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Semidirect products of ordered semigroups

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Abstract

We introduce semidirect and wreath products of finite ordered semigroups and extend some standard decomposition results to this case.

1 Introduction.

All semigroups and monoids considered in this paper are either finite or free.

The semidirect product is a powerful tool for studying finite semigroups, and it has been used in the literature to give structure theorems [8, 10, 15, 31] and classification theorems [1, 3, 4, 8, 32], especially in the context of the lattice of varieties of finite semigroups. This study in turn has deep connections with the classification of recognizable languages, and the development of formal language theory within theoretical computer science has given new motivations to this aspect of semigroup theory since the 1970s.

Our aim in this paper is to develop a body of results on the semidirect product of (finite) ordered semigroups, that can be used like the more classical results on finite unordered structures. A pioneering — and inspiring — work in this direction is briefly sketched at the end of the paper by Straubing and Thérien [29], but it long looked like an isolated attempt.

Ordered semigroups were recently applied to formal language theory. A systematic approach was developed by Pin [19, 21], followed by applications by Pin [20] and Pin and Weil [23, 25]. Applications of the results of the

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present paper to language theory are deferred to the forthcoming paper [26].

The definitions of the semidirect product and the wreath product of ordered semigroups do not pose any particular difficulty, and are analogous to the definitions in the general case.

The main results of the paper can be considered as the ordered counterparts of certain classical results on semigroups. We cover in particular two decomposition theorems which were much discussed in the literature throughout the 1980s, until they were finally proved by brilliant results of Ash [5, 6, 7]. The first of these results deals with the variety generated by inverse monoids, and the second one with the variety of so-called block-groups (see [18] for a survey). As it turns out, inverse monoids are naturally equipped with an order relation which is compatible with the product, and the variety of block-groups also contains a very natural subvariety, that of ordered monoids in which every idempotent is less than or equal to 1. The importance of the latter variety in formal language theory is detailed in the authors' work [25].

Although we were able to simplify several proofs in the ordered case, we still need the strong results mentioned above to obtain our results on ordered block-groups and on ordered inverse monoids.

The paper is organized as follows. Section 2 introduces the basic objects and tools we deal with, ordered semigroups and transformation semigroups, relational morphisms, etc. In the next section, the semidirect and the wreath product of ordered semigroups and varieties are defined. Finally Section 4 contains our main results. First we discuss the elegant decomposition result of Straubing and Thérien for ordered monoids in which the unit is the maximum element. Next we state and prove our theorems on the decomposition of naturally ordered inverse monoids and of ordered block-groups.

In order to keep the paper to a reasonable size, we have only selected those elementary results on semidirect and wreath products, which are useful for our main results. More results will be covered in forthcoming papers [22, 26].

2 Some partially ordered structures

After some general results about ordered sets, we introduce the main ingredients of this paper, namely ordered semigroups and ordered transformation semigroups. Relational morphisms, divisions and varieties are the other important definitions in this section.

2.1 Ordered sets

Let $(E, \leq)$ be a partially ordered set. An *order ideal* of $(E, \leq)$ is a subset I of E such that, if $x \leq y$ and $y \in I$, then $x \in I$. The order ideal *generated* by a subset F of E is the set

$$\downarrow F = \{x \in E \mid \text{ there exists } y \in F \text{ such that } x \leq y\}$$

An order ideal I is *principal* if $I = \downarrow x$ for some $x \in E$.

Let P and Q be two partially ordered sets. A map f from P into Q is *order-preserving* if, for all $p, p' \in P$, the condition $p \leq p'$ implies $f(p) \leq f(p')$.

2.2 Ordered semigroups

A relation $\leq$ on a semigroup S is *stable* if, for every $x, y, z \in S$, $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$. An *ordered semigroup* is a semigroup S equipped with a stable partial order $\leq$ on S. Ordered monoids are defined analogously. The notation $(S, \leq)$ will sometimes be used to emphasize the role of the order relation, otherwise the order will be implicit and the notation S will be used for semigroups as well as for ordered semigroups.

If S is a semigroup, S^1 denotes the monoid equal to S if S has an identity element and to $S \cup \{1\}$, where 1 is a new element, otherwise. In the latter case, the multiplication on S is extended by setting $s1 = 1s = s$ for every $s \in S^1$. If S is an ordered semigroup without identity, the order on S is extended to an order on S^1 by setting $1 \leq 1$, but no relation of the form $1 \leq s$ or $s \leq 1$ holds for $s \neq 1$.

Example 2.1 The set $\{0, 1\}$, equipped with the usual product on integers, is a monoid, denoted U_1. The monoid U_1, equipped with order $0 \leq 1$, is an ordered monoid, denoted U_1^+. The dual ordered monoid U_1^- is obtained by taking the opposite order $1 \leq 0$.

The next proposition shows that the notion of ordered semigroup is trivial for groups.

Proposition 2.1 *In an ordered group, the order relation is the equality.*

Proof. Let $(G, \leq)$ be an ordered group of size n, and let $g, h \in G$ with $g \leq h$. Setting $s = hg^{-1}$, we have $1 \leq s$, whence $1 \leq s \leq s^2 \leq \ldots \leq s^n = 1$. Therefore $s = 1$ and $g = h$. □

Let S and T be two ordered semigroups. A *morphism of ordered semigroups* $\varphi\colon S \to T$ is an order-preserving semigroup morphism from S into T.

An ordered semigroup S is a *quotient* of an ordered semigroup R if there exists a surjective morphism of ordered semigroups from R onto S. A *congruence* on an ordered semigroup $(S, \leq)$ is a stable preorder which is coarser than $\leq$. In particular, the order relation $\leq$ is itself a congruence. If $\preceq$ is an ordered semigroup congruence on $(S, \leq)$, then the equivalence relation $\sim$ associated with $\preceq$ is a semigroup congruence on S. Furthermore, there is a well-defined stable order on the quotient set $S/{\sim}$, given by

$$[s] \leq [t] \quad \text{if and only if } s \preceq t$$

Thus $(S/{\sim}, \leq)$ is an ordered semigroup, also denoted $S/{\preceq}$.

Given a family $(S_i)_{i \in I}$ of ordered semigroups, the product $\prod_{i \in I} S_i$ is the ordered semigroup defined on the set $\prod_{i \in I} S_i$ by the law

$$(s_i)_{i \in I}(s'_i)_{i \in I} = (s_i s'_i)_{i \in I}$$

and the order given by

$$(s_i)_{i \in I} \leq (s'_i)_{i \in I} \text{ if and only if, for all } i \in I,\ s_i \leq s'_i.$$

An *ordered subsemigroup* of S is a subsemigroup of S, equipped with the restriction of the order on S. Note that an ordered semigroup S is isomorphic to an ordered subsemigroup of an ordered semigroup T if and only if there exists a morphism of semigroups $\varphi\colon S \to T$ which is also an order embedding, that is, for every $s_1, s_2 \in S$, $\varphi(s_1) \leq \varphi(s_2)$ if and only if $s_1 \leq s_2$.

Let S and T be ordered semigroups. Then S *divides* T if S is a quotient of a subsemigroup of T.

2.3 Varieties

A *variety of semigroups* is a class of semigroups closed under taking subsemigroups, quotients and finite direct products [8]. *Varieties of ordered semigroups* are defined analogously [19]. Varieties[1] of semigroups or ordered semigroups will be denoted by boldface capital letters (e.g. $\mathbf{V}$, $\mathbf{W}$).

Given a class $\mathcal{C}$ of semigroups, the variety of semigroups *generated by* $\mathcal{C}$ is the smallest variety containing $\mathcal{C}$. It is also the class of all semigroups

[1]The varieties and identities referred to in this section are also called pseudovarieties and pseudoidentities in the literature, see [1].

dividing a product of semigroups of $\mathcal{C}$. Varieties of ordered semigroups generated by a class of ordered semigroups are defined analogously.

Varieties are conveniently defined by identities. For instance, the identity $x \leq 1$ defines the variety of ordered monoids M such that, for all $x \in M$, $x \leq 1$. This variety is denoted $[\![x \leq 1]\!]$. The notation x^ω can be considered as an abbreviation for "the unique idempotent of the subsemigroup generated by x". For instance, the variety $[\![x^\omega y = x^\omega]\!]$ is the variety of semigroups S such that, for each idempotent $e \in S$ and for each $y \in S$, $ey = e$. Precise definitions can be found in the first sections of the survey paper [21]. See also [19, 24] for more specific information. Here is a list of some of the varieties occurring in this paper:

(1) $\mathbf{J}$, the variety of $\mathcal{J}$-trivial monoids. It is well-known [8] that

$$\mathbf{J} = [\![(xy)^\omega = (yx)^\omega, x^\omega = xx^\omega]\!] = [\![(xy)^\omega x = (xy)^\omega, x(yx)^\omega = (yx)^\omega]\!]$$

(2) $\mathbf{J}^+ = [\![x \leq 1]\!]$, the "positive" counterpart of $\mathbf{J}$. It is the variety of ordered monoids in which the identity is the maximum element. The "negative" counterpart of $\mathbf{J}$ is the variety $\mathbf{J}^- = [\![1 \leq x]\!]$. One can show [21] that all ordered monoids in $\mathbf{J}^+$ (or $\mathbf{J}^-$) are $\mathcal{J}$-trivial and that $\mathbf{J}$ is the join of $\mathbf{J}^+$ and $\mathbf{J}^-$ in the lattice of varieties.

(3) $\mathbf{J_1} = [\![x^2 = x, xy = yx]\!]$, the variety of commutative and idempotent monoids. They are also called semilattices, and for this reason, the notation $\mathbf{Sl}$ is sometimes preferred in the literature. The notation $\mathbf{J_1}$ refers to the first level of a hierarchy of varieties $\mathbf{J}_n$ whose union is $\mathbf{J}$. The variety $\mathbf{J_1}$ is generated by U_1.

(4) $\mathbf{J_1^+} = [\![x^2 = x, xy = yx, x \leq 1]\!]$, the "positive" counterpart of $\mathbf{J_1}$. It is the variety of ordered monoids of $\mathbf{J_1}$ in which the identity is the maximum element. The "negative" counterpart of $\mathbf{J_1}$ is the variety $\mathbf{J_1^-} = [\![x^2 = x, xy = yx, 1 \leq x]\!]$. It is easy to see that $\mathbf{J_1^+}$ is generated by U_1^+ and that $\mathbf{J_1^-}$ is generated by U_1^-. Again, $\mathbf{J_1}$ is the join of $\mathbf{J_1^+}$ and $\mathbf{J_1^-}$.

More examples can be found in [9], where a complete description of the lattice of varieties of ordered normal bands is given.

2.4 Relational morphisms and Malcev products

The definition of a relational morphism [17] can be easily extended to ordered semigroups. If $(S, \leq)$ and $(T, \leq)$ are ordered semigroups, a *relational morphism* from S to T is a relation $\tau\colon (S, \leq) \to (T, \leq)$, i.e. a mapping from S into $\mathcal{P}(T)$ such that:

(1) $\tau(s)\tau(t) \subseteq \tau(st)$ for all $s, t \in S$,

(2) $\tau(s)$ is non-empty for all $s \in S$,

For a relational morphism between two ordered monoids $(S, \leq)$ and $(T, \leq)$, a third condition is required

(3) $1 \in \tau(1)$

Equivalently, τ is a relation whose graph

$$\operatorname{graph}(\tau) = \{ \ (s, t) \in S \times T \mid t \in \tau(s) \ \}$$

is an ordered subsemigroup (resp. submonoid if S and T are monoids) of $S \times T$, with first-coordinate projection surjective onto S.

A relational morphism $\tau\colon (S, \leq) \to (T, \leq)$ is a *division* if the conditions $s_1, s_2 \in S$, $t_1 \in \tau(s_1)$, $t_2 \in \tau(s_2)$ and $t_1 \leq t_2$ imply $s_1 \leq s_2$. In particular, a division is an *injective relation*, that is, if $\tau(s_1) \cap \tau(s_2) \neq \emptyset$, then $s_1 = s_2$. In other words, for every $t \in \tau(S)$, there is a unique $s \in S$ such that $t \in \tau(s)$. The terminology "division" is motivated by the following result.

Proposition 2.2 *Let S and T be two ordered semigroups. Then S divides T if and only if there exists a division from S into T.*

Proof. First suppose that S is a quotient of a subsemigroup R of T and let $\alpha\colon R \to S$ be a surjective morphism. Then the relation α^{-1} is a relational morphism from S into T. Suppose that $s_1, s_2 \in S$, $t_1 \in \alpha^{-1}(s_1)$, $t_2 \in \alpha^{-1}(s_2)$ and $t_1 \leq t_2$. Then $s_1 = \alpha(t_1) \leq \alpha(t_2) = s_2$. Thus α^{-1} is a division.

Conversely, let $\tau\colon S \to T$ be a division, let $R \subseteq S \times T$ be the graph of τ and let $\alpha\colon R \to S$ and $\beta\colon R \to T$ be the two projections. Then α is onto and thus S is a quotient of R. We claim that R is isomorphic to an ordered subsemigroup of T. Let $(s_1, t_1), (s_2, t_2) \in R$ and suppose that $\beta(s_1, t_1) \leq \beta(s_2, t_2)$, that is, $t_1 \leq t_2$. Then $t_1 \in \tau(s_1)$ and $t_2 \in \tau(s_2)$ by definition of R, and since τ is a division, $s_1 \leq s_2$. Therefore $(s_1, t_1) \leq (s_2, t_2)$, proving the claim. It follows that S divides T. $\square$

Let $\mathbf{W}$ be a variety of ordered semigroups. A relational morphism $\tau\colon S \to T$ is called a $\mathbf{W}$-*relational morphism* if, for every idempotent $e \in T$, the ordered semigroup $\tau^{-1}(e)$ belongs to $\mathbf{W}$.

If $\mathbf{V}$ is a variety of semigroups (resp. monoids), the class $\mathbf{W} \circledM \mathbf{V}$ of all ordered semigroups (resp. monoids) S such that there exists a $\mathbf{W}$-relational morphism from S onto a semigroup (resp. monoid) of $\mathbf{V}$ is a variety of ordered semigroups, called the *Mal'cev product* of $\mathbf{W}$ and $\mathbf{V}$. In [23], the authors gave a description of a set of identities defining $\mathbf{W} \circledM \mathbf{V}$, given a set of identities describing $\mathbf{V}$ and $\mathbf{W}$.

2.5 Ordered transformation semigroups

Let P be a partially ordered set and $(S, \leq)$ be an ordered semigroup. A *right action* from S on P is a map $P \times S \to P$, denoted $(p, s) \mapsto p{\cdot}s$, which satisfies the three following conditions, for each $s, t \in S$ and $p, q \in P$:

(1) $p \leq q$ implies $p{\cdot}s \leq q{\cdot}s$,

(2) $s \leq t$ implies $p{\cdot}s \leq p{\cdot}t$,

(3) $(p{\cdot}s){\cdot}t = p{\cdot}(st)$.

Condition (3) shows that one may use the notation $p{\cdot}st$ in the place of $(p{\cdot}s){\cdot}t$ or $p{\cdot}(st)$ without any ambiguity. We will follow this convention in the sequel.

The action is *faithful* if, given $s, t \in S$, the condition $p{\cdot}s \leq p{\cdot}t$ for all $p \in P$ implies $s \leq t$. An *ordered transformation semigroup* (P, S) is a semigroup S equipped with a faithful action of S on P.

In particular, each ordered semigroup S defines an ordered transformation semigroup (S^1, S), given by the faithful action $q{\cdot}s = qs$.

An ordered transformation semigroup (P, S) *divides* an ordered transformation semigroup (Q, T) if there exists a partial surjective order preserving function $\pi : Q \to P$ and, for every $s \in S$, an element $\hat{s} \in T$, called a *cover* of s, such that, for each $q \in \mathrm{Dom}(\pi)$, $\pi(q){\cdot}s = \pi(q{\cdot}\hat{s})$.

Proposition 2.3 *If (P, S) divides (Q, T), then S divides T. If S divides T, then (S^1, S) divides (T^1, T).*

Proof. If (P, S) divides (Q, T), every element $s \in S$ has at least one cover. Furthermore, if $\hat{s}_1$ is a cover of s_1 and $\hat{s}_2$ is a cover of s_2, then $\hat{s}_1\hat{s}_2$ is a cover of s_1s_2, since, for each $q \in \mathrm{Dom}(\pi)$,

$$\pi(q){\cdot}s_1s_2 = \pi(q{\cdot}\hat{s}_1){\cdot}s_2 = \pi((q{\cdot}\hat{s}_1){\cdot}\hat{s}_2) = \pi(q{\cdot}\hat{s}_1\hat{s}_2).$$

Therefore the relation that maps each element of S to its set of covers is a relational morphism. We claim it is a division. Indeed, if $\hat{s}_1$ covers s_1 and $\hat{s}_2$ covers s_2 with $\hat{s}_1 \leq \hat{s}_2$, then, for each $q \in \mathrm{Dom}(\pi)$,

$$\pi(q){\cdot}s_1 = \pi(q{\cdot}\hat{s}_1) \leq \pi(q{\cdot}\hat{s}_2) = \pi(q){\cdot}s_2$$

Since π is surjective and the action of S is faithful, it follows $s_1 \leq s_2$, proving the claim.

Suppose now that S divides T. By Proposition 2.2, there exists a division $\tau : S \to T$. In particular, there exists, for every $t \in \tau(S)$, a unique element $\pi(t) \in S$ such that $t \in \tau(\pi(t))$. Furthermore, if $t_1 \leq t_2$, then $\pi(t_1) \leq \pi(t_2)$.

Define $\pi(1) = 1$ if T is not a monoid. Thus π is a surjective partial order-preserving map from T^1 onto S. For every $x \in S$, choose an element $\hat{x} \in \tau(x)$. We claim that, for every $t \in \tau(S)^1$ and every $x \in S$, $\pi(t \cdot \hat{x}) = \pi(t) \cdot x$. Indeed, if $s = \pi(t)$, then $t\hat{x} \in \tau(s)\tau(x) \subseteq \tau(sx)$ and thus $\pi(t \cdot \hat{x}) = sx = \pi(t) \cdot x$. Thus (S^1, S) divides (T^1, T). $\square$

3 Semidirect product and wreath product

The semidirect product of finite semigroups was systematically studied in the literature [1, 8, 32] and is still the topic of very active research [2, 3, 4, 30, 27]. Our objective in this section is to lay the foundations for its use in the ordered case.

3.1 Semidirect product

Let S and T be ordered semigroups. We write the product in S additively to provide a more transparent notation, but it is not meant to suggest that S is commutative. A *left action* of T on S is a map $(t, s) \mapsto t \cdot s$ from $T^1 \times S$ into S such that, for all $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$,

(1) $(t_1 t_2) \cdot s = t_1 \cdot (t_2 \cdot s)$
(2) $t \cdot (s_1 + s_2) = t \cdot s_1 + t \cdot s_2$
(3) $1 \cdot s = s$
(4) if $s_1 \leq s_2$ then $t \cdot s_1 \leq t \cdot s_2$
(5) if $t_1 \leq t_2$ then $t_1 \cdot s \leq t_2 \cdot s$

If S is a monoid with identity 0, the action is *unitary* if it satisfies, for all $t \in T$,

(6) $t \cdot 0 = 0$

The *semidirect product* of S and T (with respect to the given action) is the ordered semigroup $S * T$ defined on $S \times T$ by the multiplication

$$(s, t)(s', t') = (s + t \cdot s', tt')$$

and the product order:

$$(s, t) \leq (s', t') \quad \text{if and only if } s \leq s' \text{ and } t \leq t'$$

Let us verify that $\leq$ is stable. If $(s_1, t_1) \leq (s_2, t_2)$, then by (5),

$$(s_1, t_1)(s, t) = (s_1 + t_1 \cdot s, t_1 t) \leq (s_2 + t_2 \cdot s, t_2 t)$$

and by (4),

$$(s,t)(s_1,t_1) = (s + t{\cdot}s_1, tt_1) \le (s + t{\cdot}s_2, tt_2)$$

3.2 Wreath product

Let $X = (P, S)$ and $Y = (Q, T)$ be two ordered transformation semigroups. To make the notation more readable, we shall denote the semigroup S and its action on P additively and the semigroup T and its action on Q multiplicatively. The *wreath product* of X and Y, denoted $X \circ Y$, is the ordered transformation semigroup $(P \times Q, W)$ where W consists of all pairs (f, t), with f is an order-preserving function from Q into S and $t \in T$. Since we are thinking of f as acting on the right on Q, we will use the more suitable notation $q{\cdot}f$ in place of $f(q)$. The order on W is defined by $(f, t) \le (f', t')$ if and only if $t \le t'$ and, for every $q \in Q$, $q{\cdot}f \le q{\cdot}f'$.

The partial order on $P \times Q$ is the product of the orders on P and Q, and the action of W on $P \times Q$ is given by

$$(p, q){\cdot}(f, t) = (p + q{\cdot}f, q{\cdot}t) \tag{1}$$

We claim that this action is faithful. Indeed, if $(p, q){\cdot}(f, t) \le (p, q){\cdot}(f', t')$ for all $(p, q) \in P \times Q$, then $q{\cdot}t \le q{\cdot}t'$ for all $q \in Q$ and thus $t \le t'$ since T acts faithfully on Q. On the other hand, $p + q{\cdot}f \le p + q{\cdot}f'$ for all $p \in P$ and thus $q{\cdot}f \le q{\cdot}f'$ since S acts faithfully on P. Thus $f \le f'$, proving the claim. In particular W can be considered as a subset of the semigroup of all transformations on $P \times Q$. We leave it to the reader to verify that W is closed under composition and that the product on W is defined by

$$(f, t)(f', t') = (g, tt')$$

where g is defined, for each $q \in Q$ by

$$q{\cdot}g = q{\cdot}f + (q{\cdot}t){\cdot}f'$$

Let us now verify that Formula (1) really defines an action of W on $P \times Q$. If $(p, q) \le (p', q') \in P \times Q$ and $(f, t) \in W$, we have $q{\cdot}f \le q'{\cdot}f$ since f is order preserving, and thus

$$(p, q){\cdot}(f, t) = (p + q{\cdot}f, q{\cdot}t) \le (p' + q{\cdot}f, q'{\cdot}t) \le (p' + q'{\cdot}f, q'{\cdot}t) = (p', q'){\cdot}(f, t)$$

Next, if $(f, t) \le (f', t') \in W$,

$$(p, q){\cdot}(f, t) = (p + q{\cdot}f, q{\cdot}t) \le (p + q{\cdot}f', q{\cdot}t') = (p, q){\cdot}(f', t')$$

Finally, if $(f,t),(f',t') \in W$

$$\begin{aligned}\big((p,q)\cdot(f,t)\big)\cdot(f',t') = (p+q\cdot f, q\cdot t)\cdot(f',t') &= (p+q\cdot f+(q\cdot t)\cdot f', q\cdot tt') \\ &= (p,q)\big((f,t)(f',t')\big)\end{aligned}$$

Given two ordered semigroups S and T, consider the wreath product $(S^1,S)\circ(T^1,T) = (S^1\times T^1, W)$. The ordered semigroup W is called the *wreath product* of S and T and is denoted $S\circ T$. The connection with the semidirect product is the same as in the non-ordered case. Let us denote by $OP(T^1,S)$ the monoid of all order preserving functions from T^1 to S.

Proposition 3.1 *Let S and T be ordered semigroups. Then every semidirect product of S and T is a subsemigroup of $S\circ T$. Furthermore, $S\circ T$ is a semidirect product of $OP(T^1,S)$ and T.*

Proof. Let $S*T$ be a semidirect product of S and T. Let $\varphi\colon S*T\to S\circ T$ be the function defined by $\varphi(s,t) = (f,t)$ where $f\colon T^1\to S$ is given by $t\cdot f = t\cdot s$ for every $t\in T^1$. A routine verification shows that φ is a morphism of ordered semigroups.

For the second part of the statement, define an action $(t,f)\mapsto t\cdot f$ of $OP(T^1,S)$ on T by setting $t'\cdot(t\cdot f) = (t't)\cdot f$. Then the semidirect product defined by this action is isomorphic to $S\circ T$. □

We now review some basic properties of the wreath product.

Proposition 3.2 *If (P_1,S_1) divides (Q_1,T_1) and (P_2,S_2) divides (Q_2,T_2), then $(P_1,S_1)\circ(P_2,S_2)$ divides $(Q_1,T_1)\circ(Q_2,T_2)$.*

Proof. Let $\pi_1\colon Q_1\to P_1$ and $\pi_2\colon Q_2\to P_2$ be the order preserving surjective mappings defining the divisions. Let $\pi=\pi_1\times\pi_2\colon Q_1\times Q_2\to P_1\times P_2$. For $(f,s_2)\in(P_1,S_1)\circ(P_2,S_2)$, define $\widehat{(f,s_2)} = (g,\hat{s}_2)$ by choosing a cover $\hat{s}_2$ of s_2 and, for each $q_2\in Q_2$, a cover $g(q_2)$ of $f(\pi_2(q_2))$. Now, for each $(q_1,q_2)\in Q_1\times Q_2$,

$$\begin{aligned}\pi(q_1,q_2)\cdot(f,s_2) = (\pi_1(q_1),\pi_2(q_2))\cdot(f,s_2) &= (\pi_1(q_1)\cdot f(\pi_2(q_2)), \pi_2(q_2)\cdot s_2) \\ &= (\pi_1(q_1\cdot g(q_2)), \pi_2(q_2\cdot\hat{s}_2)) = \pi(q_1\cdot g(q_2), q_2\cdot\hat{s}_2) \\ &= \pi((q_1,q_2)\cdot(g,\hat{s}_2))\end{aligned}$$

and this computation concludes the proof. □

In view of Proposition 3.2, we have the following corollary.

Corollary 3.3 *If S_1 divides T_1 and S_2 divides T_2, then $S_1 \circ S_2$ divides $T_1 \circ T_2$.*

The following proposition is analogous to the standard result in the non-ordered case, and can be proved in the same fashion, see for instance [1, p. 267].

Proposition 3.4 *The wreath product on ordered transformation semigroups is associative.*

Given two varieties of ordered semigroups $\mathbf{V}$ and $\mathbf{W}$, their *semidirect product* $\mathbf{V} * \mathbf{W}$ is the variety generated by all semidirect products of the form $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$. If $\mathbf{V}$ is a monoid variety, we assume that the action of T on S is always unitary. Equivalent definitions are gathered in the next proposition, whose proof is analogous to the proof in the non-ordered case [1, p. 269]

Proposition 3.5 *Let $\mathbf{V}$ and $\mathbf{W}$ be varieties of ordered semigroups. The semidirect product $\mathbf{V} * \mathbf{W}$ is the class of all divisors of*

(1) *the semigroups of the form $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$,*

(2) *the semigroups of wreath products of the form $(P, S) \circ (Q, T)$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$,*

(3) *the wreath products of the form $S \circ T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$.*

An important and difficult problem is to find the connections between the semidirect product and the Malcev product of two varieties. The next proposition gives a very partial answer to this question.

Proposition 3.6 *Let $\mathbf{V}$ be a variety of ordered semigroups and let $\mathbf{H}$ be a variety of groups. Then $\mathbf{V} * \mathbf{H} \subseteq \mathbf{V} \circledM \mathbf{H}$.*

Proof. Let T be a semidirect product $S * H$ of some semigroup $S \in \mathbf{V}$ by some group $H \in \mathbf{H}$. Let $\pi = T \to H$ be the morphism of ordered semigroup defined by $\pi(s, h) = h$. Then $\pi^{-1}(1) = S \in \mathbf{V}$ and thus $T \in \mathbf{V} \circledM \mathbf{G}$. Therefore, $\mathbf{V} * \mathbf{G} \subseteq \mathbf{V} \circledM \mathbf{G}$. $\square$

4 Semidirect product decompositions

Wreath products have been used to decompose semigroups into smaller pieces. The same is true for ordered semigroups. This section provides several illustrations of this idea, by giving the ordered counterparts of several important decomposition results in the structure theory of finite semigroups.

4.1 A simple decomposition

We first recall an early result of Straubing and Thérien [29] (see also [12]).

Proposition 4.1 *Every finite ordered monoid satisfying the identity $x \leq 1$ embeds in a wreath product of copies of U_1^+.*

Proof. Let $(M, \leq)$ be a finite ordered monoid satisfying the identity $x \leq 1$. Such a monoid is necessarily $\mathcal{J}$-trivial and therefore has a zero, denoted 0. Furthermore, since $0 \leq 1$, it follows $0 = 0x \leq 1x = x$ for every $x \in M$. Let x be a 0-minimal element of M, that is, a minimal element in $M \setminus \{0\}$. Then $J = \{0, x\}$ is both an ideal and an order ideal of M. We identify the Rees quotient M/J with the set $(M \setminus J) \cup \{0\}$. It is also an ordered monoid, with the order inherited from M. Denote by π the canonical morphism $(M, \leq) \to (M/J, \leq)$. The following lemma shows (by induction on $|M|$) that $(M, \leq)$ is a submonoid of $\underbrace{U_1^+ \circ (U_1^+ \circ (\cdots \circ U_1^+) \cdots)}_{|M|-1 \text{ factors}}$. $\square$

Lemma 4.2 *The ordered monoid $(M, \leq)$ is an ordered submonoid of $U_1^+ \circ (M/J, \leq)$.*

Proof. Define a function $\varphi \colon M \to U_1^+ \circ (M/J, \leq)$ by setting $\varphi(m) = (f, \pi(m))$ where f is the order-preserving map from $(M/J, \leq)$ into U_1^+ defined, for each $q \in M/J$ by

$$q{\cdot}f = \begin{cases} 1 & \text{if } qm \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

We claim that that φ is a monoid morphism. Let $m, m' \in M$. Then $\varphi(m)\varphi(m') = (f, \pi(m))(f', \pi(m')) = (g, \pi(m)\pi(m'))$, where $g \colon M/J \to U_1$ is defined by $q{\cdot}g = (q{\cdot}f)(q\pi(m)){\cdot}f'$. Then the following equivalences hold

$$\begin{aligned} q{\cdot}g = 1 &\iff q{\cdot}f = 1 \text{ and } (q\pi(m)){\cdot}f = 1 \\ &\iff qm \neq 0 \text{ and } qmm' \neq 0 \\ &\iff qmm' \neq 0 \end{aligned}$$

It follows that $\varphi(m)\varphi(m') = \varphi(mm')$, proving the claim.

Next we show φ is an order-embedding. Let $\varphi(m) = (f, \pi(m))$ and let $\varphi(m') = (f', \pi(m'))$. If $m \leq m'$, then $\pi(m) \leq \pi(m')$. Let $q \in M/J$. Since $qm \leq qm'$, the condition $qm \neq 0$ implies $qm' \neq 0$. It follows that $q{\cdot}f = 1$ implies $q{\cdot}f' = 1$ and thus $f \leq f'$. Therefore $\varphi(m) \leq \varphi(m')$. Conversely, if $\varphi(m) \leq \varphi(m')$, then $\pi(m) \leq \pi(m')$, and thus $m \leq m'$ unless $m, m' \in J = \{0, x\}$. In the latter case, $1{\cdot}f \leq 1{\cdot}f'$ and thus if $m \neq 0$ then $m' \neq 0$. It follows that $m \leq m'$. $\square$

Corollary 4.3 *The variety* $\mathbf{J}^+$ *is the smallest variety of ordered monoids closed under wreath product and containing* U_1^+*.*

Proof. Let $\mathbf{V}$ be the smallest variety of ordered monoids closed under wreath product and containing U_1^+. Corollary 4.1 shows that $\mathbf{V}$ contains $\mathbf{J}^+$. Consider a semidirect product $M * N$ of two ordered monoids of $\mathbf{J}^+$. Since the order on $M * N$ is the product order, the identity $x \leq 1$ is still valid in $M * N$. It follows that $\mathbf{J}^+$ is closed under semidirect product, and hence under wreath product. Since $\mathbf{J}^+$ contains U_1^+, $\mathbf{V}$ is equal to $\mathbf{J}^+$. $\square$

4.2 Ordered inverse semigroups

We now consider ordered inverse monoids and give ordered counterparts of several well known results, like the Vagner-Preston representation theorem, and certain decomposition results relative to inverse semigroups. Recall that a semigroup M is inverse if for every element $x \in M$, there exists a unique element $\bar{x}$ such that $\bar{x}x\bar{x} = \bar{x}$ and $x\bar{x}x = x$. We refer the reader to [16] for basic results on inverse semigroups.

Let M be an inverse monoid. It is well known that the relation $\leq$ on M defined by

$$x \leq y \text{ if and only if } x = ye \text{ for some idempotent } e \text{ of } M$$

is a stable partial order, called the *natural order* of M. The term *ordered inverse monoid* will refer to an inverse monoid, ordered by its natural order.

Let Q be a finite set. We denote by $\mathfrak{S}(Q)$ the symmetric group on Q. Similarly, we denote by $(I(Q), \leq)$ the symmetric inverse ordered monoid on Q, i.e. the ordered monoid of all injective partial functions from Q to Q under composition. In this inverse monoid, the natural order can be characterized as follows : $f \leq g$ if and only if $\mathrm{Dom}(f) \subseteq \mathrm{Dom}(g)$ and the restriction of g to $\mathrm{Dom}(f)$ is exactly f. Note that the restriction of this order to $\mathfrak{S}(Q)$ is the equality. Since every morphism of inverse semigroups preserves the natural order, the Vagner-Preston representation theorem states that every ordered inverse monoid M is an ordered submonoid of the ordered inverse monoid $(I(M), \leq)$.

We now adapt the presentation of [15] to obtain a simple decomposition of $(I(Q), \leq)$. Let $(2^Q, \leq)$ be the ordered monoid of subsets of Q under intersection, ordered by set inclusion. Note that 2^Q is an idempotent and commutative monoid and that $(2^Q, \leq) \in \mathbf{J_1^+}$. Define a left action of $\mathfrak{S}(Q)$ on $(2^Q, \leq)$ by setting for all $\sigma \in \mathfrak{S}(Q)$ and $P \subseteq Q$, $\sigma{\cdot}P = \sigma^{-1}(P)$. This action defines a semidirect product $2^Q * \mathfrak{S}(Q)$ and we can formulate our first result:

Proposition 4.4 *The ordered monoid $(I(Q), \leq)$ is a quotient of the ordered monoid $(2^Q, \leq) * \mathfrak{S}(Q)$.*

Proof. It is a well-known fact that $I(Q)$ is a quotient of $U_1 \circ \mathfrak{S}(Q)$ (see for instance [15] for a proof). Now, the proposition follows immediately from the fact that if E is a semilattice and G is a group, the natural partial order on $E * G$ is the product order. $\square$

Let **Inv** be the variety of monoids generated by all inverse monoids and let **Ecom** be the variety of ordered monoids with commuting idempotents. These varieties satisfy a well-known sequence of equalities [10].

$$\mathbf{Inv} = \mathbf{J_1} * \mathbf{G} = \mathbf{J_1} \circledM \mathbf{G} = \mathbf{Ecom} \tag{2}$$

To obtain an ordered version of these formulas, we introduce the variety of ordered monoids $\mathbf{Inv^+}$, generated by all ordered inverse monoids and $\mathbf{Ecom^+}$, the variety of ordered monoids with commuting idempotents such that $e \leq 1$ for every idempotent e.

Theorem 4.5 *The following equalities hold :*

$$\mathbf{Inv^+} = \mathbf{J_1^+} * \mathbf{G} = \mathbf{J_1^+} \circledM \mathbf{G} = \mathbf{Ecom^+}$$

Proof. First, the Vagner-Preston representation theorem shows that $\mathbf{Inv^+}$ is generated by the symmetric inverse monoids. The inclusion $\mathbf{Inv^+} \subseteq \mathbf{J_1^+} * \mathbf{G}$ now follows from Proposition 4.4.

Proposition 3.6 shows that $\mathbf{V} * \mathbf{H} \subseteq \mathbf{V} \circledM \mathbf{H}$.

Let $M \in \mathbf{J_1^+} \circledM \mathbf{G}$. Then, by definition, there exists a relational morphism $\tau \colon M \to G$ such that $\tau^{-1}(1) \in \mathbf{J_1^+}$. If e is idempotent in M, $\tau(e)$ is a non-empty subsemigroup of G and hence a subgroup of G. In particular, $1 \in \tau(e)$ and $e \in \tau^{-1}(1)$. Now since $\tau^{-1}(1) \in \mathbf{J}_1^+$, the idempotents of M commute and $e \leq 1$ for every idempotent $e \in M$. Thus $\mathbf{J}_1^+ ⓜ \mathbf{G} \subseteq \mathbf{Ecom}^+$.

We are left with the remaining inclusion $\mathbf{Ecom}^+ \subseteq \mathbf{Inv}^+$. The corresponding inclusion in Formula (2) is the hardest one and was solved by Ash [5]. Let $M \in \mathbf{Ecom}^+$. By Formula (2), M divides an inverse monoid N. That is, there exists a relational morphism $\tau\colon M \to N$ such that $x\tau \cap y\tau \neq \emptyset$ implies $x = y$. Let now $s_1, s_2 \in M$, $t_1 \in \tau(s_1)$, $t_2 \in \tau(s_2)$ and suppose that $t_1 \leq t_2$. Then $t_1 = t_2 f$ for some idempotent $f \in N$. Let e be an idempotent of the semigroup $\tau^{-1}(f)$. Then $t_1 = t_2 f \in \tau(s_2)\tau(e) \subseteq \tau(s_2 e)$. Therefore $t_1 \in \tau(s_1) \cap \tau(s_2 e)$, whence $s_1 = s_2 e$. Thus $s_1 \leq s_2$ and τ is a division of ordered monoids, as required. □

4.3 Ordered block-groups

A *block-group* is a monoid in which every $\mathcal{R}$-class and every $\mathcal{L}$-class has at most one idempotent. For instance, every inverse monoid is a block-group. In fact the inverse monoids are exactly the regular block-groups. The following theorem of [13] summarizes some equivalent formulations of the definition of block-groups. As usual $E(M)$ denotes the set of idempotents of M.

Theorem 4.6 *Let M be a monoid. The following conditions are equivalent:*

(1) *M is a block-group,*

(2) *For every regular $\mathcal{D}$-class D of M, D^0 is a Brandt semigroup,*

(3) *For all $e, f \in E(M)$, $e \mathrel{\mathcal{R}} f$ implies $e = f$ and $e \mathrel{\mathcal{L}} f$ implies $e = f$,*

(4) *For all $e, f \in E(M)$, $efe = e$ implies $ef = e = fe$.*

(5) *The submonoid generated by $E(M)$ is $\mathcal{J}$-trivial.*

Given a finite monoid M, denote by $\mathcal{P}(M)$ (resp. $\mathcal{P}'(M)$) the monoid of subsets (resp. non-empty subsets) of M under the multiplication of subsets, defined, for all $X, Y \subseteq M$, by $XY = \{xy \mid x \in X \text{ and } y \in Y\}$. Denote by $\mathcal{P}_1(M)$ the submonoid of $\mathcal{P}'(M)$ consisting of all subsets of M containing the identity. Define a partial order on $\mathcal{P}'(M)$ by setting $X \leq Y$ if and only if $Y \subseteq X$. By definition, $\mathcal{P}_1(M)$ satisfies the identity $x \leq 1$ and thus belongs to the variety $\mathbf{J}^+$. For a group G, there is a nice connection between G, $(\mathcal{P}_1(G), \leq)$ and $(\mathcal{P}'(G), \leq)$.

Proposition 4.7 *If G is a group, then $(\mathcal{P}'(G), \leq)$ is a quotient of a semidirect product of the form $(\mathcal{P}_1(G), \leq) * G$.*

Proof. Let G act on the left on $(\mathcal{P}_1(G), \leq)$ by conjugation. That is, set $g{\cdot}X = gXg^{-1}$ for all $g \in G$ and $X \in \mathcal{P}_1(G)$. This defines a semidirect product of $\mathcal{P}_1(G)$ by G. Now, the map $\pi\colon \mathcal{P}_1(G) * G \to \mathcal{P}'(G)$ defined by $\pi(X, g) = Xg$ is a surjective morphism of semigroups since $\pi(X, g)\pi(Y, h) = XgYh$ and

$$\pi((X, g)(Y, h)) = \pi(X + gY, gh) = \pi(X + gYg^{-1}, gh) = XgYg^{-1}gh = XgYh$$

Furthermore, π is order-preserving: if $(X, g) \leq (Y, h)$, then $Y \subseteq X$ and $g = h$, whence $Yh \subseteq Xg$, that is, $\pi(X, g) \leq \pi(Y, h)$. $\square$

The decomposition of block-groups has been thoroughly studied (see [11, 13, 14] or [10, 18] for a survey). Let $\mathbf{PG}$ be the variety of monoids generated by all monoids of the form $\mathcal{P}'(G)$, where G is a group and let $\mathbf{BG}$ be the variety of block-groups. These varieties satisfy the formulas

$$\mathbf{PG} = \mathbf{J} * \mathbf{G} = \mathbf{J} \,Ⓜ\, \mathbf{G} = \mathbf{BG} \tag{3}$$

Here again we prove an ordered version of these formulas. For this purpose, we introduce the variety of ordered monoids $\mathbf{PG}^+$, generated by all ordered monoids of the form $(\mathcal{P}'(G), \leq)$ and $\mathbf{BG}^+$, the variety of ordered monoids such that $e \leq 1$ for every idempotent e. It is amusing to see that this simple identity implies all the identities of $\mathbf{BG}$. Indeed, it implies that the submonoid generated by the idempotents is in $\mathbf{J}^+$, and in particular, is $\mathcal{J}$-trivial!

The variety $\mathbf{BG}^+$ plays an important role in language theory (see [20, 25]). We mention the following results because they are needed in the proof of the next theorem. The statement of these results assumes some knowledge of language theory, including the definition of the ordered syntactic monoid, which can be found in [19, 21]. Recall that a *group language* is a recognizable language whose syntactic monoid is a group. Denote by Pol($\mathcal{G}$) the class of languages which are finite unions of languages of the form $L_0a_1L_1 \cdots a_kL_k$ where the L_i's are group languages and the a_i's are letters. The following is proved in [20].

Proposition 4.8 *A language is in Pol($\mathcal{G}$) if and only if its syntactic ordered monoid belongs to* $\mathbf{J}^+ \,Ⓜ\, \mathbf{G}$.

We shall need another useful result, which occurs in the proofs of Theorem 3.2 of [14] and Proposition 6.5 in [18].

Proposition 4.9 *Every language of Pol($\mathcal{G}$) can be written as $\varphi(L)$ where L is a group language and φ is a length preserving morphism.*

Finally, we shall need a strong result of Ash [6, 7], reformulated here in a form directly adapted to our purpose. Recall that an element $\bar{x}$ is a *weak inverse* of x if $\bar{x}x\bar{x} = \bar{x}$. If M is an ordered monoid, denote by $D(M)$ the smallest submonoid of M closed under weak conjugation, that is, such that if $\bar{x}$ is a weak inverse of x, and if $s \in D(M)$, then $xs\bar{x}$ and $\bar{x}sx$ are in $D(M)$.

Theorem 4.10 *Let* $\mathbf{V}$ *be a variety of ordered monoids and let* M *be an ordered monoid. Then* $M \in \mathbf{V} \mathbin{\textcircled{\scriptsize M}} \mathbf{G}$ *if and only if* $D(M) \in \mathbf{V}$.

We are now ready to state the ordered counterpart of Formula (3).

Theorem 4.11 *The following equalities hold:*

$$\mathbf{PG}^+ = \mathbf{J}^+ * \mathbf{G} = \mathbf{J}^+ \mathbin{\textcircled{\scriptsize M}} \mathbf{G} = \mathbf{BG}^+$$

Proof. Proposition 4.7 shows that $\mathbf{PG}^+ \subseteq \mathbf{J}^+ * \mathbf{G}$. We also have $\mathbf{J}^+ * \mathbf{G} \subseteq \mathbf{J}^+ \mathbin{\textcircled{\scriptsize M}} \mathbf{G}$ by Proposition 3.6.

Some arguments of language theory are required to obtain the inclusion $\mathbf{J}^+ \mathbin{\textcircled{\scriptsize M}} \mathbf{V} \subseteq \mathbf{PG}^+$. Proposition 4.8 shows that $\mathbf{J}^+ \mathbin{\textcircled{\scriptsize M}} \mathbf{V}$ is generated by the syntactic ordered monoids of the languages of $\mathrm{Pol}(\mathcal{G})$. By Proposition 4.9, such a language can be written as $\varphi(L)$ where L is a group language of A^* and $\varphi\colon A^* \to B^*$ is a length preserving morphism. Let $\pi\colon A^* \to G$ be the syntactic morphism of L and let $P = \pi(L)$. We claim that $\varphi(L)$ is recognized by $(\mathcal{P}'(G), \leq)$. Indeed, let $\eta\colon B^* \to \mathcal{P}'(G)$ be the morphism defined by $\eta(b) = \pi(\varphi^{-1}(b))$ for each $b \in B$ and let

$$\mathcal{R} = \{Q \in \mathcal{P}'(G) \mid Q \cap P \neq \emptyset\}$$

Then $\mathcal{R}$ is an order ideal of $(\mathcal{P}'(G), \leq)$. Indeed, if $Q \in \mathcal{R}$ and $Q' \leq Q$, then $Q \subseteq Q'$ and, since $Q \cap P \neq \emptyset$, $Q' \cap P \neq \emptyset$, that is, $Q' \in \mathcal{R}$. Furthermore

$$\begin{aligned}
\eta^{-1}(\mathcal{R}) &= \{u \in B^* \mid \eta(u) \cap P \neq \emptyset\} \\
&= \{u \in B^* \mid \pi(\varphi^{-1}(u)) \cap P \neq \emptyset\} \\
&= \{u \in B^* \mid \varphi^{-1}(u) \cap \pi^{-1}(P) \neq \emptyset\} \\
&= \{u \in B^* \mid \varphi^{-1}(u) \cap L \neq \emptyset\} \\
&= \varphi(L)
\end{aligned}$$

It follows that the syntactic ordered monoid of L divides $(\mathcal{P}'(G), \leq)$, and in particular, it belongs to $\mathbf{PG}^+$. Therefore $\mathbf{J}^+ \mathbin{\textcircled{\scriptsize M}} \mathbf{G} \subseteq \mathbf{PG}^+$.

We have established so far the equalities $\mathbf{PG}^+ = \mathbf{J}^+ * \mathbf{G} = \mathbf{J}^+ \mathbin{\textcircled{\scriptsize M}} \mathbf{G}$. We now show that $\mathbf{J}^+ \mathbin{\textcircled{\scriptsize M}} \mathbf{G} \subseteq \mathbf{BG}^+$. If $M \in \mathbf{J}^+ \mathbin{\textcircled{\scriptsize M}} \mathbf{G}$, there exists by definition a relational morphism $\tau: M \to G$ such that $\tau^{-1}(1) \in \mathbf{J}^+$. Again as in the proof of Theorem 4.5, one can show that $E(M)$ is contained in $\tau^{-1}(1)$. Since $\tau^{-1}(1) \in \mathbf{J}^+$, we have $e \leq 1$ for every idempotent $e \in M$.

Finally, consider a monoid M in $\mathbf{BG}^+$. We claim that $D(M) \in \mathbf{J}^+$. Indeed, setting

$$R(M) = \{x \in M \mid x \leq 1\}$$

we observe that $R(M)$ is a submonoid of M closed under weak conjugation. Indeed, if $s \leq 1$ and if $\bar{x}$ is a weak inverse of x, then $xs\bar{x} \leq x\bar{x} \leq 1$, since $x\bar{x}$ is idempotent, and similarly, $\bar{x}sx \leq 1$. Therefore $D(M)$ is a submonoid of $R(M)$, and since $R(M) \in \mathbf{J}^+$ by construction, $D(M) \in \mathbf{J}^+$. We now conclude by Theorem 4.10 that $M \in \mathbf{J}^+ \textcircled{M} \mathbf{G}$, and thus $\mathbf{BG}^+ \subseteq \mathbf{J}^+ \textcircled{M} \mathbf{G}$. $\square$

As was mentioned above, we were not able to adapt entirely the proof of the unordered case to the ordered case. More specifically, the standard proof that $\mathbf{BG}$ is contained in $\mathbf{J} \textcircled{M} \mathbf{G}$ does not seem to be easy to adapt to the ordered case. It is interesting to note, however, that the bottleneck is not located at the same place as in the unordered case. In the unordered case, the most difficult part of the proof is the inclusion $\mathbf{J} \textcircled{M} \mathbf{G} \subseteq \mathbf{J} * \mathbf{G}$, but the inclusion $\mathbf{J}^+ \textcircled{M} \mathbf{G} \subseteq \mathbf{J}^+ * \mathbf{G}$ is easier to prove. On the other hand, the inclusion $\mathbf{BG} \subseteq \mathbf{J} \textcircled{M} \mathbf{G}$ can be proved relatively easily, but we don't know of any simple proof of the inclusion $\mathbf{BG}^+ \subseteq \mathbf{J}^+ \textcircled{M} \mathbf{G}$.

Remark. While this paper was in preparation, B. Steinberg submitted a paper [28] with the following generalization of Theorem 4.11: he proves that some of the equalities in that statement also hold if $\mathbf{G}$ is replaced by a variety of groups $\mathbf{H}$, provided that $\mathbf{H}$ satisfies certain closure conditions.

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References

[1] J. Almeida, *Finite semigroups and universal algebra*, Series in Algebra Vol 3, World Scientific, Singapore, 1994.

[2] J. Almeida, A. Azevedo and L. Teixeira, On finitely based pseudovarieties of the form $\mathbf{V} * \mathbf{D}$ and $\mathbf{V} * \mathbf{D}_n$, *J. Pure and Applied Algebra* **146** (2000) 1–15

[3] J. Almeida et P. Weil. Free profinite semigroups over semidirect products, *Izvestiya VUZ Matematika* **39** (1995) 3–31. English version, *Russian Mathem. (Iz. VUZ.)* **39** (1995) 1–28.

[4] J. Almeida and P. Weil, Profinite categories and semidirect products, *J. Pure and Applied Algebra* **123** (1998) 1–50.

[5] C.J. Ash, Finite semigroups with commuting idempotents, *J. Austral. Math. Soc., Ser. A* **43**, (1987), 81–90.

[6] C. J. Ash, Inevitable sequences and a proof of the type II conjecture, in *Proceedings of the Monash Conference on Semigroup Theory*, World Scientific, Singapore, (1991), 31–42.

[7] C. J. Ash, Inevitable Graphs: A proof of the type II conjecture and some related decision procedures, *International Journal of Algebra and Computation* **1**, (1991), 127–146.

[8] S. Eilenberg, *Automata, languages and machines*, Vol. B, Academic Press, New York, 1976.

[9] S. Emery, Varieties and pseudovarieties of ordered normal bands, *Semigroup Forum*, **58**, (1999), 348–366.

[10] K. Henckell, S. W. Margolis, J.-E. Pin and J. Rhodes, Ash's Type II Theorem, Profinite Topology and Malcev Products, *International Journal of Algebra and Computation* **1**, (1991), 411–436.

[11] K. Henckell and J. Rhodes, The theorem of Knast, the $PG = BG$ and Type II Conjectures, in J. Rhodes (ed.) *Monoids and Semigroups with Applications*, Word Scientific, (1991), 453–463.

[12] M. Kunze, Semidirect decompositions of semigroups, transformation semigroups and automata, in "*Words, Languages and Combinatorics*" (M. Ito ed.), World Scientific, Singapore, 309–325, 1992.

[13] S. W. Margolis and J.-E. Pin, Varieties of finite monoids and topology for the free monoid, in *Proceedings of the 1984 Marquette Conference on Semigroups* (K. Byleen, P. Jones and F. Pastijn eds.), Marquette University, (1984), 113–130.

[14] S. W. Margolis and J.E. Pin, Product of group languages, *FCT Conference, Lect. Notes in Comp. Sci.* **199**, (1985), 285–299.

[15] S. W. Margolis and J.E. Pin, Inverse semigroups and varieties of finite semigroups, *Journal of Algebra* **110**, (1987), 306–323.

[16] M. Petrich, *Inverse semigroups*, Wiley, New York, 1984.

[17] J.-E. Pin, *Variétés de langages formels*, Masson Paris, 1984; English translation: *Varieties of formal languages*, Plenum, New-York, 1986.

[18] J.-E. Pin, **BG** = **PG**, a success story, in NATO Advanced Study Institute *Semigroups, Formal Languages and Groups*, (J. Fountain ed.), Kluwer Academic Publishers, (1995), 33–47.

[19] J.-E. Pin, A variety theorem without complementation, *Izvestiya VUZ Matematika* **39** (1995) 80–90. English version, *Russian Mathem. (Iz. VUZ)* **39** (1995) 74–83.

[20] J.-E. Pin, Polynomial closure of group languages and open sets of the Hall topology, *Theoretical Computer Science* **169** (1996), 185–200.

[21] J.-E. Pin, Syntactic semigroups, Chapter 10 in *Handbook of formal languages*, G. Rozenberg and A. Salomaa eds., Springer, 1997.

[22] J.-E. Pin, A. Pinguet and P. Weil, Ordered categories and ordered semigroups, submitted, (1999).

[23] J.-E. Pin and P. Weil, Profinite semigroups, Malcev products and identities, *J. Algebra* **182**, (1996), 604–626.

[24] J.-E. Pin and P. Weil, A Reiterman theorem for pseudovarieties of finite first-order structures, *Algebra Universalis*, **35**, (1996), 577–595.

[25] J.-E. Pin and P. Weil, Polynomial closure and unambiguous product, *Theory Comput. Systems* **30**, (1997), 1–39.

[26] J.-E. Pin and P. Weil, The wreath product principle for ordered semigroups, submitted, (1999).

[27] B. Steinberg, Semidirect products of categories and applications, *J. Pure Appl. Algebra* **142**, (1999), 153-182.

[28] B. Steinberg, Polynomial closure and topology, *International Journal of Algebra and Computation* **10**, (2000), 603–624.

[29] H. Straubing and D. Thérien, Partially ordered finite monoids and a theorem of I. Simon, *J. of Algebra* **119**, (1985), 393–399.

[30] L. Teixeira, Estudo do produto semidirecto de pseudovariedades, Ph. D. Thesis, University Porto, 1998.

[31] B. Tilson, Chapters 11 and 12 of [8].

[32] B. Tilson, Categories as algebras: an essential ingredient in the theory of monoids, *J. Pure Appl. Algebra* **48**, (1987), 83–198.