A topological approach to transductions

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Abstract

This paper is a contribution to the mathematical foundations of the theory of automata. We give a topological characterization of the transductions τ from a monoid M into a monoid N, such that if R is a recognizable subset of N, $\tau^{-1}(R)$ is a recognizable subset of M. We impose two conditions on the monoids, which are fullfilled in all cases of practical interest: the monoids must be residually finite and, for every positive integer n, must have only finitely many congruences of index n. Our solution proceeds in two steps. First we show that such a monoid, equipped with the so-called Hall distance, is a metric space whose completion is compact. Next we prove that τ can be lifted to a map $\widehat{\tau}$ from M into the set of compact subsets of the completion of N. This latter set, equipped with the Hausdorff metric, is again a compact monoid. Finally, our main result states that τ^{-1} preserves recognizable sets if and only if $\widehat{\tau}$ is continuous.

1 Introduction

This paper is a contribution to the mathematical foundations of automata theory. We are mostly interested in the study of transductions τ from a monoid M into another monoid N such that, for every recognizable subset R of N, $\tau^{-1}(R)$ is a recognizable subset of M. We propose to call such transductions *continuous*, a term introduced in [7] in the case where M is a finitely generated free monoid.

In mathematics, the word "continuous" generally refers to a topology. The aim in this paper is to find appropriate topologies for which our use of the term *continuous* coincides with its usual topological meaning.

This problem was already solved when τ is a *mapping* from A^* into B^*. In this case, a result which goes back at least to the eighties (see [14]) states that τ is continuous in our sense if and only if it is continuous for the *profinite topology* on A^* and B^*. We shall not attempt to define here the profinite topology and the reader is referred to [4, 21, 3] for more details. This result actually extends to mappings from A^* into a residually finite monoid N, thanks to a result of [7] recalled below (Proposition 2.3).

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However, a transduction $\tau : M \to N$ is not a map from M into N, but a map from M into the set of subsets of N, which calls for a more sophisticated solution, since it does not suffice to find an appropriate topology on N. Our solution proceeds in two steps. We first show, under fairly general assumptions on M and N, which are fulfilled in all cases of practical interest, that M and N can be equipped with a metric, the Hall metric, for which they become metric monoids whose completion (as metric spaces) is compact. Next we prove that τ can be lifted to a map $\widehat{\tau}$ from M into the monoid $\mathcal{K}(\widehat{N})$ of compact subsets of $\widehat{N}$, the completion of N. The monoid $\mathcal{K}(\widehat{N})$, equipped with the Hausdorff metric, is again a compact monoid. Finally, our main result states that τ is continuous in our sense if and only if $\widehat{\tau}$ is continuous in the topological sense.

Our paper is organised as follows. Basic results on recognizable sets and transductions are recalled in Section 2. Section 3 is devoted to topology and is divided into several subsections: 3.1 is a reminder of basic notions in topology, metric monoids and the Hall metric are introduced in 3.2 and 3.3, respectively. The connections between clopen and recognizable sets are discussed in 3.5 and 3.6 deals with the monoid of compact subsets of a compact monoid. Our main result on transductions is presented in Section 4. Examples like the transductions $(x, n) \to x^n$ and $x \to x^*$ are studied in Section 5. The paper ends with a short conclusion.

2 Recognizable languages and transductions

Recall that a subset P of a monoid M is *recognizable* if there exists a finite monoid F and a monoid morphism $\varphi : M \to F$ and a subset Q of F such that $P = \varphi^{-1}(Q)$. The set of recognizable subsets of M is denoted by $\mathrm{Rec}(M)$. Recognizable subsets are closed under boolean operations, quotients and inverse morphisms. By Kleene's theorem, a subset of a finitely generated free monoid is recognizable if and only if it is rational.

The description of the recognizable subsets of a product of monoids was given by Mezei (see [5, p. 54] for a proof).

Theorem 2.1 (Mezei) *Let $M_1, \ldots, M_n$ be monoids. A subset of $M_1 \times \cdots \times M_n$ is recognizable if and only if it is a finite union of subsets of the form $R_1 \times \cdots \times R_n$, where $R_i \in \mathrm{Rec}(M_i)$.*

The following result is perhaps less known. See [5, p. 61].

Proposition 2.2 *Let $A_1, \ldots, A_n$ be finite alphabets. Then $\mathrm{Rec}(A_1^* \times A_2^* \times \cdots \times A_n^*)$ is closed under concatenation product.*

Given two monoids M and N, recall that a *transduction* from M into N is a relation on M and N, that we shall also consider as a map from M into the monoid of subsets of N. If X is a subset of M, we set

$$\tau(X) = \bigcup_{x \in X} \tau(x)$$

Observe that "transductions commute with union": if $(X_i)_{i \in I}$ is a family of subsets of M, then

$$\tau(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} \tau(X_i)$$

If $\tau : M \to N$ is a transduction, then the inverse relation $\tau^{-1} : N \to M$ is also a transduction, and if P is a subset of N, the following formula holds:

$$\tau^{-1}(P) = \{x \in M \mid \tau(x) \cap P \neq \emptyset\}$$

A transduction $\tau : M \to N$ *preserves recognizable sets* if, for every set $R \in \mathrm{Rec}(M)$, $\tau(R) \in \mathrm{Rec}(N)$. It is said to be *continuous* if τ^{-1} preserves recognizable sets, that is, if for every set $R \in \mathrm{Rec}(N)$, $\tau^{-1}(R) \in \mathrm{Rec}(M)$.

Continuous transductions were characterized in [7] when M is a finitely generated free monoid. Recall that a transduction $\tau : M \to N$ is *rational* if it is a rational subset of $M \times N$. According to [7], a transduction $\tau : A^* \to N$ is *residually rational* if, for any morphism $\varphi : N \to F$, where F is a finite monoid, the transduction $\varphi \circ \tau : A^* \to F$ is rational. We can now state:

Proposition 2.3 [7] *A transduction $\tau : A^* \to N$ is continuous if and only if it is residually rational.*

3 Topology

The aim of this section is to give a topological characterization of the transductions τ from a monoid into another monoid such that τ^{-1} preserves recognizable sets.

Even if topology is undoubtedly part of the background of the average mathematician, it is probably not a daily concern of the specialists in automata theory to which this paper is addressed. For those readers whose memories in topology might be somewhat blurry, we start with a brief overview of some key concepts in topology used in this paper.

3.1 Basic notions in topology

A *metric* d on a set E is a map from E into the set of nonnegative real numbers satisfying the three following conditions, for all $(x, y, z) \in E^3$:

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(y, x) = d(x, y)$,

(3) $d(x, z) \leqslant d(x, y) + d(y, z)$

A metric is an *ultrametric* if (3) is replaced by the stronger condition

(3′) $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$

A *metric space* is a set E together with a metric d on E. Given a positive real number ε and an element x in E, the *open ball* of center x and radius ε is the set

$$B(x, \varepsilon) = \{y \in E \mid d(x, y) < \varepsilon\}.$$

A function φ from a metric space (E, d) into another metric space (E', d') is uniformly continuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $(x, x') \in E^2$, $d(x, x') < \delta$ implies $d(\varphi(x), \varphi(x')) < \varepsilon$. It is an *isometry* if, for all $(x, x') \in E^2$, $d(\varphi(x), \varphi(x')) = d(x, x')$.

A sequence $(x_n)_{n \geqslant 0}$ of elements of E is converging to a limit $x \in E$ if, for every $\varepsilon > 0$, there exists N such that for all integers $n > N$, $d(x_n, x) < \varepsilon$. It is a *Cauchy sequence* if, for every positive real number $\varepsilon > 0$, there is an integer

N such that for all integers $p, q \geqslant N$, $d(x_p, x_q) < \varepsilon$. A metric space E is said to be *complete* if every Cauchy sequence of elements of E converges to a limit.

For any metric space E, one can construct a complete metric space $\widehat{E}$, containing E as a dense[1] subspace and satisfying the following universal property: if F is any complete metric space and φ is any uniformly continuous function from E to F, then there exists a unique uniformly continuous function $\widehat{\varphi} : \widehat{E} \to F$ which extends φ. The space $\widehat{E}$ is determined up to isometry by this property, and is called the *completion* of E.

Metric spaces are a special instance of the more general notion of topological space. A *topology* on a set E is a set $\mathcal{T}$ of subsets of E, called the *open sets* of the topology, satisfying the following conditions:

(1) $\emptyset$ and E are in $\mathcal{T}$,

(2) $\mathcal{T}$ is closed under arbitrary union,

(3) F is closed under finite intersection.

The complement of an open set is called a *closed set.* The *closure* of a subset X of E, denoted by $\overline{X}$, is the intersection of the closed sets containing X. A subset of E is *dense* if its closure is equal to E. A *topological space* is a set E together with a topology on E. A map from a topological space into another one is *continuous* if the inverse image of each open set is an open set.

A *basis* for a topology on E is a collection $\mathcal{B}$ of open subsets of E such that every open set is the union of elements of $\mathcal{B}$. The open sets of the *topology generated by* $\mathcal{B}$ are by definition the arbitrary unions of elements of $\mathcal{B}$. In the case of a metric space, the open balls form a basis of the topology.

A topological space $(E, \mathcal{T})$ is *Hausdorff* if for each $u, v \in E$ with $u \neq v$, there exist *disjoint* open sets U and V such that $u \in U$ and $v \in V$. A family of open sets $(U_i)_{i \in I}$ is said to *cover* a topological space $(E, \mathcal{T})$ if $E = \bigcup_{i \in I} U_i$. A topological space $(E, \mathcal{T})$ is said to be *compact* if it is Hausdorff and if, for each family of open sets covering E, there exists a finite subfamily that still covers E.

To conclude this section, we remind the reader of a classical result on compact sets.

Proposition 3.1 *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set E. Suppose that $(E, \mathcal{T})$ is compact and that $(E, \mathcal{T}')$ is Hausdorff. If $\mathcal{T}' \subseteq \mathcal{T}$, then $\mathcal{T}' = \mathcal{T}$.*

Proof. Consider the identity map ι from $(E, \mathcal{T})$ into $(E, \mathcal{T}')$. It is a continuous map, since $\mathcal{T}' \subseteq \mathcal{T}$. Therefore, if F is closed in $(E, \mathcal{T})$, it is compact, and its continuous image $\iota(F)$ in the Hausdorff space $(E, \mathcal{T}')$ is also compact, and hence closed. Thus ι^{-1} is also continuous, whence $\mathcal{T}' = \mathcal{T}$. $\square$

3.2 Metric monoids

Let M be a monoid. A monoid morphism $\varphi : M \to N$ *separates* two elements u and v of M if $\varphi(u) \neq \varphi(v)$. By extension, we say that a monoid N *separates* two elements of M if there exists a morphism $\varphi : M \to N$ which separates them. A

[1] see definition below

monoid is *residually finite* if any pair of distinct elements of M can be separated by a finite monoid.

Residually finite monoids include finite monoids, free monoids, free groups and many others. They are closed under direct products and thus monoids of the form $A_1^* \times A_2^* \times \cdots \times A_n^*$ are also residually finite.

A *metric monoid* is a monoid equipped with a metric for which its multiplication is uniformly continuous.

Finite monoids, equipped with the discrete metric, are examples of metric monoids. More precisely, if M is a finite monoid, the *discrete metric* d is defined by

$$d(s,t) = \begin{cases} 0 & \text{if } s = t \\ 1 & \text{otherwise} \end{cases}$$

In the sequel, we shall systematically consider finite monoids as metric monoids.

Morphisms between metric monoids are required to be *uniformly continuous*.

3.3 Hall metric

Any residually finite monoid M can be equipped with the *Hall metric* d, defined as follows. We first set, for all $(u,v) \in M^2$:

$$r(u,v) = \min\{\mathrm{Card}(N) \mid N \text{ separates } u \text{ and } v\ \}$$

Then we set $d(u,v) = 2^{-r(u,v)}$, with the usual conventions $\min \emptyset = +\infty$ and $2^{-\infty} = 0$. Let us first establish some general properties of d.

Proposition 3.2 *In a residually finite monoid M, d is an ultrametric. Furthermore, the relations $d(uw,vw) \leqslant d(u,v)$ and $d(wu,wv) \leqslant d(u,v)$ hold for every $(u,v,w) \in M^3$.*

Proof. It is clear that $d(u,v) = d(v,u)$. Suppose that $d(u,v) = 0$. Then u cannot be separated from v by any finite monoid, and since M is residually finite, this shows that $u = v$. Finally, let $(u,v,w) \in M^3$. First assume that $u \neq w$. Since M is residually finite, u and w can be separated by some finite monoid F. Therefore F separates either u and v, or v and w. It follows that $\min\{(r(u,v), r(v,w)\} \leqslant r(u,w)$ and hence $d(u,w) \leqslant \max\{d(u,v), d(v,w)\}$. This relation clearly also holds if $u = w$.

The second assertion is trivial. A finite monoid separating uw and vw certainly separates u and v. Therefore $d(uw,vw) \leqslant d(u,v)$ and dually, $d(wu,wv) \leqslant d(u,v)$. □

The next two propositions state two fundamental properties of the Hall metric.

Proposition 3.3 *Multiplication on M is uniformly continuous for the Hall metric. Thus (M,d) is a metric monoid.*

Proof. It is a consequence of the following relation

$$d(uv,u'v') \leqslant \max\{d(uv,uv'), d(uv',u'v')\} \leqslant \max\{d(v,v'), d(u,u')\}$$

which follows from Proposition 3.2. □

Proposition 3.4 *Let M be a residually finite monoid. Then any morphism from (M, d) onto a finite discrete monoid is uniformly continuous.*

Proof. Let φ be a morphism from M onto a finite monoid F. Then by definition of d, $d(u, v) < 2^{-|F|}$ implies $\varphi(u) = \varphi(v)$. Thus φ is uniformly continuous. $\square$

The completion of the metric space (M, d), denoted by $(\widehat{M}, d)$, is called the *Hall completion* of M. Since multiplication on M is uniformly continuous, it extends, in a unique way, into a multiplication onto $\widehat{M}$, which is again uniformly continuous. In particular, $\widehat{M}$ is a metric, complete monoid. Similarly, Proposition 3.4 extends to $\widehat{M}$: any morphism from $(\widehat{M}, d)$ onto a finite discrete monoid is uniformly continuous.

We now characterize the residually finite monoids M such that $\widehat{M}$ is compact.

Proposition 3.5 *Let M be a residually finite monoid. Then $\widehat{M}$ is compact if and only if, for every positive integer n, there are only finitely many congruences of index n on M.*

Proof. Recall that the completion of a metric space is compact if and only if it is *precompact*, that is, for every $\varepsilon > 0$, it can be covered by a finite number of open balls of radius ε.

Denote by C_n the set of all congruences on M of index $\leqslant n$ and let ρ_n be the intersection of all congruences of C_n.

Assume first that $\widehat{M}$ is compact and let $n > 0$. Since M is precompact, there exist a finite subset F of M such that the balls $B(x, 2^{-n})$, with $x \in F$, cover M. Let $x \in F$ and $y \in B(x, 2^{-n})$. Then $r(x, y) > n$ and thus the monoids of size $\leqslant n$ cannot separate x from y. It follows that $x \;\rho\; y$ for each $\rho \in C_n$ and thus $x \;\rho_n\; y$. Therefore ρ_n is a congruence of finite index, whose index is at most $|F|$. Now each congruence of C_n is coarser than ρ_n, and since there are only finitely many congruences coarser than ρ_n, C_n is finite.

Conversely, assume that, for every positive integer n, there are only finitely many congruences of index n on M. Given $\varepsilon > 0$, let n be an integer such that $2^{-n} < \varepsilon$. Since C_n is finite, ρ_n is a congruence of finite index on M. Let F be a finite set of representatives of the classes of ρ_n. If $x \in F$ and $x \;\rho_n\; y$, then $\varphi(x) = \varphi(y)$ for each morphism φ from M onto a monoid of size $\leqslant n$. Thus $r(x, y) > n$ and so $d(x, y) < 2^{-n} < \varepsilon$. It follows that M is covered by a finite number of open balls of radius ε. Therefore $\widehat{M}$ is compact. $\square$

An important sufficient condition is given in the following corollary.

Corollary 3.6 *Let M be a residually finite monoid. If M is finitely generated, then $\widehat{M}$ is compact.*

Proof. Let $n > 0$. There are only finitely many monoids of size n. Since M is finitely generated, there are only finitely many morphisms from M onto a monoid of size n. Now, since any congruence of index n is the kernel of such a morphism, there are only finitely many congruences on M of index n. It follows by Proposition 3.5 that $\widehat{M}$ is compact. $\square$

3.4 Hall-compact monoids

Proposition 3.5 justifies the following terminology. We will say that a monoid M is *Hall-compact* if it is residually finite and if, for every positive integer n, there are only finitely many congruences of index n on M. Proposition 3.5 can now be rephrased as follows:

"A residually finite monoid M is Hall-compact if and only if $\widehat{M}$ is compact."

and Corollary 3.6 states that

"Every residually finite and finitely generated monoid is Hall-compact."

The class of Hall-compact monoids includes most of the examples used in practice: finitely generated free monoids (resp. groups), finitely generated free commutative monoids (resp. groups), finite monoids, trace monoids, finite products of such monoids, etc.
The next proposition shows that the converse to Corollary 3.6 does not hold.

Proposition 3.7 *There exists a residually finite, non finitely generated monoid M such that $\widehat{M}$ is compact.*

Proof. Let P be the set of all prime numbers and let $M = \prod_{p \in P} \mathbb{Z}/p\mathbb{Z}$, where $\mathbb{Z}/p\mathbb{Z}$ denotes the additive cyclic group of order p. It is clear that M is residually finite. Furthermore, in a finitely generated commutative group, the subgroup consisting of all elements of finite period is finite [12]. It follows that M is not finitely generated.

Let $n > 0$ and let $\varphi : M \to N$ be a morphism from M onto a finite monoid of size n. Since M is a commutative group, N is also a commutative group. For every prime $p > n$, the order of the image of a generator of $\mathbb{Z}/p\mathbb{Z}$ must divide p and be $\leqslant n$, hence the image of this generator must be 0. Consequently, any such morphism is determined by the images of the generators of $\mathbb{Z}/p\mathbb{Z}$ for $p \leqslant n$, and so there are only finitely many of them. Therefore there are only finitely many congruences on M of index n and so $\widehat{M}$ is compact by Proposition 3.5. $\square$

3.5 Clopen sets versus recognizable sets

Recall that a *clopen* subset of a topological space is a subset which is both open and closed. A topological space is *zero-dimensional* if its clopen subsets form a basis for its topology.

Proposition 3.8 *Let M be a residually finite monoid. Then (M, d) and $(\widehat{M}, d)$ are zero-dimensional.*

Proof. The open balls of the form

$$B(x, 2^{-n}) = \{y \in M \mid d(x, y) < 2^{-n}\}$$
$$\widehat{B}(x, 2^{-n}) = \{y \in \widehat{M} \mid d(x, y) < 2^{-n}\}$$

where x belongs to M (resp. $\widehat{M}$) and n is a positive integer, form a basis of the Hall topology of M (resp. $\widehat{M}$). But these balls are clopen since

$$\{y \mid d(x,y) < 2^{-n}\} = \{y \mid d(x,y) \leqslant 2^{-(n+1)}\}$$

It follows that (M,d) and $(\widehat{M},d)$ are zero-dimensional. $\square$

Proposition 3.8 implies that if M is a Hall-compact monoid then $\widehat{M}$ is *profinite* (see [1, 3, 4, 21] for the definition of profinite monoids and several equivalent properties), but we will not use this result in this paper.

We now give three results relating clopen sets and recognizable sets. The first one is due to Hunter [9, Lemma 4], the second one summarizes results due to Numakura [13] (see also [17, 2]). The third result is stated in [3] for free profinite monoids. For the convenience of the reader, we present a self-contained proof of the second and the third results.

Recall that the *syntactic congruence* of a subset P of a monoid M is defined, for all $u, v \in M$, by

$$s \sim t \quad \text{if and only if, for all } (x,y) \in M^2,\ xuy \in P \Leftrightarrow xvy \in P.$$

It is the coarsest congruence of M which saturates P.

Lemma 3.9 (Hunter's Lemma) *In a compact monoid, the syntactic congruence of a clopen set is clopen.*

Proposition 3.10 *In a compact monoid, every clopen subset is recognizable. If M is a residually finite monoid, then every recognizable subset of $\widehat{M}$ is clopen.*

Proof. Let M be a compact monoid, let P be a clopen subset of M and let $\sim_P$ be its syntactic congruence. By Hunter's Lemma, $\sim_P$ is clopen. Thus for each $x \in M$, there exists an open neighborhood G of x such that $G \times G \subseteq \sim_P$. Therefore G is contained in the $\sim_P$-class of x. This proves that the $\sim_P$-classes form an open partition of M. By compactness, this partition is finite, and hence P is recognizable.

Suppose now that M is a residually finite monoid and let P be a recognizable subset of $\widehat{M}$. Let $\eta : \widehat{M} \to F$ be the syntactic morphism of P. Since P is recognizable, F is finite and by Proposition 3.4, η is uniformly continuous. Now $P = \eta^{-1}(Q)$ for some subset Q of F. Since F is discrete and finite, Q is a clopen subset of F and hence P is also clopen. $\square$

The last result of this subsection is a clone of a standard result on free profinite monoids (see [3] for instance).

Proposition 3.11 *Let M be a Hall-compact monoid, let P be a subset of M and let $\overline{P}$ be its closure in $\widehat{M}$. The following conditions are equivalent:*

(1) *P is recognizable,*

(2) *$P = K \cap M$ for some clopen subset K of $\widehat{M}$,*

(3) *$\overline{P}$ is clopen in $\widehat{M}$ and $P = \overline{P} \cap M$,*

(4) *$\overline{P}$ is recognizable in $\widehat{M}$ and $P = \overline{P} \cap M$.*

Proof. (1) implies (2). Let $\varphi : M \to F$ be the syntactic monoid of P and let $Q = \varphi(P)$. Since F is finite, φ is uniformly continuous by Proposition 3.4 and extends to a uniformly continuous morphism $\widehat{\varphi} : \widehat{M} \to F$. Thus $K = \widehat{\varphi}^{-1}(Q)$ is clopen and satisfies $K \cap M = P$.

(2) implies (3). Suppose that $P = K \cap M$ for some clopen subset K of $\widehat{M}$. Then the equality $P = \overline{P} \cap M$ follows from the following sequence of inclusions

$$P \subseteq \overline{P} \cap M = (\overline{K \cap M}) \cap M \subseteq \overline{K} \cap M = K \cap M = P.$$

Furthermore, since K is open and M is dense in $\widehat{M}$, $K \cap M$ is dense in K. Thus $\overline{P} = \overline{K \cap M} = \overline{K} = K$. Thus $\overline{P}$ is clopen in $\widehat{M}$.

The equivalence of (3) and (4) follows from Proposition 3.10, which shows that in $\widehat{M}$, the notions of clopen set and of recognizable set are equivalent.

(4) implies (1). Let $\widehat{\varphi} : \widehat{M} \to F$ be the syntactic monoid of $\overline{P}$ and let $Q = \widehat{\varphi}(\overline{P})$. Let φ be the restriction of $\widehat{\varphi}$ to M. Then we have $P = \overline{P} \cap M = \widehat{\varphi}^{-1}(Q) \cap M = \varphi^{-1}(Q)$. Thus P is recognizable. $\square$

3.6 The monoid of compact subsets of a compact monoid

Let M be a compact monoid, and let $\mathcal{K}(M)$ be the monoid of compact subsets of M. The *Hausdorff metric* on $\mathcal{K}(M)$ is defined as follows. For $K, K' \in \mathcal{K}(M)$, let

$$\delta(K, K') = \sup_{x \in K} \inf_{x' \in K'} d(x, x')$$

$$h(K, K') = \begin{cases} \max(\delta(K, K'), \delta(K', K)) & \text{if } K \text{ and } K' \text{ are nonempty,} \\ 0 & \text{if } K \text{ and } K' \text{ are empty,} \\ 1 & \text{otherwise.} \end{cases}$$

The last case occurs when one and only one of K or K' is empty. By a standard result of topology, $\mathcal{K}(M)$, equipped with this metric, is compact.

The next result states a property of clopen sets which will be crucial in the proof of our main result.

Proposition 3.12 *Let M be a Hall-compact monoid, let C be a clopen subset of $\widehat{M}$ and let $\varphi : \mathcal{K}(\widehat{M}) \to \mathcal{K}(\widehat{M})$ be the map defined by $\varphi(K) = K \cap C$. Then φ is uniformly continuous for the Hausdorff metric.*

Proof. Since C is open, every element $x \in C$ belongs to some open ball $B(x, \varepsilon)$ contained in C. Since $\widehat{M}$ is compact, C is also compact and can be covered by a finite number of these open balls, say $(B(x_i, \varepsilon_i))_{1 \leqslant i \leqslant n}$.

Let $\varepsilon > 0$ and let $\delta = \min\{1, \varepsilon, \varepsilon_1, \ldots, \varepsilon_n\}$. Suppose that $h(K, K') < \delta$ with $K \neq K'$. Then $K, K' \neq \emptyset$, $d(x, K') < \delta$ for every $x \in K$ and $d(x', K) < \delta$ for every $x' \in K'$. Suppose that $x \in K \cap C$. Since $d(x, K') < \delta$, we have $d(x, x') < \delta$ for some $x' \in K'$. Furthermore, $x \in B(x_i, \varepsilon_i)$ for some $i \in \{1, \ldots, n\}$. Since d is an ultrametric, the relations $d(x, x_i) < \varepsilon_i$ and $d(x, x') < \delta \leqslant \varepsilon_i$ imply that $d(x', x_i) < \varepsilon_i$ and thus $x' \in B(x_i, \varepsilon_i)$. Now since $B(x_i, \varepsilon_i)$ is contained in C, $x' \in K' \cap C$ and hence $d(x, K' \cap C) < \delta < \varepsilon$. By symmetry, $d(x', K \cap C) < \varepsilon$ for every $x' \in K' \cap C$. Hence $h(K \cap C, K' \cap C') < \varepsilon$ and φ is continuous. $\square$

4 Transductions

Let M and N be Hall-compact monoids and let $\tau : M \to N$ be a transduction. Then $\mathcal{K}(\widehat{N})$, equipped with the Hausdorff metric, is also a compact monoid. Define a map $\widehat{\tau} : M \to \mathcal{K}(\widehat{N})$ by setting, for each $x \in M$, $\widehat{\tau}(x) = \overline{\tau(x)}$.

Theorem 4.1 *The transduction τ^{-1} preserves the recognizable sets if and only if $\widehat{\tau}$ is uniformly continuous.*

Proof. Suppose that τ^{-1} preserves the recognizable sets. Let $\varepsilon > 0$. Since $\widehat{N}$ is compact, it can be covered by a finite number of open balls of radius $\varepsilon/2$, say

$$\widehat{N} = \bigcup_{1 \leqslant i \leqslant k} B(x_i, \varepsilon/2)$$

Since $\widehat{N}$ is zero-dimensional by Proposition 3.8, its clopen subsets constitute a basis for its topology. Thus every open ball $B(x_i, \varepsilon/2)$ is a union of clopen sets and $\widehat{N}$ is a union of clopen sets each of which is contained in a ball of radius $\varepsilon/2$. By compactness, we may assume that this union is finite. Thus

$$\widehat{N} = \bigcup_{1 \leqslant j \leqslant n} C_j$$

where each C_j is a clopen set contained in, say, $B(x_{i_j}, \varepsilon/2)$. It follows now from Proposition 3.11 that $C_j \cap N$ is a recognizable subset of N. Since τ^{-1} preserves the recognizable sets, the sets $L_j = \tau^{-1}(C_j \cap N)$ are also recognizable. By Proposition 3.4, the syntactic morphism of L_j is uniformly continuous and thus, there exists δ_j such that $d(u, v) < \delta_j$ implies $u \sim_{L_j} v$. Taking $\delta = \min\{\delta_j \mid 1 \leqslant j \leqslant n\}$, we have for all $(u, v) \in M^2$,

$$d(u, v) < \delta \Rightarrow \text{ for all } j \in \{1, \ldots, n\},\ u \sim_{L_j} v.$$

We claim that, whenever $d(u, v) < \delta$, we have $h(\overline{\tau(u)}, \overline{\tau(v)}) < \varepsilon$. By definition,

$$L_j = \{x \in M \mid \tau(x) \cap C_j \cap N \neq \emptyset\}$$

Suppose first that $\tau(u) = \emptyset$. Then $u \notin \bigcup_{1 \leqslant j \leqslant n} L_j$. Since $u \sim_{L_j} v$ for every j, it follows that $v \notin \bigcup_{1 \leqslant j \leqslant n} L_j$, so $\tau(v) \cap C_j \cap N \neq \emptyset$ for $1 \leqslant j \leqslant n$. Since $N = \bigcup_{1 \leqslant j \leqslant n} (C_j \cap N)$, it follows that $\tau(v) = \emptyset$. by symmetry, we conclude that $\tau(u) = \emptyset$ if and only if $\tau(v) = \emptyset$.

Thus we may assume that both $\tau(u)$ and $\tau(v)$ are nonempty. Let $y \in \tau(u)$. Then $y \in C_j \cap N$ for some $j \in \{1, \ldots, n\}$ and so $u \in L_j$. Since $u \sim_{L_j} v$, it follows that $v \in L_j$ and hence there exists some $z \in \tau(v)$ such that $z \in C_j \cap N$. Since $C_j \subseteq B(x_{i_j}, \varepsilon/2)$, we obtain $d(x_{i_j}, y) < \varepsilon/2$ and $d(x_{i_j}, z) < \varepsilon/2$, whence $d(y, z) < \varepsilon/2$ since d is an ultrametric. Thus $d(y, \overline{\tau(v)}) < \varepsilon/2$. Since $\tau(u)$ is dense in $\overline{\tau(u)}$, it follows that $d(x, \overline{\tau(v)}) \leqslant \varepsilon/2$ for every $x \in \overline{\tau(u)}$ and so

$$\delta(\overline{\tau(u)}, \overline{\tau(v)}) \leqslant \varepsilon/2 < \varepsilon.$$

By symmetry, $\delta(\overline{\tau(v)}, \overline{\tau(u)}) < \varepsilon$ and hence $h(\overline{\tau(u)}, \overline{\tau(v)}) < \varepsilon$ as required.

Next we show that if $\widehat{\tau}$ is uniformly continuous, then τ^{-1} preserves the recognizable sets. First, $\widehat{\tau}$ can be extended to a uniformly continuous mapping

$$\check{\tau} : \widehat{M} \to \mathcal{K}(\widehat{N}).$$

Let L be a recognizable subset of N. By Proposition 3.11, $L = C \cap N$ for some clopen subset C of $\widehat{N}$. Let

$$R = \{K \in \mathcal{K}(\widehat{N}) \mid K \cap C \neq \emptyset\}$$

We show that R is a clopen subset of $\mathcal{K}(\widehat{N})$. Let $\varphi : \mathcal{K}(\widehat{N}) \to \mathcal{K}(\widehat{N})$ be the map defined by $\varphi(K) = K \cap C$. By Proposition 3.12, φ is uniformly continuous and since $R = \varphi^{-1}(\{\emptyset\}^c) = [\varphi^{-1}(\{\emptyset\})]^c$, it suffices that $\{\emptyset\}$ is a clopen subset of $\mathcal{K}(\widehat{N})$. Since $B(\emptyset, 1) = \{\emptyset\}$, $\{\emptyset\}$ is open. Let $K \in \{\emptyset\}^c$. Since $\emptyset \notin B(K, 1)$, we have $B(K, 1) \subseteq \{\emptyset\}^c$ and so $\{\emptyset\}^c$ is also open. Therefore $\{\emptyset\}$ is clopen and so is R. Since $\check{\tau}$ is continuous, $\check{\tau}^{-1}(R)$ is a clopen subset of $\widehat{M}$ and so $M \cap \check{\tau}^{-1}(R)$ is recognizable by Proposition 3.11. Now

$$\begin{aligned} M \cap \check{\tau}^{-1}(R) &= \{u \in M \mid \check{\tau}(u) \in R\} \\ &= \{u \in M \mid \overline{\tau(u)} \in R\} \\ &= \{u \in M \mid \overline{\tau(u)} \cap C \neq \emptyset\} \end{aligned}$$

Since C is open, we have $\overline{\tau(u)} \cap C \neq \emptyset$ if and only if $\tau(u) \cap C \neq \emptyset$, hence

$$\begin{aligned} M \cap \check{\tau}^{-1}(R) &= \{u \in M \mid \tau(u) \cap C \neq \emptyset\} \\ &= \{u \in M \mid \tau(u) \cap L \neq \emptyset\} \\ &= \tau^{-1}(L) \end{aligned}$$

and so $\tau^{-1}(L)$ is a recognizable subset of M. Thus τ^{-1} preserves the recognizable sets. $\square$

5 Examples of continuous transductions

A large number of examples of continuous transductions can be found in the literature [20, 8, 11, 10, 18, 15, 16, 6, 7]. We state without proof two elementary results: continuous transductions are closed under composition and include constant transductions.

Proposition 5.1 *Let $L \subseteq N$ and let $\kappa_L : M \to N$ be the transduction defined by $\kappa_L(x) = L$. Then κ_L is continuous.*

Theorem 5.2 *The composition of two continuous transductions is a continuous transduction.*

Continuous transductions are also closed under product, in the following sense:

Proposition 5.3 *Let $\tau_1 : M \to N_1$ and $\tau_2 : M \to N_2$ be continuous transductions. Then the transduction $\tau : M \to N_1 \times N_2$ defined by $\tau(x) = \tau_1(x) \times \tau_2(x)$ is continuous.*

Proof. Let $R \in \mathrm{Rec}(N_1 \times N_2)$. By Mezei's Theorem, we have $R = \bigcup_{i=1}^{n} K_i \times L_i$ for some $K_i \in \mathrm{Rec}\, N_1$ and $L_i \in \mathrm{Rec}\, N_2$. Hence

$$\begin{aligned}
\tau^{-1}(R) &= \{x \in M \mid \tau(x) \cap R \neq \emptyset\} \\
&= \Big\{x \in M \mid (\tau_1(x) \times \tau_2(x)) \cap (\bigcup_{i=1}^{n} K_i \times L_i) \neq \emptyset\Big\} \\
&= \bigcup_{i=1}^{n} \big\{x \in M \mid \tau_1(x) \cap K_i \neq \emptyset \text{ and } \tau_2(x) \cap L_i \neq \emptyset\big\} \\
&= \bigcup_{i=1}^{n} \big(\tau_1^{-1}(K_i) \,\cap\, \tau_2^{-1}(L_i)\big)
\end{aligned}$$

Since τ_1 and τ_2 are continuous, each of the sets $\tau_1^{-1}(K_i)$ and $\tau_2^{-1}(L_i)$ is recognizable and thus $\tau^{-1}(R)$ is recognizable. It follows that τ is continuous. □

Further examples will be presented in a forthcoming paper. We just mention here a simple but non trivial example. An automata-theoretic proof of this result was given in [19] and we provide here a purely algebraic proof.

Proposition 5.4 *The function $\tau : M \times \mathbb{N} \to M$ defined by $\tau(x, n) = x^n$ is continuous.*

Proof. Let $R \in \mathrm{Rec}\, M$. Then

$$\tau^{-1}(R) = \{(x, n) \in M \times \mathbb{N} \mid x^n \in R\}.$$

Let $\eta : M \to F$ be the syntactic morphism of R in M and, for each $s \in F$, let $P_s = \{n \in \mathbb{N} \mid s^n \in \eta(R)\}$. Then we have

$$\begin{aligned}
\tau^{-1}(R) &= \{(x, n) \in M \times \mathbb{N} \mid x^n \in R\} \\
&= \{(x, n) \in M \times \mathbb{N} \mid \eta(x) = s \text{ for some } s \in F \text{ such that } s^n \in \eta(R)\} \\
&= \{(x, n) \in M \times \mathbb{N} \mid x \in \eta^{-1}(s) \text{ for some } s \in F \text{ such that } n \in P_s\} \\
&= \bigcup_{s \in F} \eta^{-1}(s) \times P_s.
\end{aligned}$$

Each set $\eta^{-1}(s)$ is recognizable by construction, and thus it suffices to show that $P_s \in \mathrm{Rec}\, \mathbb{N}$ for each $s \in F$. Given a finite cyclic monoid generated by a and some element b of this monoid, the set $\{n \in \mathbb{N} \mid a^n = b\}$ is either empty or an arithmetic progression. Applying this fact to the finite cyclic submonoid generated by s in F, we conclude that $P_s \in \mathrm{Rec}\, \mathbb{N}$ as required. Thus $\tau^{-1}(R) \in \mathrm{Rec}(M \times \mathbb{N})$ and hence τ is continuous. □

Corollary 5.5 *The transduction $\sigma : M \to M$ defined by $\sigma(x) = x^*$ is continuous.*

Proof. Let $\kappa_{\mathbb{N}} : M \to \mathbb{N}$ be defined by $\kappa_{\mathbb{N}}(x) = \mathbb{N}$. By Proposition 5.1, $\kappa_{\mathbb{N}}$ is continuous. Since the identity map is trivially continuous, it follows from

Proposition 5.3 that $\kappa : M \to M \times \mathbb{N}$ defined by $\kappa(x) = \{x\} \times \mathbb{N}$ is continuous. Let $\tau : M \times \mathbb{N} \to M$ be defined by $\tau(x, n) = x^n$. By Proposition 5.4, τ is continuous. Since $\sigma = \tau \circ \kappa$, it follows from Theorem 5.2 that σ is continuous. $\square$

6 Conclusion

We gave some topological arguments to call *continuous* transductions whose inverse preserve recognizable sets. It remains to see whether this approach can be pushed forward to use purely topological arguments, like fixpoint theorems, to obtain new results on transductions and recognizable sets.

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