi = \alpha^{-1}\beta$ be the canonical factorization of φ. By Corollary 1.2, there exists a continuous morphism $\rho\colon \widehat{A^+} \to R$ such that $\rho\alpha = \sigma$. By Theorem 1.1, the continuous morphism $\rho\beta$ factors through $\hat{F}_A(\mathbf{V})$ since T is pro-$\mathbf{V}$. That is, there exists a continuous morphism $\tau\colon \hat{F}_A(\mathbf{V}) \to T$ such that $\pi\tau = \rho\beta$.

$$\begin{tikzcd}
 & \widehat{A^+} \arrow[r, "\pi"] \arrow[dl, "\sigma"'] \arrow[d, "\rho"] & \hat{F}_A(\mathbf{V}) \arrow[d, "\tau"] \\
S \arrow[rr, bend right, "\varphi"'] & R \arrow[l, "\alpha"'] \arrow[r, "\beta"] & T
\end{tikzcd}$$

Then $\sigma^{-1}\pi\tau = (\alpha^{-1}\rho^{-1})(\rho\beta) = \alpha^{-1}\beta = \varphi$. □

3 Mal'cev products and pointlike sets

As we saw in Section 2, membership of a finite semigroup or monoid in a Mal'cev product of the form $\mathbf{W} \textcircled{\scriptsize M} \mathbf{V}$ involves the consideration of all the relational morphisms into the (finite) elements of $\mathbf{V}$. Our first important result is a characterization of $\mathbf{W} \textcircled{\scriptsize M} \mathbf{V}$ in terms of the properties of a specific relational morphism into a free pro-$\mathbf{V}$ object.

Theorem 3.1 *Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of semigroups. Let A be a profinite set, let S be a finite semigroup and let $\sigma\colon \widehat{A^+} \to S$ be an onto continuous morphism. Let also $\pi\colon \widehat{A^+} \to \hat{F}_A(\mathbf{V})$ be the natural projection. The following conditions are equivalent.*

(1) $S \in \mathbf{W} \textcircled{M} \mathbf{V}$.

(2) The relational morphism $\sigma^{-1}\pi$ *is a* $\mathbf{W}$*-relational morphism.*

(3) There exists a closed $\mathbf{W}$*-relational morphism from* S *into a pro-*$\mathbf{V}$ *semigroup.*

The analogous statements hold when S *is a monoid (resp.* S *is ordered) and* $\mathbf{V}$ *is a pseudovariety of monoids (resp.* $\mathbf{W}$ *is a pseudovariety of ordered semigroups).*

Proof. Let us first verify that (1) implies (2). If $S \in \mathbf{W} \textcircled{M} \mathbf{V}$, there exists a $\mathbf{W}$-relational morphism $\varphi\colon S \to T$ with $T \in \mathbf{V}$. By Proposition 2.2, there exists a continuous morphism $\tau\colon \hat{F}_A(\mathbf{V}) \to T$ such that $\varphi = \sigma^{-1}\pi\tau$. Let now e be an idempotent of $\hat{F}_A(\mathbf{V})$. Then $e\pi^{-1}\sigma \subseteq (e\tau)\tau^{-1}\pi^{-1}\sigma = (e\tau)\varphi^{-1}$. But $e\tau$ is an idempotent of T, so $e\tau\varphi^{-1} \in \mathbf{W}$, and hence $e\pi^{-1}\sigma \in \mathbf{W}$.

By Lemma 2.1, $\sigma^{-1}\pi$ is a closed relational morphism, so (2) implies (3).

There remains to verify that (3) implies (1). Let $\varphi\colon S \to T$ be a closed $\mathbf{W}$-relational morphism, with T pro-$\mathbf{V}$. Let $W_1, \ldots, W_k$ be the subsemigroups of S in $\mathbf{W}$. Our assumption implies that, for each idempotent e of T, $e\varphi^{-1}$ is contained in (in fact, is equal to) some W_i. For each $1 \le i \le k$, let $X_i = \{t \in T \mid t\varphi^{-1} \subseteq W_i\}$. Then

$$X_i = T \setminus \bigcup_{s \notin W_i} s\varphi.$$

Since φ is closed, each $s\varphi$ is closed by Lemma 2.1, and hence each X_i is open.

Since the topology of T is zero-dimensional, it has a basis of clopen sets, and it follows that for each idempotent e of T, there exists $1 \le i \le k$ and there exists a clopen set Y_e such that $e \in Y_e \subseteq X_i$. But $E(T)$ is closed and hence compact, and $E(T)$ is contained in the union of the open sets Y_e. So there exist finitely many idempotents $e_1, \ldots, e_n$ and corresponding indices $i_1, \ldots, i_n$ such that $Y_{e_j} \subseteq X_{i_j}$ for each j and $E(T) \subseteq \bigcup_{j=1}^n Y_{e_j}$.

By Proposition 1.8, the syntactic congruence of each Y_e is clopen. Therefore the intersection $\sim$ of the syntactic congruences of the Y_{e_j} $(1 \le j \le n)$ is clopen as well, and hence it has finite index by Lemma 1.7. Let $\tau\colon T \to V$ be the projection of T on its quotient by $\sim$. Since $\sim$ is closed, the projection τ is continuous. Moreover V is finite, so $V \in \mathbf{V}$.

Let now $e \in E(V)$. Then $e\tau^{-1}$ is a $\sim$-class and a closed subsemigroup of T: by compactness, it contains an idempotent. Therefore $e\tau^{-1}$ is contained in Y_{e_j} for some $1 \le j \le n$, and hence $e\tau^{-1} \subseteq X_{i_j}$. Therefore

$$e\tau^{-1}\varphi^{-1} = \bigcup_{t \in e\tau^{-1}} t\varphi^{-1} \subseteq W_{i_j}.$$

Thus $\varphi\tau\colon S \to V$ is a $\mathbf{W}$-relational morphism with $V \in \mathbf{V}$, that is, $S \in \mathbf{W} \textcircled{M} \mathbf{V}$. □

Corollary 3.2 *Let* $(\mathbf{W}_i)_{i \in I}$ *and* $\mathbf{V}$ *be pseudovarieties of semigroups. Then*

$$\Big(\bigcap_{i \in I} \mathbf{W}_i\Big) \textcircled{M} \mathbf{V} = \bigcap_{i \in I} (\mathbf{W}_i \textcircled{M} \mathbf{V}).$$

The analogous results hold for pseudovarieties of monoids, ordered semigroups and ordered monoids.

Proof. Let S be a finite semigroup, let A be a profinite set and let $\sigma\colon \widehat{A^+} \to S$ be an onto continuous morphism. With the notations of Theorem 3.1, S lies in $(\bigcap_{i\in I} \mathbf{W}_i) \textcircled{M} \mathbf{V}$ if and only if, for each idempotent e of $\hat{F}_A(\mathbf{V})$, $e\pi^{-1}\sigma$ lies in $\bigcap_{i\in I} \mathbf{W}_i$, that is, if and only if $\sigma^{-1}\pi$ is a $\mathbf{W}_i$-relational morphism for each i. This concludes the proof. □

Let now $\mathbf{V}$ be a pseudovariety of semigroups (resp. monoids). The $\mathbf{V}$*-pointlike subsets* of a finite semigroup (resp. monoid) S are the subsets X of S such that, for each relational morphism φ from S into an element T of $\mathbf{V}$, there exists $t \in T$ such that $X \subseteq t\varphi^{-1}$. The problem of the effective characterization of the $\mathbf{V}$-pointlike subsets of a finite semigroup (resp. monoid) is closely related to the membership problem of the varieties of the form $\mathbf{W} \textcircled{M} \mathbf{V}$ [9, 10, 11]. Using quite the same method as for Theorem 3.1, we obtain the following result.

Theorem 3.3 *Let $\mathbf{V}$ be a pseudovariety of semigroups, let A be a profinite set and let S be a finite semigroup. Let $\sigma\colon \widehat{A^+} \to S$ be an onto continuous morphism and let $\pi\colon \widehat{A^+} \to \hat{F}_A(\mathbf{V})$ be the natural projection. The $\mathbf{V}$-pointlike subsets of S are the $x\pi^{-1}\sigma$ ($x \in \hat{F}_A(\mathbf{V})$) and their subsets. The same results holds if we replace everywhere semigroups by monoids and $\widehat{A^+}$ by $\widehat{A^*}$.*

Proof. By Proposition 2.2, if $\varphi\colon S \to T$ is a relational morphism into a finite semigroup $T \in \mathbf{V}$, there exists a continuous morphism $\tau\colon \hat{F}_A(\mathbf{V}) \to T$ such that $\varphi = \sigma^{-1}\pi\tau$. In particular, for each $x \in \hat{F}_A(\mathbf{V})$, $x\pi^{-1}\sigma \subseteq (x\tau)\tau^{-1}\pi^{-1}\sigma = (x\tau)\varphi^{-1}$, and hence $x\pi^{-1}\sigma$ and its subsets are $\mathbf{V}$-pointlike subsets of S.

For the converse, let X be a $\mathbf{V}$-pointlike subset of S. Let us consider the (finite) family $S_1, \ldots, S_k$ of all subsets of S which are equal to $x\pi^{-1}\sigma$ for some $x \in \hat{F}_A(\mathbf{V})$. For each $1 \le i \le k$, we fix an element x_i such that $S_i = x_i\pi^{-1}\sigma$. We then follow the scheme of the proof of Theorem 3.1: The relational morphism $\sigma^{-1}\pi$ is closed by Lemma 2.1, so for each i, the set

$$X_i = \{x \in \hat{F}_A(\mathbf{V}) \mid x\pi^{-1}\sigma \subseteq S_i\} = \hat{F}_A(\mathbf{V}) \setminus \bigcup_{s\notin S_i} s\sigma^{-1}\pi$$

is open. Thus, for each $x \in \hat{F}_A(\mathbf{V})$, there exist an index $1 \le i \le n$ and a clopen set Y_x such that $x \in Y_x \subseteq X_i$. Using the compactness of $\hat{F}_A(\mathbf{V})$ it follows that there exist finitely many indices $i_1, \ldots, i_n$ and clopen sets $Y_1, \ldots, Y_n$ such that $\hat{F}_A(\mathbf{V}) = \bigcup_{j=1}^n Y_j$ and $Y_j \subseteq X_{i_j}$.

We then use Proposition 1.8 to show that there exists a clopen congruence $\sim$ on $\hat{F}_A(\mathbf{V})$ which refines the syntactic congruences of $Y_1, \ldots, Y_n$. Let $\tau : \hat{F}_A(\mathbf{V}) \to T$ be the projection onto the quotient of $\hat{F}_A(\mathbf{V})$ by $\sim$. Then T is finite and τ is continuous by Lemma 1.7, and hence $T \in \mathbf{V}$.

Now, consider the relational morphism $\sigma^{-1}\pi\tau\colon S \to T$. By definition of $\mathbf{V}$-pointlike subsets, there exists $t \in T$ such that $X \subseteq t\tau^{-1}\pi^{-1}\sigma$. But $t\tau^{-1}$ is a $\sim$-class. Therefore it is contained in some Y_j and hence in some X_{i_j}. Thus, for each $x \in t\tau^{-1}$, we have $x\pi^{-1}\sigma \subseteq S_{i_j} = x_{i_j}\pi^{-1}\sigma$, and hence

$$t\tau^{-1}\pi^{-1}\sigma = \bigcup_{x\in t\tau^{-1}} x\pi^{-1}\sigma \subseteq x_{i_j}\pi^{-1}\sigma.$$

This concludes the proof. □

We have defined the $\mathbf{V}$-pointlike subsets of S only in the unordered case, that is when $\mathbf{V}$ is a pseudovariety of semigroups (or monoids) and S is a semigroup (or monoid). Let us explain here why there is no reason to consider the analogous definition when S is ordered and $\mathbf{V}$ is a class of ordered algebras. We have observed in Section 2 that the definition of a relational morphism between ordered semigroups depends only on the algebraic structure of the semigroups, not on their order. If $\mathbf{V}$ is a pseudovariety of, say, ordered semigroups, let $\mathbf{V}'$ be the pseudovariety of semigroups generated by the semigroups S such that $(S, \leq) \in \mathbf{V}$ for some order $\leq$ on S. Then $\mathbf{V}'$ is the class of homomorphic images of elements of $\mathbf{V}$ [19, Proposition 3.6]. Thus for each alphabet A, the directed systems of A-generated elements of $\mathbf{V}$ and of A-generated elements of $\mathbf{V}'$ are cofinal, so their projective limits are equal, and hence the semigroup part of the ordered semigroup $\hat{F}_A(\mathbf{V})$ coincides with the semigroup $\hat{F}_A(\mathbf{V}')$.

Moreover, we verify that a subset X of a finite semigroup S is a $\mathbf{V}$-pointlike if and only if it is a $\mathbf{V}'$-pointlike. Suppose indeed that X is a $\mathbf{V}'$-pointlike subset. Then for each relational morphism $\varphi\colon S \to V$ with $(V, \leq) \in \mathbf{V}$ for some order $\leq$, we have $V \in \mathbf{V}'$, so there exists $v \in V$ such that $X \subseteq v\varphi^{-1}$, and hence X is a $\mathbf{V}$-pointlike. Conversely, if X is a $\mathbf{V}$-pointlike and if $\varphi\colon S \to T$ is a relational morphism with $T \in \mathbf{V}'$, there exists a morphism $\psi\colon V \to T$ with $(V, \leq) \in \mathbf{V}$ for some order $\leq$. Since $\varphi\psi^{-1}\colon S \to V$ is a relational morphism, there exists $v \in \mathbf{V}$ such that $X \subseteq v(\varphi\psi^{-1})^{-1} = (v\psi)\varphi^{-1}$. Thus, X is a $\mathbf{V}'$-pointlike. Thus Theorem 3.3 allows also the computation of pointlikes in the ordered case.

4 Identities defining a Mal'cev product

An important consequence of Theorem 3.1 is the description of a defining set of identities for the pseudovariety $\mathbf{W} \circledM \mathbf{V}$. First, we introduce the following notation. Let A be a profinite set and let B be a finite alphabet. Let also $\vec{z} = (z_b)_{b \in B} \in (\widehat{A^*})^B$. Letting $b\varphi = z_b$ for each $b \in B$ induces a continuous morphism $\varphi\colon \widehat{B^*} \to \widehat{A^*}$ by Theorem 1.1. For each $x \in \widehat{B^*}$, we write $x(\vec{z})$ for $x\varphi$. That is, $x(\vec{z})$ is the image of x by a continuous morphism which depends on $\vec{z}$. We say that $x(\vec{z})$ *is obtained from x by substituting the z_b for the variables of x.*

Let B be a finite alphabet, let $x, y \in \widehat{B^+}$ and let $\mathbf{V}$ be a pseudovariety of semigroups (resp. monoids). An identity $x' = y'$ (or $x' \leq y'$) is *obtained from the identity $x = y$ (or $x \leq y$) by $\mathbf{V}$-substitution* if there exist a profinite set A and a vector $\vec{z} = (z_b)_{b \in B} \in (\widehat{A^+})^B$ (resp. $(\widehat{A^*})^B$) such that $\mathbf{V}$ satisfies the identities $z_b = z_{b'} = z_b^\omega$ for all $b, b' \in B$, $x' = x(\vec{z})$ and $y' = y(\vec{z})$.

We can now describe a defining set of identities for a Mal'cev product.

Theorem 4.1 *Let $\mathbf{V}$ be a pseudovariety of semigroups or monoids, and let $\mathbf{W}$ be a pseudovariety of semigroups (resp. ordered semigroups). Let Σ be a defining set of identities of $\mathbf{W}$, each using a finite number of variables. Then $\mathbf{W} \circledM \mathbf{V}$ is defined by the identities of the form $x(\vec{z}) = y(\vec{z})$ (resp. $x(\vec{z}) \leq y(\vec{z})$) obtained from the elements of Σ by $\mathbf{V}$-substitution. In fact it suffices to consider those identities where the z_b use a finite number of variables.*

The proof of Theorem 4.1 follows immediately from the two following lemmas.

Lemma 4.2 *Let $\mathbf{V}$ be a pseudovariety of semigroups (resp. monoids), let B be a finite alphabet and let $x, y \in \widehat{B^+}$. Let $\mathbf{W}$ be a pseudovariety of semigroups satisfying $x = y$. Then $\mathbf{W} \circledM \mathbf{V}$ satisfies all the identities obtained from $x = y$ by $\mathbf{V}$-substitution.*

Similarly, if $\mathbf{W}$ *is a pseudovariety of ordered semigroups satisfying* $x \leq y$*, then* $\mathbf{W} \mathbin{\textcircled{\scriptsize M}} \mathbf{V}$ *satisfies all the identities obtained from* $x \leq y$ *by* $\mathbf{V}$*-substitution.*

Proof. Let $\vec{z} = (z_b)_{b\in B} \in (\widehat{A^+})^B$ be such that $\mathbf{V}$ satisfies $z_b = z_{b'} = z_b^\omega$ for all $b, b' \in B$, let $\sigma\colon \widehat{A^+} \to S$ be a continuous morphism and let $\pi\colon \widehat{A^+} \to \hat{F}_A(\mathbf{V})$ be the natural projection. We want to show that $x(\vec{z})\sigma = y(\vec{z})\sigma$. For each $b \in B$, we let $b\varphi = z_b$. This induces a continuous morphism $\varphi\colon \widehat{B^+} \to \widehat{A^+}$, and $x(\vec{z}) = x\varphi$, $y(\vec{z}) = y\varphi$. By Proposition 1.4, the hypothesis on $\vec{z}$ implies the existence of an idempotent e of $\hat{F}_A(\mathbf{V})$ such that $b\varphi\pi = z_b\pi = e$ for each $b \in B$. Since $e\pi^{-1}$ is a closed subsemigroup of $\widehat{A^+}$, it follows that $\widehat{B^+}\varphi \subseteq e\pi^{-1}$ and $\widehat{B^+}\varphi\sigma \subseteq e\pi^{-1}\sigma$.

$$\begin{tikzcd}
\widehat{B^+} \arrow[r, "\varphi"] \arrow[dr, "\varphi\sigma"'] & \widehat{A^+} \arrow[r, "\pi"] \arrow[d, "\sigma"] & \hat{F}_A(\mathbf{V}) \\
& S &
\end{tikzcd}$$

Since $S \in \mathbf{W} \mathbin{\textcircled{\scriptsize M}} \mathbf{V}$, we have $\widehat{A^+}\sigma \in \mathbf{W} \mathbin{\textcircled{\scriptsize M}} \mathbf{V}$, and hence $e\pi^{-1}\sigma \in \mathbf{W}$ by Theorem 3.1. So $e\pi^{-1}\sigma$ satisfies $x = y$ and, in particular, $x\varphi\sigma = y\varphi\sigma$, that is, $x(\vec{z})\sigma = y(\vec{z})\sigma$. Thus, S satisfies $x(\vec{z}) = y(\vec{z})$. $\square$

The proof of Theorem 4.1 will be completed by applying the following lemma to the case of a finite semigroup.

Lemma 4.3 *Let* $\mathbf{V}$ *be a pseudovariety of semigroups (resp. monoids) and let* $\mathbf{W}$ *be a pseudovariety of semigroups or ordered semigroups. Let* Σ *be a defining set of identities for* $\mathbf{W}$*, each using a finite number of variables, let* A *be a profinite set and let* $\sigma\colon \widehat{A^+} \to S$ *(resp.* $\sigma\colon \widehat{A^*} \to S$*) be an onto continuous morphism. Finally, let* $\pi\colon \widehat{A^+} \to \hat{F}_A(\mathbf{V})$ *(resp.* $\pi\colon \widehat{A^*} \to \hat{F}_A(\mathbf{V})$*) be the natural projection. If* S *satisfies all the identities obtained from the elements of* Σ *by* $\mathbf{V}$*-substitution, then* $\sigma^{-1}\pi\colon S \to \hat{F}_A(\mathbf{V})$ *is a closed pro-*$\mathbf{W}$*-relational morphism.*

Proof. Let e be an idempotent of $\hat{F}_A(\mathbf{V})$. We need to show that $e\pi^{-1}\sigma$ is pro-$\mathbf{W}$. By Corollary 1.6, this reduces to proving that $e\pi^{-1}\sigma$ satisfies Σ. Let $(x = y) \in \Sigma$, with $x, y \in \widehat{B^+}$ for some finite set B. Let φ be a continuous morphism from $\widehat{B^+}$ into $e\pi^{-1}\sigma \subseteq S$. For each $b \in B$, $b\varphi \in (e\pi^{-1})\sigma$, so we can fix an element $z_b \in e\pi^{-1}$ such that $b\varphi = z_b\sigma$. Then letting $b\psi = z_b$ for each $b \in B$ induces a continuous morphism $\psi\colon \widehat{B^+} \to \widehat{A^+}$ such that $\psi\sigma = \varphi$.

$$\begin{tikzcd}
\widehat{A^+} \arrow[r, "\pi"] \arrow[dr, "\sigma"] & \hat{F}_A(\mathbf{V}) \\
B^+ \arrow[u, "\psi"] \arrow[r, "\varphi"'] & S
\end{tikzcd}$$

If $\vec{z} = (z_b)_{b\in B}$, then $x\psi = x(\vec{z})$ and $y\psi = y(\vec{z})$. Moreover, by definition, $z_b\pi = z_{b'}\pi = e$, so, by Proposition 1.4, $\mathbf{V}$ satisfies $z_b = z_{b'} = z_b^\omega$ for all $b, b' \in B$. Now, by hypothesis, S satisfies $x(\vec{z}) = y(\vec{z})$, and hence $x(\vec{z})\sigma = y(\vec{z})\sigma$. So $x\psi\sigma = y\psi\sigma$, that is, $x\varphi = y\varphi$. Thus, $e\pi^{-1}\sigma$ satisfies $x = y$. This proves that $e\pi^{-1}\sigma$ satisfies Σ, which concludes the proof. $\square$

Lemmas 4.2 and 4.3 also prove the following characterization of the pro-($\mathbf{W} ⓜ \mathbf{V}$) semigroups.

Proposition 4.4 *Let* $\mathbf{V}$ *be a pseudovariety of semigroups (resp. monoids) and let* $\mathbf{W}$ *be a pseudovariety of semigroups semigroups. Let* $\mathbf{Z} = \mathbf{W} ⓜ \mathbf{V}$*. Then* $\widehat{\mathbf{Z}} = \widehat{\mathbf{W}} ⓜ \widehat{\mathbf{V}}$*. That is,* $\widehat{\mathbf{W}} ⓜ \widehat{\mathbf{V}}$ *is the class of pro-(*$\mathbf{W} ⓜ \mathbf{V}$*) semigroups (resp. monoids). The analogous result hods if* $\mathbf{W}$ *is a pseudovariety of ordered semigroups.*

Proof. Let us first consider a semigroup S in $\widehat{\mathbf{W}} ⓜ \widehat{\mathbf{V}}$: there exists a closed $\widehat{\mathbf{W}}$-relational morphism $\varphi\colon S \to T$ with $T \in \widehat{\mathbf{V}}$. Let $\psi\colon S \to S'$ be a continuous morphism onto a finite semigroup S'. For each idempotent e of T, the semigroup $e\varphi^{-1}\psi$ is a finite continuous quotient of $e\varphi^{-1}$, and hence $e\varphi^{-1}\psi \in \mathbf{W}$. Moreover, $\psi^{-1}\varphi$ is a closed relational morphism by Lemma 2.1. Therefore, it follows from Theorem 3.1 that $S' \in \mathbf{W} ⓜ \mathbf{V}$. Thus all the finite continuous homomorphic images of S are in $\mathbf{W} ⓜ \mathbf{V}$, and hence S is pro-($\mathbf{W} ⓜ \mathbf{V}$).

Conversely, let us assume that S is a pro-($\mathbf{W} ⓜ \mathbf{V}$) semigroup and let Σ be a set of defining identities for $\mathbf{W}$. By Corollary 1.6, S satisfies all the identities satisfied by $\mathbf{W} ⓜ \mathbf{V}$, and hence, by Theorem 4.1, S satisfies all the identities obtained from the elements of Σ by $\mathbf{V}$-substitution. By Lemma 4.3, it follows that S admits a closed $\widehat{\mathbf{W}}$-relational morphism into a pro-$\mathbf{V}$ semigroup. That is, $S \in \widehat{\mathbf{W}} ⓜ \widehat{\mathbf{V}}$. □

5 Application to the computation of Mal'cev products

We say that a class $\mathbf{V}$ of finite semigroups (resp. monoids, ordered semigroups, ordered monoids) is *decidable* if, given a finite semigroup (resp. monoid, ordered semigroup, ordered monoid) S, there is an algorithm to decide whether $S \in \mathbf{V}$. The aim of this section is to study the decidability of the pseudovarieties of the form $\mathbf{V} ⓜ \mathbf{Z}$. Let us start with the pseudovarieties of the form $\mathbf{V} ⓜ \mathbf{J}_1$. It is well known that $\mathbf{J}_1$ admits a finite free object over each finite alphabet A, namely the power set $\mathcal{P}(A)$. In particular, $\hat{F}_A(\mathbf{J}_1) = \mathcal{P}(A)$. The natural projection $\kappa\colon \widehat{A^+} \to \hat{F}_A(\mathbf{J}_1) = \mathcal{P}(A)$ is called the *content morphism*. Note that if $u \in A^*$ is a word, then $u\kappa$ is the alphabetic content of u, that is, $u\kappa$ is the set of letters occurring in u. The following result was observed in [24] in the unordered case.

Theorem 5.1 *Let* $\mathbf{V}$ *be a pseudovariety of semigroups (resp. ordered semigroups), let* A *be a finite alphabet, let* S *be a finite semigroup (resp. ordered semigroup) and let* $\sigma\colon A^+ \to S$ *be an onto morphism. Then* $S \in \mathbf{V} ⓜ \mathbf{J}_1$ *if and only if* $(B\kappa^{-1} \cap A^+)\sigma \in \mathbf{V}$ *for each subset* B *of* A*. In particular, if* $\mathbf{V}$ *is decidable, then* $\mathbf{V} ⓜ \mathbf{J}_1$ *is decidable.*

Proof. Let us first notice that the morphism $\sigma\colon A^+ \to S$ admits a unique continuous extension $\hat{\sigma}$ to $\widehat{A^+}$. By Theorem 3.1, $S \in \mathbf{V} ⓜ \mathbf{J}_1$ if and only if $B\kappa^{-1}\hat{\sigma} \in \mathbf{V}$ for each subset B of A. Since S and $\hat{F}_A(\mathbf{J}_1)$ are finite, each $B\kappa^{-1}$ and each $s\hat{\sigma}^{-1}$ ($s \in S$) is clopen. But A^+ is dense in $\widehat{A^+}$, so for each $s \in S$ and each $B \subseteq A$ such that $s \in B\kappa^{-1}\hat{\sigma}$, the set $s\hat{\sigma}^{-1} \cap B\kappa^{-1} \cap A^+$ is non empty. That is, $B\kappa^{-1}\hat{\sigma} = (B\kappa^{-1} \cap A^+)\hat{\sigma} = (B\kappa^{-1} \cap A^+)\sigma$. In order to conclude the proof, it suffices to observe that $B\kappa^{-1} \cap A^+$ is the set of all words with alphabetic content exactly B, and that $(B\kappa^{-1} \cap A^+)\sigma$ is computable (say, by induction on $|B|$: $B\kappa^{-1} \cap A^+ = \bigcup_{b\in B}((B \setminus \{b\})\kappa^{-1} \cap A^+)\sigma(bB^*)\sigma)$. □

Note. This result is in fact a special case of a more general statement regarding the products of the form $\mathbf{V} ⓜ \mathbf{Z}$ where $\mathbf{Z}$ admits a finite free object over each finite alphabet [24, Theorem 1.2]

The products of the form $\mathbf{V} \circledM \mathbf{G}$ have received particular attention in the literature [11]. More generally, let $\mathbf{H}$ be a pseudovariety of groups (that is, a pseudovariety of monoids consisting only of groups) and let S be a finite monoid. The $\mathbf{H}$-*kernel* of S is the set $K_{\mathbf{H}}(S) = \bigcap 1\tau^{-1}$, where the intersection runs over all relational morphisms $\tau\colon S \to H$ from S into a group H in $\mathbf{H}$. Even though there may exist infinitely many relational morphisms τ from a given finite monoid S into a group in $\mathbf{H}$, there are only finitely many possible values for the subset $1\tau^{-1} \subseteq S$. It is easy to deduce from this observation that there exists a single relational morphism $\tau\colon S \to H$ into a group $H \in \mathbf{H}$ such that $1\tau^{-1} = K_{\mathbf{H}}(S)$. This leads to the following well-known result [11].

Proposition 5.2 *Let $\mathbf{H}$ be a pseudovariety of groups and let $\mathbf{V}$ be a pseudovariety of semigroups or ordered semigroups. Let S be a finite monoid. Then*

$$S \in \mathbf{V} \circledM \mathbf{H} \qquad \textit{if and only if} \qquad K_{\mathbf{H}}(S) \in \mathbf{V}.$$

In particular, if $\mathbf{V}$ is decidable and if there is an algorithm to decide membership in $K_{\mathbf{H}}(S)$, then $\mathbf{V} \circledM \mathbf{H}$ is decidable.

Deciding membership in $K_{\mathbf{H}}(S)$ is a difficult question in general. Ash proved that $K_{\mathbf{G}}(S)$ is decidable for any finite semigroup S [7]. More precisely, we say that a subset T of S is closed under weak conjugacy if $sTt \subseteq T$ and $tTs \subseteq T$ for all $s, t \in S$ such that $sts = s$. Ash proved that $K_{\mathbf{G}}(S)$ is the smallest submonoid of S closed under weak conjugacy. Ribes and Zalesskiĭ gave a new proof of this result [21], and later proved the decidability of $K_{\mathbf{G}_p}(S)$ for any prime p (where $\mathbf{G}_p$ is the pseudovariety of p-groups) [22]. Recently, Margolis, Sapir and Weil proved the decidability of $K_{\mathbf{G}_{nil}}(S)$, where $\mathbf{G}_{nil}$ is the pseudovariety of nilpotent groups [14].

Let us also note the following proposition on membership in $K_{\mathbf{H}}(S)$.

Proposition 5.3 *Let S be a finite monoid, let $\sigma\colon \widehat{A^*} \to S$ be a continuous morphism and let $x \in \widehat{A^*}$. If $\mathbf{H}$ satisfies $x = 1$, then $x\sigma \in K_{\mathbf{H}}(S)$.*

Proof. Let $\pi\colon \widehat{A^*} \to \hat{F}_A(\mathbf{H})$ be the canonical projection, and let $x \in \widehat{A^*}$ such that $\mathbf{H}$ satisfies $x = 1$. Then, by Proposition 1.4, $x\pi = 1$. Let now $\varphi\colon S \to H$ be a relational morphism with $H \in \mathbf{H}$. By Proposition 2.2, there exists a continuous morphism $\tau\colon \hat{F}_A(\mathbf{H}) \to H$ such that $\varphi = \sigma^{-1}\pi\tau$. But $x\pi\tau = 1$, so $x\sigma \in 1\varphi^{-1}$.

$$\begin{tikzcd}
\widehat{A^*} \arrow[r, "\pi"] \arrow[d, "\sigma"'] & \hat{F}_A(\mathbf{H}) \arrow[d, "\tau"] \\
S \arrow[r, "\varphi"] & H
\end{tikzcd}$$

This holds for all φ, and hence $x\sigma \in K_{\mathbf{H}}(S)$. □

We now consider products of the form $\mathbf{V} \circledM \mathbf{Nil}$, where $\mathbf{Nil}$ is the pseudovariety of all finite nilpotent semigroups. A finite semigroup S is *nilpotent* if and only if S contains a single idempotent, which is a zero. That is,

$$\mathbf{Nil} = [\![x^\omega y = yx^\omega = x^\omega]\!].$$

It is known that, for any finite alphabet A, $\hat{F}_A(\mathbf{Nil}) = A^+ \cup \{0\}$, where the topology is characterized by the fact that a sequence of words $(u_n)_{n\geq 0}$ converges to 0 if and only if the length of the u_n tends to infinity.

We will give defining sets of identities for products of the form $\mathbf{V} \circledM \mathbf{Nil}$. First note the following proposition, due to Almeida and Azevedo [3, Corollary 3.4].

Proposition 5.4 *Let $x \in \widehat{A^+} \setminus A^+$. Then there exist elements $y, z, t \in \widehat{A^+}$ such that $x = yz^\omega t$.*

We can now state and prove the following theorem.

Theorem 5.5 *Let $\mathbf{V}$ be a pseudovariety of semigroups or ordered semigroups, and let Σ be a set of identities defining $\mathbf{V}$. Then $\mathbf{V} \circledM \mathbf{Nil}$ is defined by the identities obtained from Σ by substituting for each variable a the value $a_1 a_2^\omega a_3$.*

Proof. Let $x, y \in \widehat{A^+}$. Let $B = \{a_1, a_2, a_3 \mid a \in A\}$ and let $\vec{t} = (t_a)_{a\in A}$ with $t_a = a_1 a_2^\omega a_3$ ($t_a \in \widehat{B^+}$). Let S be a finite semigroup. Using Proposition 5.4 and the fact that $\mathbf{Nil}$ satisfies $u^\omega = v^\omega$, it is easily verified that S satisfies the identity $x(\vec{t}) = y(\vec{t})$ (resp. $x(\vec{t}) \leq y(\vec{t})$) if and only if it satisfies all the identities obtained from $x = y$ (resp. $x \leq y$) by $\mathbf{Nil}$-substitution. The theorem now follows from Theorem 4.1. $\square$

We derive from Theorem 5.5 the following results, the first of which is well known [2], and the third of which can be found in [25].

Corollary 5.6 *The following equalities hold.*

(1) $\mathbf{G} \circledM \mathbf{Nil} = [\![x^\omega = y^\omega]\!]$.

(2) $\mathbf{Com} \circledM \mathbf{Nil} = [\![xy^\omega z = zy^\omega x]\!]$.

(3) $[\![x = x^2]\!] \circledM \mathbf{Nil} = [\![xy^\omega z = (xy^\omega z)^2]\!] = [\![xy^\omega z = (xy^\omega z)^w]\!]$.

(4) $\mathbf{J}_1 \circledM \mathbf{Nil} = [\![xy^\omega z = zy^\omega x = (xy^\omega z)^w]\!]$.

Proof. Here, $\mathbf{G}$ is viewed as a pseudovariety of semigroups, and as such it is defined by the following identities: $\mathbf{G} = [\![x^\omega y = yx^\omega = y]\!]$. By Theorem 5.5, $\mathbf{G} \circledM \mathbf{Nil}$ satisfies $(xy^\omega z)^\omega uv^\omega w = uv^\omega w(xy^\omega z)^\omega = uv^\omega w$, so $\mathbf{G} \circledM \mathbf{Nil}$ satisfies $y^\omega v^\omega = v^\omega y^\omega = v^\omega$ (letting $x = z = y^\omega$ and $u = w = v^\omega$). By symmetry, it follows that $\mathbf{G} \circledM \mathbf{Nil}$ satisfies $y^\omega = v^\omega$. Conversely, the pseudovariety $[\![x^\omega = y^\omega]\!]$ satisfies $xy^\omega = xx^\omega = x^\omega x = y^\omega x$, and therefore it satisfies $(xy^\omega z)^\omega uv^\omega w = v^\omega uv^\omega w = uv^\omega w$. Thus $\mathbf{G} \circledM \mathbf{Nil} = [\![x^\omega = y^\omega]\!]$.

By Theorem 5.5, $\mathbf{Com} \circledM \mathbf{Nil}$ is defined by the identity

$$(xy^\omega z)(uv^\omega w) = (uv^\omega w)(xy^\omega z).$$

Therefore it satisfies $xy^\omega = y^\omega xy^\omega$. Symmetrically, it satisfies $xy^\omega = y^\omega x$ and hence

$$xy^\omega z = (xy^\omega)(y^\omega z) = (y^\omega z)(xy^\omega) = zy^\omega x.$$

Conversely, let $\mathbf{V} = [\![xy^\omega z = zy^\omega x]\!]$. Then $\mathbf{V}$ satisfies $xy^\omega = y^\omega xy^\omega = y^\omega x$, that is, the idempotents are central in $\hat{F}_A(\mathbf{V})$. It follows that $\mathbf{V}$ satisfies

$$y^\omega a_1 a_2 = a_1 y^\omega a_2 = a_2 y^\omega a_1 = y^\omega a_2 a_1,$$

and, more generally,

$$y^\omega a_1 \cdots a_n = y^\omega a_{\sigma(1)} \cdots a_{\sigma(n)}$$

for any permutation σ of $\{1, \ldots, n\}$, since any such permutation is a product of 2-cycles. In particular, $\mathbf{V}$ satisfies

$$xy^\omega zuv^\omega w = y^\omega v^\omega xzuw = y^\omega v^\omega uwxz = uv^\omega wxy^\omega z.$$

Therefore $\mathbf{Com} \circledM \mathbf{Nil} = [\![xy^\omega z = zy^\omega x]\!]$.

The calculation of $[\![x = x^2]\!] \circledM \mathbf{Nil}$ is immediate by Theorem 5.5, and that of $\mathbf{J}_1 \circledM \mathbf{Nil}$ follows from the previous ones by Corollary 3.2. $\square$

We say that a pseudovariety is *finitely based* if it can be defined by a finite set of identities, and that it is *of finite rank* if it can be defined by a set of identities using a fixed finite set of variables. Then we have the following immediate corollaries.

Corollary 5.7 *Let $\mathbf{V}$ be a pseudovariety of semigroups or ordered semigroups. If $\mathbf{V}$ is finitely based (resp. of finite rank), then so is $\mathbf{V} \circledM \mathbf{Nil}$.*

Corollary 5.8 *Let $\mathbf{V}$ be a pseudovariety of semigroups (resp. ordered semigroups), let S be a finite semigroup (resp. ordered semigroup) and let E be the set of idempotents of S. Then $S \in \mathbf{V} \circledM \mathbf{Nil}$ if and only if $SES \in \mathbf{V}$. In particular, if $\mathbf{V}$ is decidable, then so is $\mathbf{V} \circledM \mathbf{Nil}$.*

Let us now consider products of the form $\mathbf{V} \circledM \mathbf{J}$, where $\mathbf{J}$ is the pseudovariety of finite $\mathcal{J}$-trivial semigroups. It is well-known that $\mathbf{J} = \mathbf{Nil} \circledM \mathbf{J}_1$ [16]. Almeida [2] and Almeida and Azevedo [3] gave a detailed study of the structure of the free pro-$\mathbf{J}$ semigroups.

Proposition 5.9 *Let $x, y \in \widehat{A^+}$.*

(1) $\mathbf{J}$ satisfies $x^\omega = y^\omega$ if and only if $x\kappa = y\kappa$.

(2) $\mathbf{J}$ satisfies $x = x^\omega$ if and only if $x = uv^\omega w$ for some $u, v, w \in \widehat{A^+}$ such that $(uw)\kappa \subseteq v\kappa = x\kappa$.

Proof. These results are consequences of [3, Corollary 4.8], [3, Theorem 4.12] and Proposition 5.4 above. $\square$

Theorem 5.10 *Let $\mathbf{V}$ be a pseudovariety of semigroups or ordered semigroups and let Σ be a set of identities defining $\mathbf{V}$. Then $\mathbf{V} \circledM \mathbf{J}$ is defined by the set of all identities obtained from Σ by substituting, for each variable $a \in A$, an element of the form $a_1 a_2^\omega a_3$ with $a_1, a_2, a_3 \in \widehat{B^+}$ for some alphabet B, $(a_1a_3)\kappa \subseteq a_2\kappa$ and $a_2\kappa = B$ for all $a \in A$. Moreover,*

$$\mathbf{V} \circledM \mathbf{J} = \mathbf{V} \circledM (\mathbf{Nil} \circledM \mathbf{J}_1) = (\mathbf{V} \circledM \mathbf{Nil}) \circledM \mathbf{J}_1 = [\![\Sigma']\!].$$

Proof. The equality $(\mathbf{V} \circledM \mathbf{Nil}) \circledM \mathbf{J}_1 = [\![\Sigma']\!]$ is an immediate consequence of Theorem 5.5 and of Theorem 3.1 in the case of products with $\mathbf{J}_1$. Moreover

$$\mathbf{V} \circledM \mathbf{J} = \mathbf{V} \circledM (\mathbf{Nil} \circledM \mathbf{J}_1) \subseteq (\mathbf{V} \circledM \mathbf{Nil}) \circledM \mathbf{J}_1$$

[24, Lemma 1.4]. Finally, the elements of Σ' are identities obtained from Σ by $\mathbf{J}$-substitutions, so $[\![\Sigma']\!] \subseteq \mathbf{V} \circledM \mathbf{J}$, which concludes the proof. $\square$

Corollary 5.11 *If* $\mathbf{V}$ *is decidable, then so is* $\mathbf{V} \circledM \mathbf{J}$.

Proof. It is an immediate consequence of Theorem 5.1 and Corollary 5.8. □

Let $\mathbf{DG}$ be the pseudovariety of semigroups in which each regular $\mathcal{D}$-class is a group.

Corollary 5.12 *The following equalities hold.*

(1) $\mathbf{G} \circledM \mathbf{J} = \mathbf{DG}$;

(2) $\mathbf{Com} \circledM \mathbf{J} = [\![xy^\omega z = zy^\omega x \mid (xz)\kappa \subseteq y\kappa]\!]$.

(3) $[\![x = x^2]\!] \circledM \mathbf{J} = [\![xy^\omega z = (xy^\omega z)^w \mid (xz)\kappa \subseteq y\kappa]\!]$.

(4) $\mathbf{J}_1 \circledM \mathbf{J} = [\![xy^\omega z = zy^\omega x = (xy^\omega z)^w \mid (xz)\kappa \subseteq y\kappa]\!]$.

Proof. By Theorem 5.10 and Corollary 5.6, $\mathbf{G} \circledM \mathbf{J} = [\![x^\omega = y^\omega]\!] \circledM \mathbf{J}_1 = [\![x^\omega = y^\omega \mid x\kappa = y\kappa]\!]$. But the latter pseudovariety is equal to $\mathbf{DG}$ by [2, Theorem 8.1.7].

The other computations are immediate applications of Theorem 5.10. □

Our last example deals with the pseudovariety $\mathbf{B}_1$, which plays an important role in the study of the dot-depth hierarchy [19]. By definition,

$$\mathbf{B}_1 = [\![(x^\omega sy^\omega tx^\omega)^\omega x^\omega sy^\omega vx^\omega (x^\omega uy^\omega vx^\omega)^\omega = (x^\omega sy^\omega tx^\omega)^\omega (x^\omega uy^\omega vx^\omega)^\omega]\!].$$

Proposition 5.13 $\mathbf{B}_1 \circledM \mathbf{Nil} = \mathbf{B}_1$ *and* $\mathbf{B}_1 \circledM \mathbf{J}_1 = \mathbf{B}_1 \circledM \mathbf{J}$.

Proof. It is clear that $\mathbf{B}_1$ is contained in $\mathbf{B}_1 \circledM \mathbf{Nil}$. Conversely, by Theorem 5.5, $\mathbf{B}_1 \circledM \mathbf{Nil}$ satisfies the defining identity of $\mathbf{B}_1$ where x, y, s, t, u, v have been replaced respectively by $x_1x_2^\omega x_3$, $y_1y_2^\omega y_3$, $s_1s_2^\omega s_3$, $t_1t_2^\omega t_3$, $u_1u_2^\omega u_3$ and $v_1v_2^\omega v_3$. Letting $x_1 = x_2 = x_3 = x^\omega$, $y_1 = y_2 = y_3 = y^\omega$, $s_2 = s_3 = u_2 = u_3 = y^\omega$ and $t_2 = t_3 = v_2 = v_3 = x^\omega$, it follows that $\mathbf{B}_1 \circledM \mathbf{Nil}$ satisfies

$$(x^\omega s_1y^\omega t_1x^\omega)^\omega x^\omega s_1y^\omega v_1x^\omega (x^\omega u_1y^\omega v_1x^\omega)^\omega = (x^\omega s_1y^\omega t_1x^\omega)^\omega (x^\omega u_1y^\omega v_1x^\omega)^\omega,$$

so that $\mathbf{B}_1 \circledM \mathbf{Nil} = \mathbf{B}_1$.

The equality $\mathbf{B}_1 \circledM \mathbf{J}_1 = \mathbf{B}_1 \circledM \mathbf{J}$ then follows from Theorem 5.10. □

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