Monoids of upper-triangular matrices

Jean-Eric Pin[1], Howard Straubing[2]

Abstract

We study the variety **W** generated by monoids of upper-triangular boolean matrices. First, we present **W** as a natural extension of the variety **J** of finite $\mathcal{J}$-trivial monoids and we give a description of the family of recognizable languages whose syntactic monoids are in **W**. Then we show that **W** can be described in terms of the generalized Schützenberger product of finite monoids. We also show that **W** is generated by the power monoids of members of **J**. Finally we consider the membership problem for **W** and the connection with the dot-depth hierarchy in language theory. Although the majority of our results are purely " semigroup-theoretic" we use recognizable languages constantly in the proofs.

Résumé

Nous étudions la variété **W** engendrée par les monoïdes de matrices booléennes triangulaires supérieures. Nous présentons tout d'abord **W** comme une extension naturelle de la variété **J** des monoïdes finis $\mathcal{J}$-triviaux et nous décrivons l'ensemble des langages reconnaissables dont le monoïde syntactique est dans **W**. Puis nous montrons que **W** peut être décrite en termes de produit de Schützenberger généralisé de monoïdes finis. Nous montrons également que **W** est engendrée par les monoïdes des parties associés aux éléments de **J**. Finalement, nous étudions le problème de l'appartenance à **W** et les relations qui existent entre **W** et les hiérarchies de concaténation en théorie des langages. Bien que la majorité de nos résultats soient de pure théorie des semigroupes, nous utilisons constamment les langages reconnaissables dans les démonstrations.

The subject of the present paper belongs to the theory of varieties of finite monoids and recognizable languages. We refer the reader to the books by Eilenberg [2] and Lallement [3] for the elements of this theory. Here we study the variety generated by monoids of upper-triangular boolean matrices. This is a subvariety of the variety of finite aperiodic monoids and, as we show in this paper, it appears in a number of different contexts.

In this first section we present the variety **W** generated by monoids of upper-triangular boolean matrices as a natural extension of the variety **J** of finite $\mathcal{J}$-trivial monoids (which was studied in detail by Simon [12, 13]). We give a

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description of the family of recognizable languages whose syntactic monoids are in $\mathbf{W}$.

In the second section we show that the variety $\mathbf{W}$ can be described in terms of the generalized Schützenberger product of finite monoids (introduced in [16]). More exactly, we prove that a monoid M belongs to $\mathbf{W}$ if and only if M divides an n-fold Schützenberger product $\Diamond_n(M_1, \ldots, M_n)$ where $M_1, \ldots, M_n$ are idempotent and commutative finite monoids.

In the third section we show that $\mathbf{W}$ can be described in terms of power sets: if $\mathbf{V}$ is a variety of finite monoids, then $\mathbf{PV}$ denotes the variety generated by $\{\mathcal{P}(M) \mid M \in \mathbf{V}\}$, where $\mathcal{P}(M)$ is the power set of M. The operation $\mathbf{V} \to \mathbf{PV}$ has been studied by several authors [4, 5, 6, 7, 9, 14]. Here we use these earlier results to prove that $\mathbf{W} = \mathbf{PJ}$.

In the final section we consider the membership problem for $\mathbf{W}$. That is, we would like an algorithm for determining, given the multiplication table of a finite monoid M, whether or not $M \in \mathbf{W}$. We are able to give an effective necessary condition for membership in $\mathbf{W}$, but we do not yet know if our condition is sufficient. In the same section we cite an unpublished result of Straubing[1] which connects the variety $\mathbf{W}$ to the dot-depth hierarchy of Brzozowski.

Although the majority of our results are purely "semigroup theoretic" — in the sense that they make no reference to recognizable languages or the theory of automata — we use recognizable languages constantly in the proofs. In essence we are exploiting the correspondence between varieties of monoids and varieties of languages, as described in Eilenberg [2]. This provides us with a powerful tool for proving theorems about varieties of finite monoids.

1 The variety generated by monoids of upper-triangular matrices

Let $n \geqslant 1$. We denote by M_n the set of all $n \times n$ matrices over the boolean semiring $\mathbb{B} = \{0, 1\}$ (in which $1+1 = 1$) and by K_n the set of all upper triangular matrices in M_n all of which diagonal entries equal 1. That is,

$$K_n = \{m \in M_n \mid m_{i,j} = 0 \text{ for } 1 \leqslant j < i \leqslant n \text{ and } m_{i,i} = 1 \text{ for } 1 \leqslant i \leqslant n\}.$$

K_n is closed under multiplication of matrices and is thus a submonoid of the multiplicative monoid M_n. We define

$$\mathbf{U} = \{M \mid M \prec K_n \text{ for some } n.\}.$$

That is, $\mathbf{U}$ consists of all monoids (necessarily finite) which are divisors of K_n for a certain n. (We say that a monoid M divides a monoid M' if M is a quotient of a submonoid of M' — see [2]).

The family $\mathbf{U}$ is evidently closed under division. It is also closed under direct product: to see this observe that there is an injective morphism $\varphi : M_m \times M_n \to$

[1] This result has later been published in [17].

M_{m+n} defined by

$$(p,q)\varphi_{i,j} = \begin{cases} 0 & \text{if } i \leqslant m \text{ and } j > m \\ 0 & \text{if } i > m \text{ and } j \leqslant m \\ p_{i,j} & \text{if } i \leqslant m \text{ and } j \leqslant m \\ q_{i-m,j-m} & \text{if } i > m \text{ and } j > m, \end{cases}$$

and that φ embeds $K_m \times K_n$ into K_{m+n}. Since $N_1 \prec M_1$ and $N_2 \prec M_2$ implies $N_1 \times N_2 \prec M_1 \times M_2$, it follows that $\mathbf{U}$ contains the direct product of any two of its members. Thus $\mathbf{U}$ is a *variety of finite monoids* in the sense used by Eilenberg [2, Chapter V].

If $\mathbf{V}$ is a variety of finite monoids and A is a finite alphabet, then we denote by $A^*\mathcal{V}$ the family of recognizable languages in A^* whose syntactic monoids belong to $\mathbf{V}$. Eilenberg has shown [2, Chapter VII] that every variety of finite monoids is generated by the syntactic monoids it contains. Thus if $\mathbf{V}_1$ and $\mathbf{V}_2$ are varieties of finite monoids, $\mathbf{V}_1 \subset \mathbf{V}_2$ if and only if $A^*\mathcal{V}_1 \subset A^*\mathcal{V}_2$ for every alphabet A. This enables us to show that two varieties are equal by showing that the corresponding families of recognizable languages are equal.

In the case of the variety $\mathbf{U}$ defined above, Straubing [15] showed that $A^*\mathcal{U}$ is the boolean closure of the family of languages of the form

$$A^*a_1A^* \cdots a_kA^*, \text{ where } a_1, \ldots, a_k \in A.$$

A deep result of Simon [13] asserts that this is precisely the family of languages whose syntactic monoids belong to the variety $\mathbf{J}$ of finite $\mathcal{J}$-trivial monoids. (A monoid is $\mathcal{J}$-trivial if and only if the $\mathcal{J}$-relation is the identity). We thus have

Theorem 1. $\mathbf{U} = \mathbf{J}$. *(That is, a finite monoid M is $\mathcal{J}$-trivial if and only if it divides K_n for some n).*

We now consider the family T_n of all $n \times n$ matrices over the semiring $\mathbb{B}$ which are *upper triangular.* T_n is a submonoid of M_n which contains K_n. We define

$$\mathbf{W} = \{M \mid M \prec T_n \text{ for some } n\}.$$

Once again it is evident that $\mathbf{W}$ is closed under division, and that the morphism: $\varphi : M_m \times M_n \to M_{m+n}$ maps $T_m \times T_n$ into T_{m+n}. Thus $\mathbf{W}$ is a variety of finite monoids. This variety is the principal concern of the present paper.

We begin by describing the family of recognizable languages corresponding to $\mathbf{W}$:

Theorem 2. *$A^*\mathcal{W}$ is the boolean closure of the family of languages of the form*

$$A_0^*a_1A_1^* \cdots a_kA_k^*$$

where $k \geqslant 0$, $a_1, \ldots, a_k \in A$ and $A_0, \ldots, A_k$ are (possibly empty) subsets of A.

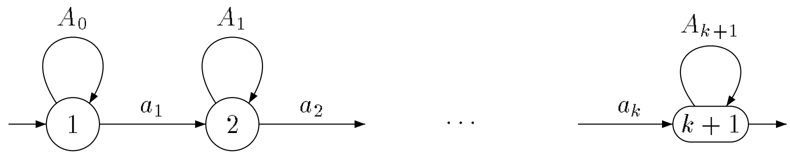
Remark. If $A_i = \emptyset$ then $A_i^* = \{1\}$.

Proof. Let $\mathcal{F}$ denote the boolean closure of the family of languages of the form $A_0^*a_1A_1^* \cdots a_kA_k^*$, where $k \geqslant 0$, $a_1, \ldots, a_k \in A$ and $A_0, \ldots, A_k \subset A$. We first show that $\mathcal{F} \subset A^*\mathcal{W}$.

For this it suffices to show that the syntactic monoid of any language of the form $L = A_0^* a_1 A_1^* \cdots a_k A_k^*$ is in $\mathbf{W}$, since it is known [2, Chapter VII] that $A^*\mathcal{V}$ is closed under boolean operations for any variety $\mathbf{V}$. We will show that L is *recognized* by the monoid T_{k+1}: that is, there exists a morphism $\psi : A^* \to T_{k+1}$ and a set $X \subset T_{k+1}$ such that $X\psi^{-1} = L$. This implies [2, Chapter VII] that $M(L) \prec T_{k+1}$ and thus $M(L) \in \mathbf{W}$. The morphism ψ is defined by

$$(a\psi)_{i,j} = \begin{cases} 1 & \text{if } i = j \text{ and } a \in A_{i-1} \\ 1 & \text{if } j = i+1 \text{ and } a = a_i \\ 0 & \text{otherwise} \end{cases}$$

for all $a \in A$, $i, j \in \{1, \ldots, k+1\}$. It is easy to verify that if $w \in A^*$ then $(w\psi)_{i,j} = 1$ if and only if there is a path labeled w from state i to state j in the nondeterministic automaton pictured below



In particular, $(w\psi)_{1,k+1} = 1$ if and only if $w \in A_0^* a_1 A_1^* \cdots a_k A_k^* = L$. Thus $L = X\psi^{-1}$, where $X = \{m \in T_{k+1} \mid m_{1,k+1} = 1\}$. This proves that $\mathcal{F} \subset A^*\mathcal{W}$.

To prove the opposite inclusion, suppose that $L \in A^*\mathcal{W}$. Then $M(L) \in \mathbf{W}$ and consequently L is recognized by T_n for some $n \geqslant 1$. Thus there exists a morphism $\eta : A^* \to T_n$ and a subset X of T_n such that $L = X\eta^{-1}$. We need to show that $X\eta^{-1} \in \mathcal{F}$. Since $X\eta^{-1} = \bigcup_{x \in X} x\eta^{-1}$ and since $\mathcal{F}$ is closed under boolean operations, it suffices to show that $x\eta^{-1} \in \mathcal{F}$ for each $x \in T_n$. Now

$$\begin{aligned} x\eta^{-1} &= \bigcap_{1 \leqslant i,j \leqslant n} \{w \mid (w\eta)_{i,j} = x_{i,j}\} \\ & \bigcap_{\{(i,j) \mid x_{i,j}=1\}} \{w \mid (w\eta)_{i,j} = 1\} \setminus \bigcap_{\{(i,j) \mid x_{i,j}=0\}} \{w \mid (w\eta)_{i,j} = 0\} \end{aligned}$$

Thus it suffices to show that each set of the form $\{w \mid (w\eta)_{i,j} = 1\}$ belongs to $\mathcal{F}$. Let $A_{k,\ell} = \{a \in A \mid (a\eta)_{k,\ell} = 1\}$ and let $Q_{i,j}$ be the set of all *strictly increasing* sequences $(i_0, \ldots, i_t)$ such that $i_0 = i$ and $i_t = j$. (If $i > j$ then $Q_{i,j}$ is empty. If $i = j$, then $Q_{i,j}$ consists of the single sequence (i)). Then

$$\{w \mid (w\eta)_{i,j} = 1\} = \bigcup_{(i_0,\ldots,i_t) \in Q_{i,j}} A_{i_0,i_0}^* A_{i_0,i_1} A_{i_1,i_1}^* \cdots A_{i_{k-1},i_k} A_{i_k}^*$$

Since each language $A_{i_0,i_0}^* A_{i_0,i_1} A_{i_1,i_1}^* \cdots A_{i_{k-1},i_k} A_{i_k}^*$ is a finite union of languages of the form $A_{i_0,i_0}^* a_1 A_{i_1,i_1}^* \cdots a_k A_{i_k}^*$ it follows that $\{w \mid (w\eta)_{i,j} = 1\} \in \mathcal{F}$. This completes the proof of Theorem 2. $\square$

From Theorem 2 we can deduce that the monoids T_n are *aperiodic* — that is, they contain no notrivial groups. Let $\mathbf{Ap}$ denote the variety of finite aperiodic monoids, and $A^*\mathcal{A}p$ the family of recognizable languages in A^* whose syntactic monoids are in $\mathbf{Ap}$. According to a theorem of Schützenberger [10], $A^*\mathcal{A}p$ is the smallest family of languages in A^* which contains all the languages $\{a\}$, $a \in A$, and which is closed under boolean operations and product. Now for each subset B of A, $B^* \in A^*\mathcal{A}p$ because $M(B^*)$ is either the two element aperiodic monoid $U_1 = \{0,1\}$ (if $\emptyset \neq B \neq A$) or $M(B)$ is trivial (if $B = \emptyset$ or $B = A$). Since $A^*\mathcal{A}p$ contains the letters and is closed under products, it follows that every language of the form $A_0^*a_1A_1^* \cdots a_kA_k^*$ is in $A^*\mathcal{A}p$. Since $A^*\mathcal{A}p$ is closed under boolean operations, it follows from Theorem 2 that $A^*\mathcal{W} \subset A^*\mathcal{A}p$. Thus $\mathbf{W} \subset \mathbf{Ap}$. In particular the monoids T_n are aperiodic.

2 Connection with the Schützenberger product

In [10] Schützenberger introduced a binary product on finite monoids to study the product operation on recognizable languages. This was generalized to an n-fold product by Straubing [16]. Here we recall the definition of this product and some of its basic properties.

Let $M_1, \ldots, M_n$ be finite monoids and consider the set $\mathcal{P}(M_1 \times \cdots M_n)$ of all subsets of $M_1 \times \cdots M_n$. Multiplication in the direct product $M_1 \times \cdots M_n$ is extended to $\mathcal{P}(M_1 \times \cdots M_n)$ by the formula $XY = \{xy \mid x \in X, y \in Y\}$ for all $X, Y \in \mathcal{P}(M_1 \times \cdots M_n)$. Addition in $\mathcal{P}(M_1 \times \cdots M_n)$ is defined by $X + Y = X \cup Y$. With these operations $\mathcal{P}(M_1 \times \cdots M_n)$ is a semiring (with $\{(1, \ldots, 1)\}$ as the multiplicative identity and $\emptyset$ as the additive identity) and we can thus consider the monoid $\mathcal{M}$ of all $n \times n$ matrices over $\mathcal{P}(M_1 \times \cdots M_n)$. The *Schützenberger product* $\diamondsuit_n(M_1, \ldots, M_n)$ is the submonoid of $\mathcal{M}$ consisting of all matrices P such that:

(1) $P_{i,j} = 0$ if $i > j$,

(2) $P_{i,i} = \{(1, \ldots, 1, s_i, 1, \ldots, 1)\}$ for some $s_i \in M_i$,

(3) $P_{i,j} \subseteq \{(s_1, \ldots, s_n) \in M_1 \times \cdots \times M_n \mid s_k = 1 \text{ if } k < i \text{ or if } k > j\}$ if $i < j$,

The following property of the Schützenberger product was proved by Reutenauer [9] in the case $n = 2$ and by Pin [8] in general:

If $L \subset A^*$ is recognized by $\diamondsuit_n(M_1, \ldots, M_n)$ then L belongs to the boolean closure of the family of languages of the form $L_0a_1L_1 \cdots a_kL_k$ where $k \geqslant 0$, $a_1, \ldots, a_k \in A$ and $M(L_j) \prec M_{i_j}$ for some sequence $1 \leqslant i_0 < i_1 < \cdots < i_k \leqslant n$.

If $\mathbf{V}$ is a variety of finite monoids we denote by $\diamondsuit\mathbf{V}$ the smallest variety which contains all the Schützenberger products of the form $\diamondsuit_n(M_1, \ldots, M_n)$, where $M_1, \ldots, M_n \in \mathbf{V}$.

In Section 1 we showed that $T_m \times T_n$ is a submonoid of T_{m+n}. An identical argument shows that $\diamondsuit_m(M_1, \ldots, M_m) \times \diamondsuit_n(M_1', \ldots, M_n')$ is a submonoid of $\diamondsuit_{m+n}(M_1, \ldots, M_n, M_1', \ldots, M_n')$. It follows that $M \in \diamondsuit\mathbf{V}$ if and only if M divides a Schützenberger product $\diamondsuit_k(M_1, \ldots, M_k)$ where $M_1, \ldots, M_k$ all belong to $\mathbf{V}$.

As above, we denote by $\mathbf{J}$ the variety of $\mathcal{J}$-trivial monoids. $\mathbf{J_1}$ denotes the variety of idempotent and commmutative monoids, $\mathbf{R}$ the variety of $\mathcal{R}$-trivial monoids and $\mathbf{R}^r$ the variety of $\mathcal{L}$-trivial monoids. Finally, $\mathbf{DA}$ denotes the variety of aperiodic monoids with the property that every regular $\mathcal{J}$-class is closed under multiplication. We have the inclusions $\mathbf{J_1} \subset \mathbf{J} \subset \mathbf{R} \subset \mathbf{DA}$ and $\mathbf{J_1} \subset \mathbf{J} \subset \mathbf{R}^r \subset \mathbf{DA}$.

Theorem 3. $\mathbf{W} = \diamondsuit\mathbf{J_1} = \diamondsuit\mathbf{J} = \diamondsuit\mathbf{R} = \diamondsuit\mathbf{R}^r = \diamondsuit\mathbf{DA}$

Proof. In light of the inclusions cited above it suffices to prove that $\mathbf{W} \subset \Diamond\mathbf{J_1}$ and that $\Diamond\mathbf{DA} \subset \mathbf{W}$. We prove the first of these inclusions by showing that T_n is a submonoid of $\Diamond_n(U_1, \ldots, U_1)$, where U_1 is the two-element monoid $\{0, 1\}$. (Since $U_1 \in \mathbf{J_1}$, this implies that $\mathbf{W} \subset \Diamond\mathbf{J_1}$.) Indeed, if $m \in T_n$, let $m\varphi$ be the element of $\Diamond_n(U_1, \ldots, U_1)$ defined by

$$(m\varphi)_{i,j} = \begin{cases} \emptyset & \text{if } m_{i,j} = 0 \\ \{(1, \ldots, 1)\} & \text{if } m_{i,j} = 1 \end{cases}$$

For example, if $n = 3$ and $m = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ then

$$m\varphi = \begin{pmatrix} \{(1,1,1)\} & \{(1,1,1)\} & \emptyset \\ \emptyset & \emptyset & \{(1,1,1)\} \\ \emptyset & \emptyset & \{(1,1,1)\} \end{pmatrix}$$

It is now easy to verify that $\varphi : T_n \to \Diamond_n(U_1, \ldots, U_1)$ is an injective morphism.

To prove the inclusion $\Diamond\mathbf{DA} \subset \mathbf{W}$ we make use of the following result of Schützenberger [11]. If $L \subset A^*$ is a recognizable language such that $M(L) \in \mathbf{DA}$, then L is a finite disjoint *union* of languages of the form

$$A_0^* a_1 A_1^* \cdots a_k A_k^*$$

where $A_0, \ldots, A_k \subset A$, $a_1, \ldots, a_k \in A$, and where the product $A_0^* a_1 A_1^* \cdots a_k A_k^*$ is unambiguous. From this result and the property of the Schützenberger product cited above, we conclude that every language whose syntactic monoid is in $\Diamond\mathbf{DA}$ is in the boolean closure of the family of languages of the form $B_0^* b_1 B_1^* \cdots b_m B_m^*$ where $B_0, \ldots, B_m \subset A$ and $b_1, \ldots, b_m \in A$. (We have used neither the unambiguity of the product $A_0^* a_1 A_1^* \cdots a_k A_k^*$ or the fact that the union is disjoint — only the fact that the product of languages distributes over union). It follows from Theorem 2 that $M(L) \in \mathbf{W}$. Thus $\Diamond\mathbf{DA} \subset \mathbf{W}$. This completes the proof of Theorem 3. $\square$

Corollary 4. *Let M be a finite monoid. The following conditions are equivalent:*

(1) *$M \prec T_n$ for some $n \geqslant 1$.*

(2) *$M \prec \Diamond_m(U_1, \ldots, U_1)$ for some $m \geqslant 1$*

Proof. We showed in the proof of Theorem 3 that T_n is a submonoid of $\Diamond_n(U_1, \ldots, U_1)$ and thus (1) implies (2). Conversely, if $M \prec \Diamond_m(U_1, \ldots, U_1)$, then $M \in \Diamond\mathbf{J_1}$ and, by Theorem 3, $M \in \mathbf{W}$. Thus $M \prec T_n$ for some $n \geqslant 1$ and consequently (2) implies (1). $\square$

3 Connection with power sets

If M is a finite monoid then $\mathcal{P}(M)$, the set of subsets of M, is a finite monoid with respect to the operation

$$XY = \{xy \mid x \in X, y \in Y\} \quad \text{for all } X, Y \in \mathcal{P}(M)$$

If $\mathbf{V}$ is a variety of finite monoids then we denote by $\mathbf{PV}$ the smallest variety which contains $\{\mathcal{P}(M) \mid M \in \mathbf{V}\}$. Thus $M \in \mathbf{PV}$ if and only if there exist $M_1, \ldots, M_k \in \mathbf{V}$ such that $M \prec \mathcal{P}(M_1) \times \cdots \times \mathcal{P}(M_k)$. The operation $\mathbf{V} \to \mathbf{PV}$ has been studied by several authors [4, 5, 6, 7, 9, 14]. We cite the following result, which appeared in [14]:

Let A be a finite alphabet and let $A^*\mathcal{PV}$ be the family of recognizable languages whose syntactic monoids belong to $\mathbf{PV}$. $A^*\mathcal{PV}$ is the boolean closure of the family of languages of the form $L\varphi$, where $L \subset B^*\mathcal{V}$ for some finite alphabet B, and where $\varphi : B^* \to A^*$ is a length-preserving morphism (that is, $B\varphi \subset A$).

Theorem 5. $\mathbf{W} = \mathbf{PJ} = \mathbf{PR} = \mathbf{PR}^r = \mathbf{PDA}$

Proof. As in the proof of Theorem 3, it suffices to show that $\mathbf{W} \subset \mathbf{PJ}$ and that $\mathbf{PDA} \subset \mathbf{W}$.

We begin by showing that $\mathbf{PDA}$ is contained in $\mathbf{W}$: let A and B be finite alphabets and let $\varphi : B^* \to A^*$ be a length-preserving morphism. If $L \subset B^*$ is a recognizable language such that $M(L) \in \mathbf{DA}$, then, by the result of Schützenberger cited in Section 2, L is a union of languages of the form

$$B_0^* b_1 B_1^* \cdots b_k B_k^*$$

where $B_0, \ldots, B_k \subset B$ and $b_1, \ldots, b_k \in B$. It follows that L is a union of languages of the form

$$A_0^* a_1 A_1^* \cdots a_k A_k^* \qquad (*)$$

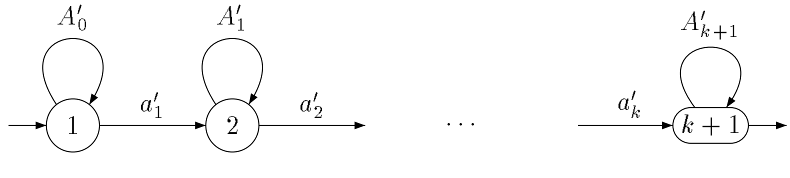
where $A_0, \ldots, A_k \subset A$ and $a_1, \ldots, a_k \in A$. The result cited above on the variety $\mathbf{PV}$ implies that if $L \subset A^*$ is a language such that $M(L) \in \mathbf{PDA}$, then L belongs to the boolean closure of the family of languages of the form $(*)$. It follows from Theorem 2 that $M(L) \in \mathbf{W}$. Thus $\mathbf{PDA} \subset \mathbf{W}$.

To prive that $\mathbf{W} \subset \mathbf{PJ}$ it suffices (by Theorem 2 and the fact that $A^*\mathcal{PJ}$ is closed under boolean operations) to show that each language of the form $L = A_0^* a_1 A_1^* \cdots a_k A_k^*$ belongs to $A^*\mathcal{PJ}$. For each $i = 0, \ldots, k$ we consider a copy A_i' of A_i, and for each $j = 1 \ldots, k$ a copy a_j' of a_j such that the sets $A_0', \ldots, A_k', \{a_1'\}, \ldots, \{a_k'\}$ are pairwise disjoint. Let B be the union of these sets. Let $\varphi : B \to A$ be the map which sends each $b \in A_i'$ to the corresponding letter in A_i and each a_j' to a_j. The map φ extends to a length-preserving $\varphi : B^* \to A^*$, and we have

$$L = (A_0')^* a_1' (A_1')^* \cdots a_k' (A_k')^*.$$

In light of the result on the operation $\mathbf{V} \to \mathbf{PV}$ cited above, it remains to show that $L' = (A_0')^* a_1' (A_1')^* \cdots a_k' (A_k')^*$ belongs to $B^*\mathcal{J}$.

L' is recognized by the automaton



Since $a_i' \notin A_{i-1}'$ for $i = 1, \ldots, k$, this automaton is deterministic (though not complete) and reduced, and consequently $M(L')$, the syntactic monoid of L', is the monoid of partial transformations on the states induced by the words of B^*. Now if i is a state of the automaton and $x, y, z \in B^*$ are such that $ixyz = ix = j$, then $(yz) \in (A_j')^*$, consequently $y \in (A_j')^*$, and thus $ixy = ix$. It follows that in $M(L')$, $m_1 m_2 m_3 = m_1$ implies $m_1 m_2 = m_1$ and thus $M(L')$ is $\mathcal{R}$-trivial. The identical argument shows that the syntactic monoid of $(A_k')^* a_k \cdots a_1 (A_1')^*$, which is the reversal of $M(L')$, is $\mathcal{R}$-trivial. Thus $M(L')$ is $\mathcal{L}$-trivial as well, and thus $\mathcal{J}$-trivial. Hence $L' \in B^*\mathcal{J}$. This completes the proof.

Corollary 6. *$M \in \mathbf{W}$ if and only if $M \prec \mathcal{P}(K_n)$ for some $n > 1$.*

Proof. By Theorem 1, $\mathcal{P}(K_n) \in \mathbf{PJ}$ and by Theorem 5, $\mathbf{PJ} = \mathbf{W}$. Thus $M \prec \mathcal{P}(K_n)$ implies $M \in \mathbf{W}$.

In [11] it is proved that if $\mathbf{V}$ is a nontrivial variety of finite monoids, then $\mathbf{PV}$ is generated by the monoids $\{\mathcal{P}'(M) \mid M \in \mathbf{V}\}$, where $\mathcal{P}'(M)$ denotes the monoid of *nonempty* subsets of M. Thus if $M \in \mathbf{PJ}$ there exists $M_1, \ldots, M_r \in \mathbf{J}$ such that $M \prec \mathcal{P}'(M_1) \times \cdots \times \mathcal{P}'(M_r)$. Now it is easy to see that the map $(X_1, \ldots, X_r) \to X_1 \times \ldots \times X_r$ is an injective morphism embedding from $\mathcal{P}'(M_1) \times \cdots \times \mathcal{P}'(M_r)$ into $\mathcal{P}(M_1 \times \cdots \times M_r)$. Thus $M \prec \mathcal{P}(M_1 \times \cdots \times M_r)$. Now $M_1 \times \cdots \times M_r \in \mathbf{J}$, and, by Theorem 1, $M_1 \times \cdots \times M_r \prec K_n$ for some $n \geqslant 1$. Since $M' \prec M''$ implies $\mathcal{P}(M') \prec \mathcal{P}(M'')$, we obtain $M \prec \mathcal{P}(K_n)$. □

4 Further results and open problems

The varieties $\mathbf{J}$ and $\mathbf{W} = \mathbf{PJ}$ play a role in the dot-depth hierarchy of Brzozowski (see [1] and [2, Chap. 9] for the definitions). Let $\mathbf{V}_k$ be the variety generated by the syntactic semigroups of languages of dot-depth less than or equal to k. In [12] it is whown that

$$\mathbf{V}_1 = \mathbf{J} * \mathbf{D}$$

that is, the variety generated by semidirect products of the form $M * S$, where $M \in \mathbf{J}$ and S is a definite semigroup (see [2, Chap. 5]). More generally, it is shown in [17] that for all $k > 1$ the variety $\mathbf{V}_k$ is of the form $\mathbf{V}_k' * \mathbf{D}$ where $\mathbf{V}_k'$ is a variety of finite monoids. Furthermore $\mathbf{V}_2' = \mathbf{PJ}$, hence

$$\mathbf{V}_2 = \mathbf{PJ} * \mathbf{D}$$

Thus $\mathbf{J}$ and $\mathbf{PJ}$ are the first two levels in an infinite hierarchy of varieties of finite monoids, whose union is the variety of all *aperiodic* monoids.

The most important open problem concerning the variety $\mathbf{W} = \mathbf{PJ}$ is the decision problem: is there an algorithm to determine whether or not a finite monoid M, given by its multiplication table, belongs to $\mathbf{W}$? (Such an algorithm exists for the variety $\mathbf{J}$, because we can write down the $\mathcal{J}$-classes of M once we possess the multiplication table). We have not found such an algorithm — however, we do have an effective necessary condition for membership in $\mathbf{W}$: if M is a finite monoid and $e \in M$ is an idempotent, then we denote by M_e the subsemigroup of M generated by the elements of M which are greater than or equal to e in the $\mathcal{J}$-ordering on M. We can then form the subsemigroup $eM_e e$, which is a monoid whose identity element is e. Our necessary condition is:

Theorem 7. *If $M \in \mathbf{W}$, then $eM_e e \in \mathbf{J}$ for every idempotent e in M.*

Proof. It is easy to show that for any variety $\mathbf{V}$ of finite monoids, the family $\{M \mid eM_e e \in \mathbf{V}\}$ is also a variety (see [8]). Thus, by the corollary to Theorem 3, it suffices to show that if $M = \Diamond_n(U_1, \ldots, U_1)$ then $eM_e e \in \mathbf{J}$. In [16] it is shown that the Schützenberger product $\Diamond_n(M_1, \ldots, M_n)$ has the following property:

There exists a surjective morphism $\pi : \Diamond_n(M_1, \ldots, M_n) \to M_1 \times \cdots \times M_n$ such that for each idempotent $f \in M_1 \times \cdots \times M_n$, the semigroup $S = f\pi^{-1}$ is locally $\mathcal{J}$-trivial — that is, $eSe \in \mathbf{J}$ for each idempotent $e \in S$.

Now let us consider an idempotent $e \in M = \Diamond_n(U_1, \ldots, U_1)$ and an element a of M_e. Then $a = a_1 \cdots a_k$ where $e \leqslant_{\mathcal{J}} a_i$ for $i = 1, \ldots, k$. Since $U_1 \times \cdots \times U_1$ is idempotent and commmutative, $(eae)\pi = (e\pi)(a_1\pi)(e\pi)\cdots(e\pi)(a_k\pi)(e\pi)$. In an idempotent and commmutative semigroup, $s \leqslant_{\mathcal{J}} t$ implies $st = s$, thus $(e\pi)(a_i\pi) = e\pi$ for $i = 1, \ldots, k$, and consequently $eae \in (e\pi)\pi^{-1}$. Thus $eM_e e = e[eMe]e \subset e[(e\pi)\pi^{-1}]e \in \mathbf{J}$. Thus $eM_e e \in \mathbf{J}$.

We do not know if the converse of Theorem 7 is true — if it were, it would provide an effective method for testing membership in $\mathbf{W}$.

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