A Reiterman theorem for pseudovarieties of finite first-order structures[1]

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Résumé: Nous étendons le théorème de Reiterman aux structures du premier ordre : une classe de structures du premier ordre finies est une pseudovariété si et seulement si elle est définie par un ensemble d'identités dans une structure profinie relativement libre (pseudoidentités).

Abstract: We extend Reiterman's theorem to first-order structures: a class of finite first-order structures is a pseudovariety if and only if it is defined by a set of identities in a certain relatively free profinite structure (pseudoidentities).

A well-known result of Birkhoff states that a class of algebras is a *variety*, that is, is closed under taking subalgebras, homomorphic images and direct products, if and only if it is *equational*, i.e. it is defined by a set of equations on the corresponding free structures. This result was then extended to first-order structures [5, 9]: in this framework, varieties are defined by universal positive Horn sentences, i.e. by relational identities (see Section 1.2).

Birkhoff's original statement was generalized in another direction by Reiterman [14]. Reiterman's theorem states that a class of *finite* algebras is a *pseudovariety* (that is, it is closed under taking subalgebras, homomorphic images and *finitary* direct products) if and only if it is defined by a set of equations in the appropriate free profinite structures. This result has led to a large body of consequences, in particular in finite semigroup theory (see in particular Almeida [1]).

The aim of this paper is to complete the picture by extending Reiterman's theorem to first-order structures. We prove that, under certain natural conditions of finiteness and non-emptiness, pseudovarieties of finite first-order structures are defined by relational identities in some relatively free profinite structures.

There are actually some differences with Reiterman's original statement. First, Reiterman's theorem was stated in terms of implicit operations. We have preferred, following the exposition of Almeida and Weil [2], to state our result in terms of profinite completions. However, in order to allow the reader to confront the two points of view, implicit operations are treated in Section 4.

The second difference is more technical, and was introduced to treat the important example of ordered algebras, already considered by Bloom [4]. It turns out that ordered

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algebras do not constitute a variety of first-order structures since, for instance, the transitivity of a binary relation cannot be expressed by relational identities. However, they form a quasivariety of structures, and our result can be applied in this context: under the same conditions of finiteness and non-emptiness as above, the classes of finite ordered algebras which are defined by relational identities in the appropriate free profinite structures are exactly the classes of ordered algebras which are closed under taking subalgebras, finite direct products and homomorphic images by non decreasing morphisms. These classes are also the intersections of pseudovarieties with the axiomatised class of ordered algebras. This observation of Bloom [4] extends to other axiomatised classes than ordered algebras, such as relational algebras (first-order structures where the operations preserve the relations).

This work was originally motivated by the study of so-called pseudovarieties of finite ordered semigroups, which were recently proved to be an important tool in the theory of rational languages, with far-reaching applications [12, 13]. There are of course many other well-known examples of first-order structures in mathematics and in theoretical computer science. For instance, (relational) databases can be represented by first-order structures with respect to a first-order language without function symbols [8]. Algebraic specification of abstract data type [6, 15] form another possible field of applications.

1 Preliminaries

For basic notions on first-order structures, we refer the readers to Burris and Sankappanavar [5] and Keisler [9]. Note that first-order structures are also known as *algebraic systems* (Mal'cev [10]).

1.1 $\mathcal{L}$-structures, morphisms and congruences

A *first-order language* $\mathcal{L}$ consists of a set $\mathcal{R}$ of *relation symbols* and a set $\mathcal{F}$ of *function symbols*, and associated to each element $r \in \mathcal{R}$ (resp. $f \in \mathcal{F}$), of a positive (resp. non-negative) integer called the *arity* of r (resp. f). For each integer n, we let $\mathcal{R}_n$ (resp. $\mathcal{F}_n$) be the set of relation (resp. function) symbols of arity n. The elements of $\mathcal{F}_0$ are called *constants*.

A *first-order structure* relative to $\mathcal{L}$, or $\mathcal{L}$*-structure*, is a pair (S, L) where S is a non-empty set and L consists of a family r^S of relations on S indexed by $\mathcal{R}$ and of a family f^S of operations on S indexed by $\mathcal{F}$ such that the arity of r^S (resp. f^S) is equal to the arity of the relation symbol r (resp. the function symbol f).

Let $\mathcal{L}$ be a *first-order language* and let (S, L) be an $\mathcal{L}$*-structure*. When there is no ambiguity, we write r and f for r^S and f^S, for each function symbol f and each relation symbol r in $\mathcal{L}$. We also write S for (S, L).

We say that an $\mathcal{L}$-structure S is an $\mathcal{L}$*-relational algebra* if the operations on S preserve the relations on S: that is if, for each $r \in \mathcal{R}_n$, for each $f \in \mathcal{F}_m$ and for all $s_{i,j} \in S$ ($1 \leq i \leq m$, $1 \leq j \leq n$),

$$(s_{i,1}, \ldots, s_{i,n}) \in r \text{ for all } 1 \leq i \leq m \quad \Longrightarrow \quad (f(s_{1,1}, \ldots, s_{m,1}), \ldots, f(s_{1,n}, \ldots s_{m,n})) \in r.$$

Example. If $\mathcal{R} = \emptyset$, an $\mathcal{L}$-structure is an *algebra of signature* $\mathcal{F}$, or $\mathcal{F}$*-algebra*. If $\mathcal{F} = \emptyset$, an $\mathcal{L}$-structure is called a *relational structure*. An *ordered* $\mathcal{F}$*-algebra* is an $\mathcal{L}$-relational algebra S, where $\mathcal{L}$ consists of the function symbol set $\mathcal{F}$ and of a relation symbol set $\mathcal{R}$ containing exactly one binary relation symbol r, such that r^S is a partial order relation on S.

In general, we can always view an $\mathcal{L}$-structure S as an $\mathcal{F}$-algebra, by forgetting about the relational part of $\mathcal{L}$. This algebra will be denoted S as well. That is, whenever convenient, we will consider the $\mathcal{L}$-structure S as an $\mathcal{F}$-algebra.

Let S and T be $\mathcal{L}$-structures. A (set) mapping $\varphi: S \to T$ is a *morphism of $\mathcal{L}$-structures* if

- $\varphi(f(s_1, \ldots, s_n)) = f(\varphi(s_1), \ldots, \varphi(s_n))$ for each $f \in \mathcal{F}_n$ and for each $s_1, \ldots, s_n \in S$;
- $(s_1, \ldots, s_n) \in r$ implies $(\varphi(s_1), \ldots, \varphi(s_n)) \in r$ for each $r \in \mathcal{R}_n$ and for each $s_1, \ldots,$ $s_n \in S$.

If, in addition, φ is bijective and φ^{-1} is a morphism, we say that φ is an isomorphism. Whenever convenient, we identify isomorphic $\mathcal{L}$-structures.

Let A be a set. We say that an $\mathcal{L}$-structure S is *A-generated* if there exists a mapping $\sigma: A \to S$ such that the algebra S is generated by $\sigma(A)$.

An equivalence relation $\sim$ on an $\mathcal{L}$-structure S is said to be a *congruence* if it is a congruence for the underlying algebraic structure, that is, if for each $f \in \mathcal{F}_n$ $(n \geq 1)$ and for all $(s_1, \ldots, s_n), (t_1, \ldots, t_n) \in S^n$ such that $s_i \sim t_i$ $(1 \leq i \leq n)$, we have $f(s_1, \ldots, s_n) \sim f(t_1, \ldots, t_n)$. If $\varphi: S \to T$ is a morphism of $\mathcal{L}$-structures, then

$$s \sim_\varphi s' \quad \Longleftrightarrow \quad \varphi(s) = \varphi(s')$$

defines a congruence on S.

The following result extends to $\mathcal{L}$-structures the standard First Homomorphism Theorem. The proof is immediate and is left to the reader. If $\varphi: S \to T$ is a morphism and $n \geq 1$, we denote again by φ the mapping defined on S^n by $\varphi(s_1, \ldots, s_n) = (\varphi(s_1), \ldots, \varphi(s_n))$.

Proposition 1.1 *Let S, T and U be $\mathcal{L}$-structures and let $\varphi: S \to T$ and $\psi: S \to U$ be morphisms of $\mathcal{L}$-structures. Let $\sim_\varphi$ and $\sim_\psi$ be the corresponding congruences of S. Then $\sim_\varphi$ is contained in $\sim_\psi$ if and only if there exists a morphism of $\mathcal{F}$-algebras $\chi: T \to U$ such that $\chi \circ \varphi = \psi$.*

Moreover, χ is a morphism of $\mathcal{L}$-structures if and only if, for each $r \in \mathcal{R}$, $\varphi^{-1}(r^T) \subseteq \psi^{-1}(r^U)$.

1.2 Varieties and pseudovarieties of $\mathcal{L}$-structures

Let S and T be $\mathcal{L}$-structures. We say that S is a *substructure* of T if there exists an injective morphism of $\mathcal{L}$-structures from S into T. The substructures of T can be seen as the subsets of T which, when equipped with the restriction of the operations and of the relations of T, are $\mathcal{L}$-structures themselves. This in turn is ensured as soon as these subsets are closed under the operations on T. That is, the substructures of T are exactly the sub-$\mathcal{F}$-algebras of T, equipped with the restrictions of the relations of T.

If $(S_i)_{i \in I}$ is a family of $\mathcal{L}$-structures, the direct product $\prod_{i \in I} S_i$ is the $\mathcal{L}$-structure given by

$$\begin{aligned} f((s_{1,i})_i, \ldots, (s_{n,i})_i) \quad &= \quad (f(s_{1,i}, \ldots, s_{n,i}))_i, \\ ((s_{1,i})_i, \ldots, (s_{n,i})_i) \in r \quad &\Longleftrightarrow \quad (s_{1,i}, \ldots, s_{n,i}) \in r \text{ for all } i \in I \end{aligned}$$

for all $f \in \mathcal{F}_n$ and $r \in \mathcal{R}_n$ ($n \geq 0$). The canonical projections $\pi_i: \prod_i S_i \to S_i$ onto the coordinate components are morphisms of $\mathcal{L}$-structures.

Finally, if $(S_i)_{i\in I}$ is a family of $\mathcal{L}$-structures and F is an ultrafilter over I, we consider the relation $=_F$ on $\prod_{i\in I} S_i$ defined by

$$(s_i)_i =_F (t_i)_i \quad \iff \quad \{i \in I \mid s_i = t_i\} \in F.$$

Then $=_F$ is a congruence on $\prod_{i\in I} S_i$. We define the *ultraproduct* $\prod_F S_i$ to be the quotient set $(\prod_{i\in I} S_i) / =_F$ and we let π_F be the canonical projection of $\prod_{i\in I} S_i$ onto $\prod_F S_i$. For each relation symbol r, we define r on $\prod_F S_i$ by

$$(x_1, \ldots, x_n) \in r \quad \iff \quad \exists y_1 \in \pi_F^{-1}(x_1) \ \ldots \ \exists y_n \in \pi_F^{-1}(x_n) \ (y_1, \ldots, y_n) \in r$$

(where n is the arity of r). Then the ultraproduct $\prod_F S_i$ is an $\mathcal{L}$-structure and π_F is a morphism of $\mathcal{L}$-structures. More details can be found in [5, 6].

We say that a class of $\mathcal{L}$-structures is a *quasivariety* if it is closed under taking substructures, direct products and ultraproducts. It is a *variety* if it is closed under taking substructures, homomorphic images and direct products. Finally, a class of finite $\mathcal{L}$-structures is called a *pseudovariety* if it is closed under taking substructures, homomorphic images and finitary direct products.

Observe that, as in the classical result of Birkhoff on classes of algebras [5, Theorem 10.12], each quasivariety of $\mathcal{L}$-structures contains a *free object* over each set. More precisely, if $\mathcal{L}$ is a first-order language and A is a set such that either $\mathcal{F}_0 \neq \emptyset$ or $A \neq \emptyset$, and if $\mathcal{Q}$ is a quasivariety of $\mathcal{L}$-structures, then $\mathcal{Q}$ contains an A-generated element $F_A(\mathcal{Q})$, unique up to isomorphism, such that $F_A(\mathcal{Q})$ is equipped with a mapping $\imath: A \to F_A(\mathcal{Q})$ and, for every mapping $\varphi: A \to S$ with $S \in \mathcal{Q}$, there exists a unique morphism $\bar{\varphi}: F_A(\mathcal{Q}) \to S$ satisfying $\bar{\varphi} \circ \imath = \varphi$. This free object $F_A(\mathcal{Q})$ may be constructed as a subdirect product of the collection of A-generated elements of $\mathcal{Q}$. It is also worth observing that if $\mathcal{Q}$ is non-trivial, then $\imath$ is one-to-one.

It is known that varieties are defined by positive universal Horn sentences, that is, relational identities [9, Theorem 5.10]. Birkhoff's classical theorem on varieties of algebras is the special case of this statement corresponding to a first-order language $\mathcal{L}$ without relation symbols. Similarly, quasivarieties are exactly those classes which are defined by universal Horn sentences, that is, implications of relational identities [5, Theorem 2.23].

Examples. The class of all $\mathcal{L}$-structures is a variety (defined by an empty set of relational identities).

The class of all $\mathcal{L}$-relational structures is a quasivariety, defined by the following implications of relational identities (indexed by integers n, m and by elements r of $\mathcal{R}_n$ and f of $\mathcal{F}_m$):

$$\bigwedge_{i=1}^{m} (x_{i,1}, \ldots, x_{i,n}) \in r \quad \Longrightarrow \quad (f(x_{1,1}, \ldots, x_{m,1}), \ldots, f(x_{1,n}, \ldots x_{m,n})) \in r.$$

Since the reflexivity, anti-symmetry and transitivity of a binary relation as well as the monotonicity of an operation symbol can be expressed by implications of relational identities, ordered $\mathcal{F}$-algebras form a quasivariety.

Let $\mathcal{L} = \{\leq, \cdot, 1\}$ be the language of ordered monoids and let $\mathcal{Q}$ be the quasivariety of ordered monoids satisfying the relational identity $x \leq 1$. Let A be a set. Then $F_A(\mathcal{Q})$ is the usual free monoid A^*, equipped with the so-called subword order, given by $u \leq v$ ($u, v \in A^*$) if and only if $v = a_1 \cdots a_n$ ($n \geq 0$, $a_i \in A$) and there exist $u_0, \ldots, u_n \in A^*$ such that $u = u_0 a_1 u_1 \cdots a_n u_n$.

1.3 Varieties and pseudovarieties of axiomatized structures

Let $\mathcal{Q}$ be a quasivariety of $\mathcal{L}$-structures. The examples we have in mind in this section are those of $\mathcal{L}$-relational algebras and of ordered $\mathcal{F}$-algebras. We say that a class of $\mathcal{L}$-structures (resp. finite $\mathcal{L}$-structures) of $\mathcal{Q}$ is a *variety* (resp. *pseudovariety*) *of elements of* $\mathcal{Q}$ if it is closed under taking substructures, direct products (resp. finitary direct products) and homomorphic images, for homomorphisms between $\mathcal{L}$-structures in $\mathcal{Q}$. We observe that, in general, $\mathcal{Q}$ is not closed under taking homomorphic images.

Thus, varieties or pseudovarieties of elements of $\mathcal{Q}$ are not varieties or pseudovarieties of $\mathcal{L}$-structures in general. Our choice of terminology is however justified by the following easy observation.

Proposition 1.2 *Let $\mathcal{Q}$ be a quasivariety of $\mathcal{L}$-structures and let $\mathcal{V}$ be a subclass of $\mathcal{Q}$. The following conditions are equivalent.*

(i) $\mathcal{V}$ is a variety (resp. pseudovariety) of elements of $\mathcal{Q}$

(ii) $\mathcal{V}$ is the intersection of $\mathcal{Q}$ with the variety (resp. pseudovariety) it generates.

(iii) $\mathcal{V}$ is the intersection of $\mathcal{Q}$ with some variety (resp. pseudovariety) of $\mathcal{L}$-structures.

Proof. It is immediate that, if $\mathcal{W}$ is a variety of $\mathcal{L}$-structures, then $\mathcal{W} \cap \mathcal{Q}$ is a variety of elements of $\mathcal{Q}$. Conversely, let us assume that $\mathcal{V}$ is a variety of elements of $\mathcal{Q}$ and let $\mathcal{W}$ be the variety of $\mathcal{L}$-structures generated by $\mathcal{V}$. It is known that $\mathcal{W}$ is exactly the class of all homomorphic images of elements of $\mathcal{V}$ [5, Theorem 9.5]. It follows that $\mathcal{V} = \mathcal{W} \cap \mathcal{Q}$. The statements regarding pseudovarieties are proved in the same fashion. $\square$

This observation already appears in Bloom [4] (see also Wechler [15]), where the special case of varieties of ordered algebras is studied.

1.4 Topological $\mathcal{L}$-structures

An $\mathcal{L}$-structure S is said to be a *topological $\mathcal{L}$-structure* if the set S is equipped with a topology such that:

- for each $f \in \mathcal{F}_n$ ($n \geq 1$), the mapping $(s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n)$, from S^n into S, is continuous, and
- for each $r \in \mathcal{R}_n$ ($n \geq 1$), the subset r of S^n is closed.

In the sequel, all topological spaces are assumed to be Hausdorff.

Let A be a finite set. We say that the topological $\mathcal{L}$-structure S is *A-generated* if there exists a mapping $\sigma: A \to S$ such that the subalgebra generated by σA is dense in S.

Let $(S_i)_{i\in I}$ be a family of topological $\mathcal{L}$-structures, and let the direct product $\prod_{i\in I} S_i$ be equipped with the product topology. By definition, a basis of open sets for this topology is given by the sets of the form

$$V_{i_1} \times \cdots \times V_{i_n} \times \prod_{\substack{j\in I\\ j\neq i_1,\ldots,i_n}} S_j,$$

where $n \geq 1$, $i_1, \ldots, i_n$ are distinct elements of I and V_{i_h} is an open subset of S_{i_h}. The coordinate projections of $\prod_i S_i$ onto the S_i are trivially continuous.

Proposition 1.3 *The direct product of a family of topological $\mathcal{L}$-structures is a topological $\mathcal{L}$-structure.*

Proof. Let $(S_i)_{i\in I}$ be a family of topological $\mathcal{L}$-structures, let $S = \prod_{i\in I} S_i$ and, for each $i \in I$, let π_i be the corresponding (continuous) coordinate projection. Let $n \geq 1$. The bijection

$$\begin{aligned} \delta_n\colon S^n &\longrightarrow \prod_{i\in I}(S_i^n) \\ (s_1,\ldots,s_n) &\longmapsto ((\pi_i(s_1),\ldots,\pi_i(s_n)))_{i\in I} \end{aligned}$$

is easily seen to be continuous. Let now $f \in \mathcal{F}_n$: then $f^S = (f^{S_i})_{i\in I} \circ \delta_n$, so f^S is continuous.

Similarly, if $r \in \mathcal{R}_n$, then $r^S = \bigcap_i \pi_i^{-1}(r^{S_i})$. But the π_i are continuous and the r^{S_i} $(i \in I)$ are closed, so r^S is closed. $\square$

The following elementary result, which complements Proposition 1.1, will be used several times in the sequel.

Proposition 1.4 *Let $\varphi\colon S \to T$ be a continuous morphism between topological $\mathcal{L}$-structures. Then the congruence $\sim_\varphi$ is a closed subset of $S \times S$.*

If $\varphi\colon S \to T$ and $\psi\colon S \to U$ are continuous morphisms such that $\sim_\varphi$ is contained in $\sim_\psi$ and if S is compact, then there exists a continuous morphism of $\mathcal{F}$-algebras $\chi\colon T \to U$ such that $\chi \circ \varphi = \psi$.

χ is a continuous morphism of $\mathcal{L}$-structures if and only if, in addition, $\varphi^{-1}(r^T) \subseteq \psi^{-1}(r^U)$ for each $r \in \mathcal{R}$.

Proof. The first statement is immediate since T is Hausdorff. As for the second statement, Proposition 1.1 shows that the hypothesis on the congruences $\sim_\varphi$ and $\sim_\psi$ implies the existence of a morphism of $\mathcal{F}$-algebras $\chi\colon T \to U$ such that $\chi \circ \varphi = \psi$. Let now X be a closed subset of U: since ψ is continuous, $\psi^{-1}(X)$ is closed in S, and hence compact. But φ is continuous as well, so $\chi^{-1}(X) = \varphi(\psi^{-1}(X))$ is compact, and hence closed. Thus χ is continuous.

The last statement is immediate. $\square$

2 Profinite $\mathcal{L}$-structures

Pseudovarieties of finite $\mathcal{L}$-structures do not, in general, contain free objects. However, certain profinite $\mathcal{L}$-structures can be shown to play much the same role as free objects.

2.1 Projective limits

Recall that a partially ordered set (poset) I is *directed* if any two elements of I admit a common upper bound. A *directed system of $\mathcal{L}$-structures* $(S_i)_{i\in I}$ is a family of $\mathcal{L}$-structures indexed by a directed poset I such that, for each pair (i,j) of elements of I verifying $i \geq j$, there exists a morphism of $\mathcal{L}$-structures $\varphi_{i,j}\colon S_i \to S_j$, and such that, for all $i \geq j \geq k$ in I, we have $\varphi_{i,i} = \mathrm{id}_{S_i}$ and $\varphi_{j,k} \circ \varphi_{i,j} = \varphi_{i,k}$.

By definition, the *projective limit* of a directed system of $\mathcal{L}$-structures $(S_i)_{i\in I}$ is the following substructure of the direct product $\prod_{i\in I} S_i$:

$$\varprojlim (S_i)_{i\in I} = \{(x_i)_{i\in I} \in \prod_{i\in I} S_i \mid \forall i \geq j,\ \varphi_{i,j}(x_i) = x_j\}.$$

The *canonical morphisms* $\pi_i\colon \varprojlim (S_i)_{i\in I} \to S_i$ are defined to be the restrictions of the coordinate projections to $\varprojlim (S_i)_{i\in I}$. Observe that $\varprojlim (S_i)_{i\in I}$ is also equal to $\varprojlim (\pi_i S)_{i\in I}$, so that whenever necessary, we may assume the canonical projections π_i to be onto.

A *directed system of morphisms* defined on an $\mathcal{L}$-structure S is a family of morphisms $(\rho_i\colon S \to S_i)_{i\in I}$ indexed by a directed poset I such that:

- there exist morphisms $\varphi_{i,j}\colon S_i \to S_j$ for all pairs (i,j) of elements of I verifying $i \geq j$, and these morphisms make $(S_i)_i$ a directed system of algebras,
- for all $i \geq j$ in I, we have $\rho_j = \varphi_{i,j} \circ \rho_i$.

The following universal property of projective limits can be found in any textbook of category theory.

Proposition 2.1 *Let $(\rho_i\colon S \to S_i)_{i\in I}$ be a directed system of morphisms defined on an $\mathcal{L}$-structure S. Then there exists a unique morphism $\rho\colon S \to \varprojlim (S_i)_{i\in I}$ satisfying $\rho_i = \pi_i \circ \rho$ for all i. This property defines $\varprojlim (S_i)_i$ up to an isomorphism.*

We say that the morphism ρ described above is *induced* by the directed system $(\rho_i)_i$. Notice that ρ is described by $\rho(s) = (\rho_i(s))_i$.

Let us now assume that the S_i are topological $\mathcal{L}$-structures, and that the morphisms $\varphi_{i,j}$ are continuous. Let us consider the direct product $\prod_i S_i$ to be equipped with the product topology. Then one verifies that $\varprojlim (S_i)_i$ is a closed substructure of $\prod_i S_i$. In particular, if the S_i are compact, then so is their direct product (by Tikhonov's theorem), and hence so is their projective limit.

Note. Using the fact that the poset I is directed, one may verify that a basis of open subsets of $\varprojlim (S_i)_{i\in I}$ is given by the sets of the form $\pi_i^{-1}(V_i)$ where $i \in I$ and V_i is an open subset of S_i.

Moreover, if $(\rho_i\colon S \to S_i)_{i\in I}$ is a directed system of continuous morphisms defined on a topological $\mathcal{L}$-structure, then the induced morphism $\rho\colon S \to \varprojlim (S_i)_i$ is easily seen to be continuous.

Let $\mathcal{C}$ be a class of topological $\mathcal{L}$-structures. We say that an $\mathcal{L}$-structure is *pro-$\mathcal{C}$* if it is a projective limit of $\mathcal{L}$-structures in $\mathcal{C}$. In particular, an $\mathcal{L}$-structure is *profinite* if it is a projective limit of finite $\mathcal{L}$-structures (considered as equipped with the discrete topology). Such $\mathcal{L}$-structures are therefore compact and totally disconnected.

A very important property of projective limits of compact $\mathcal{L}$-structures is the following.

Theorem 2.2 *Let $\mathcal{L}$ be a first-order language such that the set $\mathcal{R}$ of relation symbols is finite. Let $S = \varprojlim(S_i)_{i\in I}$ be a projective limit of compact $\mathcal{L}$-structures such that, for each $r \in \mathcal{R}$ and each $i \in I$, r^{S_i} is non empty, and let $\pi_i\colon S \to S_i$ $(i \in I)$ be the canonical morphisms. Let T be a finite $\mathcal{L}$-structure and let $\varphi\colon S \to T$ be a continuous morphism. Then there exists $i \in I$ and a continuous morphism $\varphi_i\colon S_i \to T$ such that $\varphi = \varphi_i \circ \pi_i$.*

Proof. Let $\sim$ be the congruence on S induced by the morphism φ, and let $\sim_i$ $(i \in I)$ be that induced by π_i. All these congruences are closed, since φ and the π_i are continuous. In fact, since T is finite, $\sim$ is clopen and hence the complement C of $\sim$ in $S \times S$ is closed.

Moreover, $\bigcap_i \sim_i$ is equal to the diagonal of S, by definition of the projective limit. Thus we have

$$C \cap \bigcap_{i\in I} \sim_i \;=\; \emptyset.$$

By compactness, the intersection of some finite subfamily of these closed sets is empty. But the intersection $\bigcap_{i\in I} \sim_i$ is the diagonal and hence is non-empty, so there exists $i_1, \ldots, i_k \in I$ such that

$$C \cap \bigcap_{j=1}^{k} \sim_{i_j} \;=\; \emptyset,$$

that is, $\bigcap_{j=1}^{k} \sim_{i_j}$ is contained in $\sim$. Now the S_i constitute a directed system, so there exists $i_0 \in I$ such that, for all $i \in I$ such that $i \geq i_0$, the congruence $\sim_i$ is contained in $\sim$.

Let us now consider a relation symbol r of arity n $(n \geq 1)$. The existence of the morphism φ implies that $r^S \subseteq \varphi^{-1}(r^T)$. By definition of S, $r^S = \bigcap_{i\in I} \pi_i^{-1}(r^{S_i})$. Moreover, $\varphi^{-1}(r^T)$ and the $\pi_i^{-1}(r^{S_i})$ are closed subsets of the compact space S^n. As above, the finiteness of T implies that $\varphi^{-1}(r^T)$ is in fact clopen, and hence the complement D of $\varphi^{-1}(r^T)$ is closed. So the following intersection of closed sets is empty:

$$D \cap \bigcap_{i\in I} \pi_i^{-1}(r^{S_i}) = \emptyset.$$

Observe that, because the S_i constitute a directed system, our assumption implies that each finite intersection of $\pi_i^{-1}(r^{S_i})$ is non empty, so that $\bigcap_{i\in I} \pi_i^{-1}(r^{S_i}) \neq \emptyset$ by compactness. By compactness again, it follows that there exist $i_1, \ldots, i_k \in I$ such that

$$D \cap \bigcap_{j=1}^{k} \pi_{i_j}^{-1}(r^{S_{i_j}}) = \emptyset,$$

that is, $\bigcap_{j=1}^{k} \pi_{i_j}^{-1}(r^{S_{i_j}}) \subseteq \varphi^{-1}(r^T)$. Again, we use the fact that the S_i constitute a directed system: if $i, j \in I$ and $i > j$, then there exists a morphism of $\mathcal{L}$-structures from S_i into S_j, so that $\pi_i^{-1}(r^{S_i})$ is contained in $\pi_j^{-1}(r^{S_j})$. In particular, there exists $j_r \in I$ such that, for all $i \geq j_r$, $\pi_i^{-1}(r^{S_i}) \subseteq \varphi^{-1}(r^T)$.

Consider now $i \in I$ such that $i \geq i_0$ and $i \geq j_r$ for each $r \in \mathcal{R}$. For such an index i, we have simultaneously $\sim_i \subseteq \sim$ and $\pi_i^{-1}(r^{S_i}) \subseteq \varphi^{-1}(r^T)$ for each $r \in \mathcal{R}$. By Proposition 1.4, it follows that there exists a continuous morphism of $\mathcal{L}$-structures $\chi\colon S_i \to T$ such that $\chi \circ \pi_i = \varphi$. □

Note. If the first-order language $\mathcal{L}$ is finite, then the topology of profinite $\mathcal{L}$-structures can be defined by a metric. Indeed, there are only countably many finite $\mathcal{L}$-structures (up to isomorphism), so that any profinite structure S can be constructed as the projective limit of a countable system $(S_n)_{n\geq 0}$ of finite $\mathcal{L}$-structures, where in addition, the index set is ordered by the usual order relation on non negative integers. Let π_n be the canonical morphism from S into S_n $(n \geq 0)$. For all $x, y \in S$, let

$$d(x,y) = 2^{-\min\{n\geq 0 | \pi_n x \neq \pi_n y\}}$$

It is not difficult to verify that d is an ultrametric distance function, which defines the topology of S.

2.2 Relatively free profinite $\mathcal{L}$-structures

We now construct relatively free profinite $\mathcal{L}$-structures, thus laying the groundwork for the extension of Reiterman's theorem to first-order structures.

Let $\mathbf{V}$ be a pseudovariety of $\mathcal{L}$-structures and let A be a profinite non empty set (equivalently, A is a non empty compact totally disconnected topological space). We say that a topological pro-$\mathbf{V}$ structure F, equipped with a continuous mapping $\imath: A \to F$, is the *free pro-*$\mathbf{V}$ structure over A if $\imath(A)$ generates a dense subalgebra of F, and if every continuous mapping $\sigma: A \to S$ into a pro-$\mathbf{V}$ topological structure S induces a continuous morphism $\hat{\sigma}: F \to S$ such that $\hat{\sigma} \circ \imath = \sigma$.

For simplicity, we now assume that A is finite (see Section 6). Let $\mathbf{V}_A$ be the category of all A-generated elements of $\mathbf{V}$. Formally, the objects of $\mathbf{V}_A$ are the mappings $\sigma: A \to S_\sigma$ such that $S_\sigma \in \mathbf{V}$ and the subalgebra generated by $\sigma(A)$ is S_σ itself. The morphisms of $\mathbf{V}_A$ from $\sigma: A \to S_\sigma$ to $\tau: A \to S_\tau$ are the morphisms of $\mathcal{L}$-structures $\varphi: S_\sigma \to S_\tau$ such that $\varphi \circ \sigma = \tau$. Observe that, by definition of $\mathbf{V}_A$, there is at most one morphism of $\mathbf{V}_A$ from σ to τ. We now define a partial order on the objects of $\mathbf{V}_A$ by letting $\tau \leq \sigma$ if and only if either $\sigma = \tau$, or there exists a morphism of $\mathbf{V}_A$ from σ to τ which is not an isomorphism. For all objects σ, τ of $\mathbf{V}_A$, let $\varphi_{\sigma,\sigma} = \mathrm{id}_{S_\sigma}$ and, if $\tau \leq \sigma$ and $\tau \neq \sigma$, let $\varphi_{\sigma,\tau}$ be the (uniquely determined) morphism of $\mathbf{V}_A$ from σ to τ. It is immediate that the family $(S_\sigma)_\sigma$ indexed by the objects of $\mathbf{V}_A$, together with the $\varphi_{\sigma,\tau}$, constitutes a directed system.

Let $\hat{F}_A(\mathbf{V})$ be the projective limit of the directed system $(S_\sigma)_\sigma$. The mapping $\imath: A \to \hat{F}_A(\mathbf{V})$ given by $\imath(a) = (\sigma(a))_\sigma$ is called the *natural mapping* from A into $\hat{F}_A(\mathbf{V})$. If $\mathbf{V}$ is non trivial, that is, if $\mathbf{V}$ contains an element with cardinality at least 2, then $\imath$ is easily seen to be one-to-one, so we may view A as a subset of $\hat{F}_A(\mathbf{V})$.

The following observation will be useful. Let $\mathcal{V}$ be a non-trivial quasivariety of $\mathcal{L}$-structures containing $\mathbf{V}$, and let $F_A(\mathcal{V})$ be its free object over A — equipped with the mapping $\jmath: A \to F_A(\mathcal{V})$. Since $\hat{F}_A(\mathbf{V})$ is a substructure of a direct product of elements of $\mathcal{V}$, $\hat{F}_A(\mathbf{V})$ lies in $\mathcal{V}$ and hence $\imath$ induces a morphism $\hat{\imath}: F_A(\mathcal{V}) \to \hat{F}_A(\mathbf{V})$ such that $\imath = \hat{\imath} \circ \jmath$.

$$\begin{tikzcd}
A \arrow[r, "\jmath"] \arrow[dr, "\imath"'] & F_A(\mathcal{V}) \arrow[d, "\hat{\imath}"] \\
 & \hat{F}_A(\mathbf{V})
\end{tikzcd}$$

Note that the substructure of $\hat{F}_A(\mathbf{V})$ generated by $\imath(A)$ is exactly $\hat{\imath}(F_A(\mathcal{V}))$, independently of the choice of $\mathcal{V}$.

The fundamental universal property of $\hat{F}_A(\mathbf{V})$ is the following.

Theorem 2.3 *Let A be a finite set and let $\mathbf{V}$ be a pseudovariety of $\mathcal{L}$-structures. Then $\hat{F}_A(\mathbf{V})$ (together with the mapping $\imath$) is the free pro-$\mathbf{V}$ $\mathcal{L}$-structure over A.*

Proof. Let H be the subalgebra of $\hat{F}_A(\mathbf{V})$ generated by $\imath(A)$. We first verify that H is dense in $\hat{F}_A(\mathbf{V})$. Let $\mathcal{V}$ be a quasivariety of $\mathcal{L}$-structures containing $\mathbf{V}$, and let $F_A(\mathcal{V})$ be its free object over A, equipped with the mapping $\jmath: A \to F_A(\mathcal{V})$. For each object $\sigma: A \to S_\sigma$ of $\mathbf{V}_A$, let $\pi_\sigma: \hat{F}_A(\mathbf{V}) \to S_\sigma$ be the corresponding canonical morphism. Then $\sigma = \pi_\sigma \circ \imath = \pi_\sigma \circ \hat{\imath} \circ \jmath$.

$$\begin{tikzcd} A \arrow[r, "\jmath"] \arrow[d, "\sigma"'] \arrow[dr, "\imath"] & F_A(\mathcal{V}) \arrow[d, "\hat{\imath}"] \\ S_\sigma & \hat{F}_A(\mathbf{V}) \arrow[l, "\pi_\sigma"] \end{tikzcd}$$

By definition of the topology on $\hat{F}_A(\mathbf{V})$, a basis of neighborhoods of an element $x \in \hat{F}_A(\mathbf{V})$ consists of all $\pi_\sigma^{-1}(\pi_\sigma(x))$. But for any given σ, the set $\sigma(A)$ generates S_σ, so $\pi_\sigma \circ \hat{\imath}$ is onto. Therefore there exists $u \in F_A(\mathcal{V})$ such that $\pi_\sigma(x) = \pi_\sigma(\hat{\imath}(u))$. But $\hat{\imath}(u) \in \hat{\imath}(F_A(\mathcal{V})) = H$, so H is dense in $\hat{F}_A(\mathbf{V})$.

Let now S be an A-generated topological pro-$\mathbf{V}$ $\mathcal{L}$-structure: there exists a mapping $\sigma: A \to S$ such that the subalgebra generated by $\sigma(A)$ is dense in S, that is, there exists a morphism $\sigma: F_A(\mathcal{V}) \to S$ whose range is dense in S. Moreover there exists a directed system $(S_i)_i$ of elements of $\mathbf{V}$ such that $S = \varprojlim (S_i)_i$. Let $\pi_i: S \to S_i$ $(i \in I)$ be the corresponding canonical morphisms, which we may assume to be onto. By definition of projective limits, each element x of S is of the form $x = (\pi_i(x))_i$. Let $\sigma_i = \pi_i \circ \sigma: F_A(\mathcal{V}) \to S_i$. Then the σ_i are onto, so that the S_i are A-generated $\mathcal{L}$-structures in $\mathbf{V}$. Therefore, considering the canonical morphisms associated with the projective limit defining $\hat{F}_A(\mathbf{V})$, we find that there exist continuous morphisms $\varphi_i: \hat{F}_A(\mathbf{V}) \to S_i$ such that $\varphi_i \circ \hat{\imath} = \sigma_i = \pi_i \circ \sigma$. We use the universal property of projective limits once more to see that these morphisms induce an onto continuous morphism $\varphi: \hat{F}_A(\mathbf{V}) \to S$ such that $\pi_i \circ \varphi = \varphi_i$. So $\pi_i \circ \varphi \circ \hat{\imath} = \pi_i \circ \sigma$ for each i, and hence $\varphi \circ \hat{\imath} = \sigma$.

$$\begin{tikzcd} F_A(\mathcal{V}) \arrow[r, "\sigma"] \arrow[dr, "\sigma_i"'] \arrow[rr, bend left, "\hat{\imath}"] & S \arrow[d, "\pi_i"] & \hat{F}_A(\mathbf{V}) \arrow[l, "\varphi"'] \arrow[dl, "\varphi_i"] \\ & S_i & \end{tikzcd}$$

Notice that the choice of $\mathcal{V}$ plays no role in this proof: the σ_i are introduced only in order to verify that the S_i are A-generated $\mathcal{L}$-structures, thus ensuring the existence of the φ_i, and ultimately of φ. $\square$

This yields the following important corollaries.

Corollary 2.4 *Let $\mathbf{V}$ be a pseudovariety of $\mathcal{L}$-structures, let $\mathbf{W}$ be a pseudovariety of $\mathcal{L}$-structures contained in $\mathbf{V}$, let A and B be finite sets and let $\imath$ be the natural mapping* *from A into $\hat{F}_A(\mathbf{V})$. If $\varphi: A \to \hat{F}_B(\mathbf{W})$ is a mapping, then φ induces a unique continuous morphism $\hat{\varphi}: \hat{F}_A(\mathbf{V}) \to \hat{F}_B(\mathbf{W})$ such that $\hat{\varphi} \circ \imath = \varphi$. In particular, the identity function of A induces an onto continuous morphism from $\hat{F}_A(\mathbf{V})$ onto $\hat{F}_A(\mathbf{W})$.*

In the sequel of this paper, we consider only first-order languages for which the set of relation symbols is finite. For such a language $\mathcal{L}$, we say that an $\mathcal{L}$-structure S is *admissible* if $r^S \neq \emptyset$ for each $r \in \mathcal{R}$. A class of $\mathcal{L}$-structures is admissible if all its elements are. Thus, Theorem 2.2 can be applied to projective limits of directed systems of admissible structures.

Corollary 2.5 *Let A be a finite set and let $\mathbf{V}$ be an admissible pseudovariety of $\mathcal{L}$-structures. Then a finite A-generated $\mathcal{L}$-structure is in $\mathbf{V}$ if and only if it is a continuous homomorphic image of $\hat{F}_A(\mathbf{V})$.*

Proof. This is an immediate consequence of the definition of $\hat{F}_A(\mathbf{V})$ and of Theorem 2.2. $\square$

3 Reiterman's theorem for $\mathcal{L}$-structures

We are now ready to state and prove the extension of Reiterman's theorem for $\mathcal{L}$-structures. Let $\mathcal{L}$ be a first-order language with a finite set of relation symbols, let $\mathbf{V}$ be an admissible pseudovariety of $\mathcal{L}$-structures and let A be a finite set. The *$\mathcal{L}$-pseudoidentities* (or simply *pseudoidentities*) on $\mathbf{V}$ on the set A (or, in $|A|$ variables) are defined as follows:

- a *pure* pseudoidentity is a pair of elements (u, v) of the free pro-$\mathbf{V}$ $\mathcal{L}$-structure $\hat{F}_A(\mathbf{V})$, which we write $u = v$,
- for each $r \in \mathcal{R}_n$ $(n \geq 1)$, an r-pseudoidentity is an n-tuple $(u_1, \ldots, u_n)$ of elements of $\hat{F}_A(\mathbf{V})$, which we write $(u_1, \ldots, u_n) \in r$.

Let S be a pro-$\mathbf{V}$ $\mathcal{L}$-structure. Recall that each mapping $\sigma: A \to S$ induces a continuous morphism $\sigma: \hat{F}_A(\mathbf{V}) \to S$. We say that S satisfies $u = v$ (resp. $(u_1, \ldots, u_n) \in r$) if, for each map $\sigma: A \to S$, we have $\sigma(u) = \sigma(v)$ (resp. $(\sigma(u_1), \ldots, \sigma(u_n)) \in r^S$). The next propositions describe the pseudoidentities satisfied respectively by a projective limit and by a sub-pseudovariety of $\mathbf{V}$.

Proposition 3.1 *Let S be the projective limit of a directed system $(S_i)_i$ of elements of $\mathbf{V}$, such that the projections $\pi_i: S \to S_i$ are onto. Then the pseudoidentities satisfied by S are exactly the pseudoidentities which are satisfied by all the S_i.*

Proof. Let π_i $(i \in I)$ be the canonical projections of S onto the S_i. Let $r \in \mathcal{R}_n$ $(n \geq 1)$ and let $u, v, u_1, \ldots u_n \in \hat{F}_A(\mathbf{V})$. If all the S_i satisfy $u = v$ (resp. $(u_1, \ldots, u_n) \in r$), then for any continuous morphism $\sigma: \hat{F}_A(\mathbf{V}) \to S$, each $\pi_i \circ \sigma$ $(i \in I)$ is a continuous morphism into S_i, so $\pi_i(\sigma u) = \pi_i(\sigma v)$ (resp. $(\pi_i(\sigma u_1), \ldots, \pi_i(\sigma u_n)) \in r$) for each $i \in I$. Thus $\sigma u = \sigma v$ ($(\sigma u_1, \ldots, \sigma u_n) \in r$) and hence S satisfies $u = v$ (resp. S satisfies $(u_1, \ldots, u_n) \in r$). Conversely, suppose that S satisfies $u = v$ (resp. $(u_1, \ldots, u_n) \in r$) and let $i \in I$. Let $\sigma: \hat{F}_A(\mathbf{V}) \to S_i$ be a continuous morphism. For each $a \in A$, let τa be chosen in $\pi_i^{-1}(\sigma a)$: By Proposition 2.1, these choices induce a continuous morphism $\tau: \hat{F}_A(\mathbf{V}) \to S$ such that $\pi_i \circ \tau = \sigma$. But $\tau u = \tau v$ (resp. $(\tau u_1, \ldots, \tau u_n) \in r$), so $\sigma u = \sigma v$ (resp. $(\sigma u_1, \ldots, \sigma u_n) \in r$). That is, S_i satisfies $u = v$ (resp. S_i satisfies $(u_1, \ldots, u_n) \in r$). $\square$

Proposition 3.2 *Let $\mathbf{W}$ be a sub-pseudovariety of $\mathbf{V}$ and let $\pi\colon \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{W})$ be the canonical projection. Then the set of pure pseudoidentities satisfied by all the elements of $\mathbf{W}$ is equal to the congruence $\sim_\pi$ and, for each $r \in \mathcal{R}$, the set of r-pseudoidentities satisfied by all the elements of $\mathbf{W}$ is equal to $\pi^{-1}(r)$.*

Proof. By Proposition 3.1, we need only to verify that the set $\Sigma_=$ of pure pseudoidentities satisfied by $\hat{F}_A(\mathbf{W})$ is equal to $\sim_\pi$ and that, for each $r \in \mathcal{R}$, the set Σ_r of r-pseudoidentities satisfied by $\hat{F}_A(\mathbf{W})$ is equal to $\pi^{-1}(r)$. It is clear that $\Sigma_=$ refines $\sim_\pi$ and that Σ_r is contained in $\pi^{-1}(r)$. Let now $u, v \in \hat{F}_A(\mathbf{V})$ be such that $\pi(u) = \pi(v)$. Let $\sigma\colon \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{W})$ be a continuous morphism. Then by Corollary 2.4, there exists a continuous morphism $\tau\colon \hat{F}_A(\mathbf{W}) \to \hat{F}_A(\mathbf{W})$ such that $\tau \circ \pi = \sigma$. Therefore $\sigma(u) = \sigma(v)$ and hence $\hat{F}_A(\mathbf{W})$ satisfies $u = v$. The analogous statement regarding r-pseudoidentities is proved in the same fashion. $\square$

Let now Σ be a set of pseudoidentities on $\mathbf{V}$ (not necessarily involving a bounded number of variables), and let $[\![\Sigma]\!]_{\mathbf{V}}$ be the class of all elements of $\mathbf{V}$ satisfying all pseudoidentities in Σ. We can now state the analogue of Reiterman's theorem for $\mathcal{L}$-structures.

Theorem 3.3 *Let $\mathcal{L}$ be a first-order language with a finite set of relation symbols, let $\mathbf{V}$ be an admissible pseudovariety of $\mathcal{L}$-structures and let $\mathbf{W} \subseteq \mathbf{V}$ be a subclass of $\mathbf{V}$. Then $\mathbf{W}$ is a sub-pseudovariety if and only if there exists a set Σ of pseudoidentities on $\mathbf{V}$ such that $\mathbf{W} = [\![\Sigma]\!]_{\mathbf{V}}$.*

Proof. It is not difficult to verify that, if Σ is a class of pseudoidentities on $\mathbf{V}$, then $[\![\Sigma]\!]_{\mathbf{V}}$ is closed under taking substructures, quotients and finitary direct products. That is, $[\![\Sigma]\!]_{\mathbf{V}}$ is a sub-pseudovariety of $\mathbf{V}$.

We now consider a sub-pseudovariety $\mathbf{W}$ of $\mathbf{V}$, and the set Σ of all pseudoidentities on $\mathbf{V}$ which are satisfied by all elements of $\mathbf{W}$. In fact, Σ consists of the sets $\Sigma_=$ of all pure pseudoidentities satisfied by the elements of $\mathbf{W}$, and of the sets Σ_r (indexed by $\mathcal{R}$) of all r-pseudoidentities satisfied by the elements of $\mathbf{W}$. Let S be a finite $\mathcal{L}$-structure in $\mathbf{V}$, satisfying all pseudoidentities in Σ. We need to show that $S \in \mathbf{W}$.

Since S is finite, there exists an onto continuous morphism $\sigma\colon \hat{F}_A(\mathbf{V}) \to S$ for some finite set A. Let also π be the natural projection of $\hat{F}_A(\mathbf{V})$ onto $\hat{F}_A(\mathbf{W})$. By Proposition 3.2, $\sim_\pi$ is exactly the subset of Σ consisting of all pure pseudoidentities on $\mathbf{V}$ on the set A which are satisfied by the elements of $\mathbf{W}$. Thus, the hypothesis made on S implies that $\sim_\pi$ is contained in $\sim_\sigma$. Similarly, for each $r \in R$, our assumption implies that $\pi^{-1}(r) \subseteq \sigma^{-1}(r)$. Therefore, by Proposition 1.4, σ factors through π, that is, there exists an onto continuous morphism of $\mathcal{L}$-structures $\tau\colon \hat{F}_A(\mathbf{W}) \to S$ such that $\sigma = \tau \circ \pi$.

$$
\begin{tikzcd}
\hat{F}_A(\mathbf{V}) \arrow[r, "\pi"] \arrow[d, "\sigma"'] & \hat{F}_A(\mathbf{W}) \arrow[dl, "\tau"] \\
S &
\end{tikzcd}
$$

By Corollary 2.5, this implies $S \in \mathbf{W}$. $\square$

4 Implicit operations

Reiterman's original approach was in terms of implicit operations. Implicit operations are defined on any non-void class of finite $\mathcal{L}$-structures, and in particular on any pseudovariety $\mathbf{V}$. Let A be a finite set. An *A-ary implicit operation x on* $\mathbf{V}$ is a family $x = (x_S)$ indexed by the elements S of $\mathbf{V}$, such that x_S is a mapping from S^A into S, and such that, for each morphism of $\mathcal{L}$-structures $\varphi: S \to T$ between elements of $\mathbf{V}$, the following diagram is commutative,

$$\begin{tikzcd} S^A \arrow[r, "x_S"] \arrow[d, "\varphi^A"'] & S \arrow[d, "\varphi"] \\ T^A \arrow[r, "x_T"] & T \end{tikzcd}$$

that is, such that $\varphi x_S((s_a)_a) = x_T((\varphi s_a)_a)$ for all $(s_a)_{a\in A}$ in S^A.

Let $\overline{\Omega}_A\mathbf{V}$ be the set of all A-ary operation on $\mathbf{V}$. Then $\overline{\Omega}_A\mathbf{V}$ is an $\mathcal{L}$-structure for the operations and relations defined as follows. Let $n \geq 1$, $f \in \mathcal{F}_n$, $r \in \mathcal{R}_n$ and $x^{(1)}, \ldots, x^{(n)} \in \overline{\Omega}_A\mathbf{V}$.

$$\begin{aligned} \left(f(x^{(1)}, \ldots, x^{(n)})\right)_S(s) &= f(x_S^{(1)}(s), \ldots, x_S^{(n)}(s)) \\ &\qquad \text{for each } S \in \mathbf{V} \text{ and } s \in S^A, \text{ and} \\ (x^{(1)}, \ldots, x^{(n)}) \in r &\iff (x_S^{(1)}(s), \ldots, x_S^{(n)}(s)) \in r \\ &\qquad \text{for each } S \in \mathbf{V} \text{ and } s \in S^A. \end{aligned}$$

For each letter $b \in A$, let $\jmath b$ be the implicit operation defined, for each $S \in \mathbf{V}$, by $(\jmath b)_S((s_a)_a) = s_b$. The substructure of $\overline{\Omega}_A\mathbf{V}$ generated by the $\jmath a$ ($a \in A$) is called the substructure of *explicit operations*. It is in fact dense in $\overline{\Omega}_A\mathbf{V}$, as a consequence of the following fact (whose proof is similar to that of [2, Proposition 1.10] and is included here for completeness).

Proposition 4.1 *Let* $\mathbf{V}$ *be an admissible pseudovariety of* $\mathcal{L}$*-structures. Then there is an isomorphism* $\psi: \overline{\Omega}_A\mathbf{V} \to \hat{F}_A(\mathbf{V})$ *such that* $\psi\jmath a = \imath a$ *for each* $a \in A$.

Proof. For each continuous onto morphism $\sigma: \hat{F}_A(\mathbf{V}) \to S$ with $S \in \mathbf{V}$, we consider the mapping $\psi_\sigma: \overline{\Omega}_A\mathbf{V} \to S$ given by $\psi_\sigma x = x_S((\sigma a)_{a\in A})$. It is easy to verify that ψ_σ is a morphism of $\mathcal{L}$-structures. We let $\psi: \overline{\Omega}_A\mathbf{V} \to \hat{F}_A(\mathbf{V})$ be the morphism induced by the ψ_σ (see Proposition 2.1). Observe that for each $a \in A$, $\psi_\sigma \jmath a = \sigma a$, so that $\psi\jmath a = (\sigma a)_\sigma = \imath a$.

We now construct the reciprocal morphism χ of ψ. For each element $x \in \hat{F}_A(\mathbf{V})$ and for each $S \in V$, we let $(\chi x)_S$ be the mapping from S^A into S defined for each $s = (s_a)_{a\in A} \in S^A$, by

$$(\chi x)_S(s) = \sigma_s x,$$

where $\sigma_s: \hat{F}_A(\mathbf{V}) \to S$ is the continuous morphism induced by $\sigma_s a = s_a$ for each $a \in A$. We then let $\chi x = ((\chi x)_S)_S$. In order to verify that χx is an implicit operation, we consider a morphism $\varphi: S \to T$ between two $\mathcal{L}$-structures in $\mathbf{V}$. Let $s = (s_a)_a \in S^A$, let $t = (\varphi s_a)_a \in T^A$ and let σ_s (resp. σ_t) be the continuous morphism from $\hat{F}_A(\mathbf{V})$ into S (resp. T) induced by $\sigma_s a = s_a$ (resp. $\sigma_t a = \varphi s_a$) for each $a \in A$. Then σ_t and $\varphi \circ \sigma_s$ coincide on A, so they coincide on $\hat{F}_A(\mathbf{V})$ by continuity. Thus $\varphi x_S(s) = \varphi(\sigma_s x) = \sigma_t x = x_T(\varphi^A s)$.

Thus χ is a mapping from $\hat{F}_A(\mathbf{V})$ into $\overline{\Omega}_A\mathbf{V}$. It is not difficult to verify that χ is a morphism and that it is the reciprocal mapping of ψ. $\square$

5 Pseudovarieties of axiomatised structures

Let us go back to the observation of Section 1.3. Let $\mathcal{L}$ be a first-order language with finitely many relation symbols, let $\mathcal{Q}$ be a quasi-variety of $\mathcal{L}$-structures (for instance, the class of ordered algebras of a given signature) and let $\mathbf{V}$ be an admissible pseudovariety of elements of $\mathcal{Q}$. Let $\mathbf{W}$ be the pseudovariety of $\mathcal{L}$-structures generated by $\mathbf{V}$. Then we know that $\mathbf{V} = \mathbf{W} \cap \mathcal{Q}$, and that $\mathbf{W}$ is the class of homomorphic images of elements of $\mathbf{V}$. It follows that $\mathbf{W}$ is admissible as well. For the same reason, if A is a finite set and $\mathbf{V}_A$ and $\mathbf{W}_A$ are the directed systems of A-generated elements of $\mathbf{V}$ and $\mathbf{W}$ (see Section 2.2), then $\mathbf{V}_A$ and $\mathbf{W}_A$ are cofinal systems, so that $\varprojlim \mathbf{V}_A = \varprojlim \mathbf{W}_A$. In particular, $\varprojlim \mathbf{W}_A \in \mathcal{Q}$, since it is a substructure of a direct product of elements of $\mathcal{Q}$. Thus it makes sense to talk of $\hat{F}_A(\mathbf{V})$, even though $\mathbf{V}$ is not in itself a pseudovariety of $\mathcal{L}$-structures. In this context, it still holds true that any mapping from A into an element S of $\mathbf{V}$ induces a unique continuous morphism from $\hat{F}_A(\mathbf{V})$ into S, and that the A-generated elements of $\mathbf{V}$ are exactly the continuous homomorphic images of $\hat{F}_A(\mathbf{V})$ *which lie in* $\mathcal{Q}$ (Theorem 2.3 and Corollary 2.5). Similarly, a class of finite elements of $\mathcal{Q}$ is a pseudovariety of elements of $\mathcal{Q}$ if and only if it is exactly the class of all finite elements of $\mathcal{Q}$ satisfying a given set of pseudoidentities. That is, Theorem 3.3 can be stated "in the framework of $\mathcal{Q}$". For instance, a pseudovariety of ordered semigroups can be defined by a set of pseudoinequalities, that is, of relational pseudoidentities of the form $x \leq y$. In such an expression, x and y may be regarded as lying in some relatively free profinite structure of the language $\mathcal{L} = \{\leq, \cdot\}$ of ordered semigroups, or in some relatively free profinite ordered semigroup.

Note. For a related construction, see Gorbunov and Tumanov [7].

6 Remarks

In applications, especially in theoretical computer science, we sometimes need to consider many-sorted algebras and many-sorted first-order structures rather than algebras. Important examples can be drawn from the field of algebraic specification [6, 15]. More recently, applications of many-sorted algebras to the study of recognizable languages of infinite words and to logic were given by Wilke [16] and Perrin and Pin [11]. The results reported above can also be adapted to this more general framework.

For a different extension of these results, observe that the constructions of $\hat{F}_A(\mathbf{V})$, which was presented here in the case of a finite set A, can be given also in the case of a profinite set A: as in the finite set case, $\hat{F}_A(\mathbf{V})$ is defined to be the projective limit of the directed system $\mathbf{V}_A$ of all A-generated elements of $\mathbf{V}$, but we consider only the continuous mappings $\sigma: A \to S_\sigma$ such that $S \in \mathbf{V}$ and $\sigma(A)$ generates S_σ. Then the analogues of the results of Section 2.2 holds. Recent work by Almeida and Weil [3] showed that it is sometimes necessary to consider such relatively free profinite structures with a profinite set of generators.

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