A Logical Account for Linear Partial Differential Equations

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Abstract
Differential Linear Logic (DiLL), introduced by Ehrhard and Regnier, extends linear logic with a notion of linear approximation of proofs. While DiLL is a classical logic, classical models of it in which this notion of differentiation corresponds to the usual one of functional analysis were missing. We solve this issue by constructing a model, without higher order, based on nuclear topological vector spaces and distributions with compact support. This interpretation sheds a new light on the rules of DiLL as we are able to understand them as the computational steps for the resolution of Linear Partial Differential Equations. We thus introduce D-DiLL, a deterministic refinement of DiLL with a D-exponential, for which we exhibit a cut-elimination procedure, and a categorical semantics. When $D$ is a Linear Partial Differential Operator with constant coefficients, then the D-exponential is interpreted as the space of distributions acting on the parameters $f$ of the differential equation $Dg = f$. The inference rules represent the computational steps for the construction of the solution $g$. We recover linear logic and its differential extension DiLL as a particular case.

Keywords Differential Linear Logic, Linear Partial Differential Equations, Functional Analysis, Categorical semantics

1 Introduction
A Partial Differential Equation (PDE) is an equation $Dg = f$, where $D$ is a Partial Differential Operator, that is a (possibly non-linear) combination of partial derivatives, with smooth functions as coefficients. The study of Partial Differential Equations (PDEs) through theoretical, numerical and computational methods is one of the most active areas of modern mathematics. Most effort concentrate on non-linear equations such as Navier-Stokes equation. Programs are used to approximate solutions to these, and applied mathematicians work at finding quick and efficient algorithms to do so.

Linear PDEs (LPDEs) are easier to solve theoretically, and when they have constant coefficients a universal method was found separately by Malgrange and Ehrenpreis. Examples of LPDEs with constant coefficients (LPDEcc) include fundamental examples such as the Laplacian equation or the heat equation:

$$\sum_i \frac{\partial^2 q}{\partial x_i^2} = f \quad \text{or} \quad \frac{\partial q}{\partial t} - \alpha \left( \frac{\partial^2 q}{\partial x_1^2} + \frac{\partial^2 q}{\partial x_2^2} + \frac{\partial^2 q}{\partial x_3^2} \right) = f.$$

In this paper, we construct a proof syntax, with cut-elimination, whose denotational model correspond to the resolution of LPDEs in the theory of distributions. We also prove that the categorical constructions for a model of DiLL match the constructions of the theory of distributions. This builds an new and strong bridge between Logic and Mathematical Physics, by extending the Proof/Function part of the Curry-Howard-Lambek correspondance to LPDEs. We understand this result as a first step towards a more general computational theory encompassing non-linear PDEs. On a more practical level, we believe D-DiLL could lead to a type system for the verification of numerical programs.

From linear to non-linear proofs and back. Linear Logic (LL) was introduced by Girard [13] as a proof theory where a distinction is made between linear deductions of $B$ under the hypothesis $A$, and non-linear ones. The former is represented by the sequent $A \vdash B$, while the latter is represented by $!A \vdash B$. The intuition is that a linear proof will make use of $A$ exactly once: thus, $!A$ is traditionally interpreted as a collection of all finite copies of $A$. The inference rules for the exponential connective $!$ of LL then represent a calculus of resources. Among these rules, the dereliction rule $\delta$ allows to deduce $!A \vdash B$ from $A \vdash B$: thus linear proofs can always be considered as non-linear ones.

DiLL was introduced by Ehrhard and Regnier [9], as a refinement of LL without its promotion rules but with dual exponential rules. It features in particular a coderection rule $\delta$ allowing to deduce from a sequent $!A \vdash B$ a linear approximation of it: $A \vdash B$. This second sequent is considered as the differentiation of the first sequent. Both LL and DiLL are first presented under a classical form: sequents are monolateral $\vdash A \vdash B$, and formulas are equivalent to their double linear-negation $A^{\bot\bot}$. A sequent $!A \vdash B$ is then rewritten $?A^{\bot\bot} \vdash B$, where $?$ is the "why not" modality.

Thus DiLL, and after that differential or quantitative $\lambda$-calculi, are traditionally understood as a logic and as calculi of approximations, as they account a syntactic variant of the Taylor Formula [28]. In this paper, we change this point of view and consider it as the basis for a calculus of Partial Differential Equations.

The equation solved by DiLL, and its generalisation. The fundamental idea behind this paper is that $A^{\bot\bot}$ is the type of $\psi$ such that $\delta(\psi) = \phi$, for $\phi$ of type $!A$.

This is true as the level of functions: a function $q$ is linear, i.e. of type $A^{\bot}$, if and only if there is $f : A^{\bot}$ such that the differential at $0$ of $f$ corresponds to $q$. The previous statement extends this at the level of linear duals of spaces of functions, that is spaces of distributions.
We generalize this idea into a new connective !\(D\), and a new codereliction rule \(\frac{!A}{\frac{\Delta}{\check{\psi}D(\check{\psi}D)} \frac{\psi}{\check{\psi}D(\check{\psi}D)} \frac{\Gamma}{\check{\psi}D(\check{\psi}D)}}{\check{\psi}D(\check{\psi}D)}\).

The fact that we work in a classical setting is central here, as is allows to understand \(d : \Delta \rightarrow !\) and to generalize it as \(d : \Delta \rightarrow !\). DiLL corresponds to a special case where \(!\gamma \equiv \text{id}\).

We then construct a new sequent calculus D-DiLL which refines DiLL, and modelises the resolution of Linear Partial Differential Equations:

\[
\frac{\Gamma \vdash A}{\frac{\Gamma, !A}{!A}} \quad \frac{\Gamma \vdash \Delta}{\frac{\Gamma, !\Delta}{\Delta}}
\]

In a categorical setting, the cut-elimination procedure of D-DiLL translates into

\[
\check{\psi}D(\check{\psi}D(\check{\psi}D)) = \phi
\]

for \(\phi \) of type \(!A\). In the syntax, this says that the solution \(\psi\) to the equation \(\check{\psi}D(\check{\psi}D) = \phi\) is exactly \(\check{\psi}D(\check{\psi}D(\check{\psi}D))\). As we will see, this is interpreted in models of D-DiLL exactly as the resolution of a Linear Partial Differential Equation in the theory of distributions [17].

A classical and smooth semantics. Denotational semantics is the study of proofs and programs through their interpretation as denotations (functions) between spaces. In a denotational model of LL, there are spaces \(\mathcal{L}(E, F)\) of linear functions from \(E\) to \(F\), spaces \(C(E, F)\) of non-linear ones, and a way to understand non-linear functions on \(E\) as linear functions on \(!E\). \(\mathcal{L}(\mathcal{L}(E, F), \mathcal{L}(E, F)) \equiv C(E, F)\).

In a model of DiLL, functions must also be smooth, that is able to be iteratively differentiatied everywhere. We write \(C^\infty(E, F)\) the space of all smooth functions between \(E\) and \(F\).

The first models of DiLL introduced by Ehrhard [5, 6] have a discrete basis: non-linear proofs are interpreted as power series between spaces of sequences. In order to get a better understanding of the differential nature of DiLL rules, one is bound to search for a denotational model of DiLL where functions are interpreted as the smooth functions of differential geometry or functional analysis. But to account for linearity of functions, and for the classical setting of DiLL, one needs to interpret formulas as some topological vector spaces which are reflexive: denoting \(E' = \mathcal{L}(E, \mathbb{R})\), we need \(E \cong E''\).

The requirements for reflexivity to be preserved by the connectives of LL and the ones for having smooth functions work as opposite forces. More precisely:

- One needs a monoidal category of reflexive spaces, that is spaces which are isomorphic to their bidual and such that this property is preserved by tensor product and internal hom-sets. This is true for euclidean spaces, but fails in general when considering infinite dimensional spaces: it is false in particular for Banach spaces.
- One needs a cartesian closed category of smooth functions: we want \(C^\infty(E \times F, G) \cong C^\infty(E, C^\infty(F, G))\). These structures are notably scarce in analysis, but are fundamental in the semantics of LL as it accounts for the possibility to curry programs.

Solutions to the first point are for instance models based on spaces of sequences [5, 6], or topological vector spaces with very coarse topologies [19]. Solutions to the second point were constructed by Fröhlicher, Kriegl and Michor [21], leading to models of Intuitionnistic DiLL [20], [2]. The attempt by Girard to interpret LL in Banach spaces [12] fails, as the requirement of a norm on power series is to strong to allow a good cartesian closed category. We propose here a classical and smooth model of DiLL without promotion, while another one with a more intricate structure and interpreting promotion was recently exposed in [3].

Computing with distributions. Distributions are naturally in the quest for a model of LL. On the one hand, consider a model of DILL made of \(\mathbb{K}\)-vector spaces, with spaces of linear functions \(\mathcal{L}(E, F)\), and spaces of smooth functions \(C^\infty(E, F) \equiv \mathcal{L}(E, F)\). Then as these spaces are reflexive we have a precise idea of the interpretation of the exponential and its negation.

\[!E \cong C^\infty(E, \mathbb{K})' \quad \text{and} \quad ?E \cong C^\infty(E, \mathbb{K})\]

Thus \(!E\) is a space of linear forms acting on a certain space of smooth function, i.e. a space of distributions.

On the other hand, one of the central isomorphism in the categorical semantics of LL is the Seely’s isomorphism: \(!A \otimes B \equiv (A \otimes B)\). It translates immediately into the Schwartz’s Kernel theorem [25], written here for distributions with compact support: \(C^\infty(\mathbb{R}^n, \mathbb{K})' \otimes C^\infty(\mathbb{R}^m, \mathbb{K}) \equiv C^\infty(\mathbb{R}^{n+m}, \mathbb{K})\). Based on these intuitions, we find a classical semantics of DiLL in the theory of Nuclear spaces and distributions. The language of distributions has been used for a while in Linear Logic, and this work can be seen as a way to ground this fact.

Nuclear spaces, Fréchet space, and distributions: a model of Smooth DiLL. Typical examples of nuclear spaces are either euclidean spaces as \(\mathbb{R}^n\) or \(\mathbb{R}^m\), either spaces of (test) function \(\mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \) or their duals, spaces of distributions \(\mathcal{D}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)', \mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)'\). Moreover, a nuclear Fréchet space (that is, nuclear, complete and metrisable space) is reflexive, and while it is not preserved by duality, this condition is preserved by tensor product. We translate this structure in the syntax (section 4) obtaining a polarized Smooth DiLL with a distinction between finitary and smooth formulas.

Modelizing D-DiLL by Linear Partial Differential Equations. Our definition of \(D = \text{DILL}\) is justified by the fact that for \(D\) any linear partial differential operator (LPDO) with constant coefficients, we have a model of D-DiLL.

\([\mathbb{R}^n]\) is then interpreted as the space of distribution with compact support \(\mathcal{E}(\mathbb{R}^n), D\) as a LPDO, and

\[!D\mathbb{R}^n := D(\mathcal{E}(\mathbb{R}^n))'\],

\[\check{\psi}D(\check{\psi}D) = \phi\]
Consider $D_0$ the operator mapping a function $f \in C^\omega(\mathbb{R}, \mathbb{R})$ to its differential at 0. Then $D_0((E(\mathbb{R})) = (\mathbb{R})')$ and $D_0(\mathbb{R}) = (\mathbb{R})'' = \mathbb{R}$. The fact that we work in a classical setting, and thus with reflexive spaces, is central here, as is allows to understand the interpretation of $\tilde{e}_D : x \in \mathbb{R}^n \mapsto f \mapsto \text{Diff}(f)(x)(0)$ as operator matching $\phi \in (\mathbb{R})''$ to $\phi \circ D_0 \in C^\omega(\mathbb{R}, \mathbb{R})''$, and to generalize it. The codereication $\tilde{d}_D : !D \rightarrow !$ is then postcomposition\(^1\) by $D$: $\tilde{d}_D(\phi) = \phi \circ D$.

The coweakening $\tilde{w}_D$ is then interpreted as the input of a fundamental solution $E_D$, solution to $\psi \circ D = \delta_0$. We prove in particular that while $E_D$ is not a distribution with compact support in general, it is an element of the interpretation of $!D A$. The co-contraction $\tilde{c}_D$ is interpreted by the convolution between a solution in $!D A$ and a distribution in $!A$, producing another solution in $!D A$. Following the rules in the sequent calculus, we have, for every $\psi \in \mathbb{R}^n$, for every $f \in E(\mathbb{R})$:

$$\tilde{c}_D(\tilde{w}_D, \psi)(D(f)) = D(\tilde{c}_D(\tilde{w}_D, \psi)(f)) = f.$$ 

That is, the solution $g$ to $Dg = f$ is $\tilde{c}_D(\tilde{w}_D, f)$.

**Contributions**

This paper:

- defines a Polarized Smooth variant of DILL with a distinction between smooth and finitary formulas, and its categorical models.
- constructs a denotational model for it, based on the idempotent adjunction between Nuclear Fréchet and Nuclear DF spaces, and the construction of the exponential as a space of compact support distributions. This solve partially the search for a smooth and categorical model of DILL.
- defines a Polarized D-Differential Linear Logic, which refines Smooth DILL with an indexed exponential $!D$ whose rules represent the computation of a solution to a partial differential equation. We define a cut-elimination procedure for D-DILL.
- shows that we have a model of Polarized D-DILL for any LPDOcc.

**Related work**

There is a major research effort towards the understanding and the semantics of probabilistic programming [4, 11, 16]. Our work bears similarity with these, if only because we use the same language of distributions and kernels. More generally, this works takes place in a global understanding of continuous data-types and computations: machine-learning, which uses gradients to optimize the computations, is one example. The change of paradigm, allowing to go from a discrete point-of-view on resource-sensitive programs to solutions of Differential Equations, relates to recent work on continuous probability distributions in probabilistic programming [8]. Notice however that models of probabilistic programming are in general models of Differential Linear Logic.

\[^1\]To avoid early confusion, we recall that for a distribution $\psi$, $D(\phi) = D(\phi)$. See section 5.4.

**Organisation of the paper**

We first introduce in section 2 the rules, cut-elimination procedure and categorical semantics of DILL. Then in section 3 we give an overview of the functional analysis necessary to the paper. We barely recall any proofs, but show examples and precise references for our claims. Section 4 is quite short, it formalizes syntactically and categorically the content of section 3 into the definition of Smooth DILL. Section 5 defines D-DILL, its syntax, rules, cut-elimination procedure and its categorical semantics. We also show in this section that for any $D$ LPDOcc, we have a model of D-DILL.

## 2 DILL

In this section, we recall the rules of DILL, the categorical structure needed to interpret the rules of DILL and its cut-elimination procedure. We explain the denotational intuitions behind these rules.

### 2.1 The formulas and proofs of DILL

The formulas of Differential Linear Logic are constructed according to the same grammar as LL, see figure 1, with additive and multiplicative disjunction and conjunction connective. The negation of a formula $A$ is denoted $A^\perp$ and defined as follows:

$$A^\perp = \top$$

We recall the rules for the exponential connectives ($\otimes, !$) of DILL in figure 1. The other rules corresponds to the MALL group of LL ($\otimes, \forall, \exists, \times$) as described by Girard [13].

![Figure 1. Syntax for the formulas and proof-trees of DILL](image)

Let us detail the denotational intuitions behind these rules. We interpret a sequent $E \vdash F$ as a linear function between $E$ and $F$, and choose to denote by the same letter a formula and its interpretation, which one is which will be clear from the context. The codereication $\tilde{d}$ allows then, by precomposition on a function from $E$ to $F$, that is by a cut rule on a sequent $\vdash E \otimes F$ to find the differential at 0 of $f$, that is a sequent $\vdash E^\perp, F$. The fact that we take here the differential at 0 must be understood as the necessity...
to isolate one single copy of $E$ in $!E$, while the others are not used, that is replaced by an empty hypothesis $0$. Then, in order to be able to differentiate at any point, the cocontraction $\tilde{c}$ is introduced, which at function level corresponds semantically to the convolution on functions:

$$f \ast g : x \mapsto \int f(y)g(x-y)dy.$$ 

Finally, the coweakening $\tilde{w}$ is interpreted as $\delta_0 : f \mapsto f(0)$. It is in particular the neutral for the convolution.

**Definition 2.1.** Proofs of DiLL are finite sums of proof-trees generated by these rules. In particular, there is of an empty proof tree denoted by $0$.

The cut-elimination procedure follows the one of LL for the MALL connectives, and the one for the exponential group are detailed by Ehrhard [7]. They follow the intuitions for the differentiation in euclidean spaces. We recall them semantically, through commutative diagrams in figure 2.

### 2.2 Categorical models of DiLL

There are two point of view: the first one is to extend the notion of Seely Model of Linear Logic with a biproduct and an interpretation for the codereliction [10], and the second one consider first models of DiLL without prom, and then extend this definition [7]. We adopt the first point of view, but make use of the numerous details and diagrams exposed by Ehrhard [7]. The following definitions are those of Fiore [10], sometimes adapted to the classical setting.

**Definition 2.2.** A biproduct on a category $\mathcal{L}$ is a monoidal structure $(\otimes, I)$ together with natural transformations:

such that $(A, u, \nabla)$ is a commutative monoid and $(A, n, \Delta)$ is a commutative comonoid.

**Definition 2.3.** A *-autonomous category is a symmetric monoidal closed category $(\mathcal{L}, \otimes, 1)$ with an object $\perp$ giving an equivalence of categories $(\cdot)' = [\cdot, \perp]_{\mathcal{L}} : \mathcal{L}^{op} \rightarrow \mathcal{L}$ with the canonical map $ev_E : E \rightarrow E''$ being a natural isomorphism.

**Definition 2.4.** A model of DiLL with promotion is consists of a symmetrical monoidal closed category $(\mathcal{L}, \otimes, 1)$ with a *-autonomous structure, a biproduct structure $(\varepsilon, \top)$, and an interpretation for the codereliction $(\epsilon, \nabla)$, and the notion of Seely Model of Linear Logic with a biproduct one requires the biproduct structure, the strong monoidality of $!$ to $(\mathcal{L}, \otimes)$, and a natural transformation $d : Id \rightarrow !$ satisfying strength and comonad diagrams [10].

**Remark 1.** As shown by Fiore, from the biproduct structure follows the fact that the category $\mathcal{L}$ is enriched over commutative monoids. This induces an additive law $+$ on hom-sets, which is necessary to interpret the sums of proofs-trees of DiLL which stems from cut-elimination.

$$f + g : E \xrightarrow{\varepsilon(f,g)} F \circ F \xrightarrow{\top} F.$$ 

### 2.3 Interpreting DiLL in its categorical model.

We briefly recall how to interpret a sequent of DiLL as morphism in $\mathcal{L}$, detailing only the action of exponential rules. The connectives $\otimes$, $\varepsilon$, $\otimes$ are interpreted respectively by $\otimes$ and its dual, and by the coproduct and product deduced from $\otimes$. We have $\perp = 1$ by strong monoidality of $!$.

We write $m_{E,F} : (E \otimes F) \rightarrow !E \otimes F$ the isomorphism resulting from the monoidality of $!$, and $d : !Id \rightarrow Id$ the co-unit of $!$. Then:

- from $f : E \rightarrow F$ one construct $f \circ d_E : !E \rightarrow !F$ and from $g : !E \rightarrow F$ one construct $g \circ d : !F \rightarrow F$.
- one construct $w : !Id \rightarrow !Id$ as $w_E = !u$ and $\tilde{w} : 1 \rightarrow !w$ as $\tilde{w}_E = !u$.
- one construct the natural transformation $c : ! \rightarrow !\otimes$ as $c_A = m_{A,A} \circ \Delta_A$ and $\tilde{c} : !\otimes \rightarrow !\otimes$ as $\tilde{c}_A = \nabla_A \circ m_{A,A}^{-1}$.

It should be clear then that in order to interpret the exponential rules of DiLL one requires the biproduct structure, the strong monoidality of $!$ and an interpretation for $d$ and $d$. The co-monadic structure of $!$ is used only for the interpretation of the promotion rule, and enforces the definition of $d$. We will make use of that statement in section 4 when we relax the co-monad requirement on $!$.

### 3 Topological vector spaces

In this section, we give technical accounts on some specific classes of topological vector spaces, on distribution theory and the theory of Linear Partial Differential Operators. (LPDO). We refer mainly to the books by Jarchow
Definition 3.1. A topological vector space (tvs) is a vector space endowed with a topology, that is a recovering class of open sets closed by infinite union and finite intersection, making the scalar multiplication and the addition continuous. A tvs is said to be Hausdorff if for any two distinct point $x$ and $y$ one can find two disjoint open sets an open set containing $x$ and $y$ respectively. It is locally convex if every point is contained in a convex open set.

From now on we work with locally convex separated topological vector spaces and denote them by lctvs. Examples of lctvs includes all euclidean spaces $\mathbb{R}^n$, normed spaces and metric spaces. For the rest of the section we consider $E$ and $F$ two lctvs.

Notation 1. We will write $E \simeq F$ for the equality between $E$ and $F$ as vector spaces, and $E \cong F$ for the equality between $E$ and $F$ as topological vector spaces.

Definition 3.2. Consider $U \subset E$ and $x \in U$, then $U$ is said to be a neighborhood of $x$ if $U$ contains an open set containing $x$. A set $B \subset E$ is bounded if for every $U$ neighborhood of 0, there is $\lambda \in \mathbb{R}$ such that $B \subset U^\lambda$.

Definition 3.3. For two lctvs $E$ and $F$ we consider $L_b(E, F)$ the lctvs of all linear continuous functions between $E$ and $F$ and endow it with the topology of uniform convergence on bounded subsets of $E$. We write $E' = L_b(E, \mathbb{R})$ for the dual of $E$.

Definition 3.4. A lctvs is reflexive if $E \cong E''$ through the transpose of the evaluation map in $E'$:

$$\delta : \begin{cases} E \\ x \end{cases} \mapsto \delta_x : (f \mapsto f(x))$$

Typically, all euclidean spaces are reflexive, as they are isomorphic to their dual. This is also true for every Hilbert spaces, but as soon as we generalize to Banach spaces we encounter the famous examples of $l_1$ and $l_\infty$.

The restriction to reflexive spaces is not preserved by tensor product nor linear hom-sets: typically, the space $L(H, H)$ is not reflexive when $H$ is a Hilbert space $^2$.

Definition 3.5. Consider $E$ and $F$ two lctvs. The projective tensor product$^3$ $E \otimes_\pi F$ is the algebraic tensor product, endowed with the finest topology making the canonical bilinear map $E \times F \rightarrow E \otimes F$ continuous. Then $E \otimes_\pi F$ is a lctvs. The completion of $E \otimes_\pi F$ is called the completed projective tensor product and denoted $E\hat{\otimes}_\pi F$

3.1 (F)-spaces and (DF)-spaces

Definition 3.6. If $F$ is a metrisable space, we write $\hat{F}$ its completion. Fréchet spaces are very common in analysis, but are not preserved by duality: the dual of a Fréchet space is not necessarily metrisable. In particular, the dual $C^\infty(\mathbb{R}, R')$ of the space of smooth scalar functions, as described in section 3.2, is not metrisable.

Definition 3.7. A (DF)-space is a lctvs $E$ admitting a countable basis of bounded sets $\mathcal{A} = (A_n)_n$ and such that if $(U_n)_n$ is a sequence of closed and disked neighborhoods of 0 whose intersection $U$ is bornivorous, then $U$ is a neighborhood of 0.

Although this definition may seem obscure, it is the right one for interpreting the dual and pre-dual of (F)-spaces.

Proposition 3.8 ([15] IV.3.1). • If $F$ is metrisable, then its strong dual $F'$ is a (DF)-space.
• If $E$ is a (DF)-space and $F$ and $(F)$-space, then $L_b(E, F)$ is an (F)-space. In particular, $F''$ is an (F)-space.

Proposition 3.9 ([18] 12.4.2 and 15.6.2). The class of (DF)-spaces is preserved by countable inductive limits, countable direct sums, quotient and completions. The class of (F)-spaces is stable with the construction of products and completed projective tensor products $\hat{\otimes}_\pi$.

3.2 Distributions with compact support

We refer to the exposition of distributions by Hörmander [17] for proofs and details.

Definition 3.10. Consider $n \in \mathbb{N}$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. The function $f$ is said to be smooth if it is differentable at every point $x \in \mathbb{R}^n$, and if at every point its differential is smooth.

The theory of distribution is traditionally introduced by considering the space $\mathcal{D}(\mathbb{R}^n) := C^\infty(\mathbb{R}^n)$ of test functions, i.e. the space of scalars smooth functions on $\mathbb{R}^n$ with compact support, and define distributions as elements of its dual $\mathcal{D}'(\mathbb{R}^n)$. But because of the dereliction rule $d$ we are going to consider linear continuous function as a particular case of smooth function, we work with the following:

Definition 3.11. We consider $\mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n, R)$ the space of all scalar smooth functions on $\mathbb{R}^n$, endowed with the topology of uniform convergence of all differentials on all compact subsets of $\mathbb{R}^n$. Its dual is called the space of distributions with compact support and denoted $\mathcal{E}'(\mathbb{R}^n)$.

Proposition 3.12. For any $n \in \mathbb{N}$, $\mathcal{E}(\mathbb{R}^n)$ is an (F)–space and $\mathcal{E}'(\mathbb{R}^n)$ is a complete (DF)–space.

Example 3.13. A distribution must be considered as a generalized function, and acts as such. The key idea is that, if $f \in C^\infty(\mathbb{R}^n)$ then on defines a compact distribution by $g \in C^\infty(\mathbb{R}^n) \mapsto \int f(x)g(x)dx$.

Typical examples of distributions which do not follow this pattern are the dirac distributions: for $x \in \mathbb{R}^n$ one defines the dirac at $x$ as: $\delta_x : f \in \mathcal{E}(\mathbb{R}^n) \mapsto f(x)$.

$^2$The author thanks Marc Bagnol for this clarifying example

$^3$Many topologies can be defined on the vector space resulting from the tensor product of two lctvs. The later restriction to Nuclear spaces will de facto identify all reasonable topological tensor product to the projective one.
Definition 3.14. Consider \( \phi \in \mathcal{E}'(\mathbb{R}^n) \) and \( f \in \mathcal{E}(\mathbb{R}^n) \). Then one defines the convolution between a distribution and a function as \( \phi * f \in \mathcal{E}(\mathbb{R}^n) \) as: \( \phi * f : x \mapsto \phi(y \mapsto f(x - y)) \).

This definition is extended to a convolution product between distributions. Consider \( \psi \in \mathcal{E}'(\mathbb{R}^n) \). Then \( \phi * \psi \) is the unique distribution in \( \mathcal{E}'(\mathbb{R}^n) \) such that:

\[
\forall f \in \mathcal{E}(\mathbb{R}^n), \phi * \psi(f) = \phi(\psi * f).
\]

Although the above is not a symmetric definition, one proves easily that the convolution is commutative and associative.

Example 3.15. Let us note that \( \delta_0 \) defined in 3.13 acts as neutral element for the convolution law.

The central theorem of the theory of distributions is the Kernel Theorem:

Theorem 3.16 ([26] 51.6). For any \( n, m \in \mathbb{N} \) we have:

\[
\mathcal{E}'(\mathbb{R}^{m+n}) = \mathcal{E}'(\mathbb{R}^m) \hat{\otimes} \mathcal{E}'(\mathbb{R}^n) = \mathcal{L}(\mathcal{E}'(\mathbb{R}^m), \mathcal{E}(\mathbb{R}^n))
\]

This theorem is proved on the spaces of functions by showing the density of smooth functions of the kind \( f \otimes g, f \in \mathcal{E}(\mathbb{R}^m), g \in \mathcal{E}(\mathbb{R}^n) \), and then that the topology induced by \( \mathcal{E}(\mathbb{R}^{m+n}) \) on \( \mathcal{E}(\mathbb{R}^{m+n}) \) \( \mathcal{Y} \mathcal{E}(\mathbb{R}^m) \) is indeed the projective topology of the tensor product. This, and particularly the fact that \( \mathcal{Y} = \hat{\otimes} \mathcal{Y} \), is justified by the theory of Nuclear spaces, which is recalled below.

3.3 Nuclear spaces
The theory of nuclear spaces will allow us to interpreted the idempotent negation of \( \mathfrak{D} \), and as the same time the theory of exponentials as distributions

Definition 3.17. An linear map \( f \) between a lctvs \( E \) and a Banach \( X \) is said to be nuclear if there is an equicontinuous sequence \( (\alpha_n) \) in \( E' \), a bounded sequence \( (y_n) \) in \( X \), and a sequence \( (\lambda_n) \in l_1 \) such that for all \( x \in E \):

\[
f(x) = \sum_n \lambda_n \alpha_n(x) y_n.
\]

Definition 3.18. Consider \( E \) a lctvs. We say that \( E \) is nuclear every continuous linear map of \( E \) into any Banach space is nuclear.

Proposition 3.19 ([18] 21.2.3). The class of nuclear spaces is closed with respect to the formation of completion, cartesian products, countable direct sums, projective tensor products, subspaces and quotients.

An important property of nuclear spaces is that as soon as they are normed, they are finite dimensional. In other word, if a Hilbert or Banach or simply normed space is nuclear, then it is isomorphic to \( \mathbb{R}^n \) for a certain \( n \).

Example 3.20. Typical examples of nuclear spaces are euclidean spaces \( \mathbb{R}^n \), spaces of smooth functions \( C^\infty_c(\mathbb{R}^n, \mathbb{R}) \), \( C^\infty(\mathbb{R}^n, \mathbb{R}) \) and their duals \( \mathcal{D}'(\mathbb{R}^n) \) and \( \mathcal{E}'(\mathbb{R}^n) \).

Theorem 3.21. An \( (F) \)-space \( F \) which is also nuclear is reflexive.

\textbf{Proof.} It is enough to prove that \( F \) is semi-reflexive, that is that \( F = F'' \), as the equality between the topologies will follow from the metrisability of \( F \). Indeed, when \( F \) is metrisable \( E \)-equicontinuous sets and \( E \)-weakly bounded sets corresponds in \( E' \) [18, 8.5.1]. Now we have that every bounded set of a nuclear space is precompact [24, III.7.2.2]. Thus as \( F \) is nuclear and complete, its bounded sets are compact, and \( F' \) is endowed with the Arens-topology of uniform convergence on absolutely convex compact subsets of \( F \). By the Mackey-Arens theorem, this makes \( F \) semi-reflexive.

As a consequence, \( \mathcal{E}(\mathbb{R}^n) \) and \( \mathcal{E}'(\mathbb{R}^n) \) are reflexive.

Proposition 3.22. \( \bullet \) Consider \( E \) a lctvs which is either an \( (F) \)-space or a \( (DF) \)-space. Then \( E \) is nuclear if and only if \( E' \) is nuclear [14, Chap II, 2.1, Thm 7].

\( \bullet \) IF \( E \) is a complete \( (DF) \)-space and if \( F \) is nuclear, then \( \mathcal{L}(E,F) \) is nuclear. If moreover \( F \) is an \( (F) \)-space or a \( (DF) \)-space, then \( \mathcal{L}(E,F') \) is nuclear [14, Chapter II, 2.2, Thm 9, Cor. 3]. As a corollary, the dual of a nuclear \( (DF) \)-space is a nuclear \( (F) \)-space.

Proposition 3.23 ([14] Chapter II, 2.2, Thm 9). If \( E \) and \( F \) are nuclear \( (DF) \)-spaces, then so is \( E \otimes_F F \).

A central result of the theory of nuclear spaces, explaining for the form of the Kernel theorem 3.16, is the following proposition. It is proved by applying the hypothesis that \( E \) is reflexive and thus \( E' \) is complete and barrelled, and thus applying the hypothesis of Treves’ book [26].

Proposition 3.24 ([26] prop. 50.5). Consider \( E \) a Fréchet nuclear space, and \( F \) a complete space. Then \( E \otimes_F F \simeq \mathcal{L}(E',F) \).

3.4 Linear Partial Operators
We recall the very first steps in the theory of LPDEs\(^5\). For \( \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n \) we write \( \partial^\alpha \) the smooth function:

\[
f \in C^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto x \in \mathbb{R}^n \mapsto \frac{\partial^\alpha f}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}(x)
\]

Definition 3.25. Consider, for \( \alpha \in \mathbb{N}^n \) smooth functions \( a_\alpha \in C^\infty(\mathbb{R}^n, \mathbb{R}) \). Then a Linear Partial Differential Operator (LPDO) is defined as an operator \( D : \mathcal{E}(\mathbb{R}^n) \rightarrow \mathcal{E}(\mathbb{R}^n) : D = \sum a_\alpha \partial^\alpha \).

A LPDO with constant coefficients is a LPDO \( D \) such that the \( a_\alpha \) are constants.

The definition of \( D \) is extended to \( \mathcal{E}'(\mathbb{R}^n) \) as follows:

\[
D(\phi) = f \mapsto \sum a_\alpha (-1)^\alpha a_\alpha \partial^\alpha (f),
\]

so that for \( g \in C^\infty(\mathbb{R}^n) \): \( D(f \mapsto \int fg) = f \mapsto \int f D(g) \).

\(^5\) We are not considering in this paper border conditions, regularity of the solutions to equations with non-constant coefficients, nor modern research subjects in the theory non-linear equations.
The weak resolution of the Linear Partial Differential Equation consists then, when $\phi \in \mathcal{E}'(\mathbb{R}^n)$, of finding $\psi$ such that, for all $f \in \mathcal{E}(\mathbb{R}^n)$\footnote{This definition is particular to the paper, and needed to be coherent with the notation of DiLL. As described in the literature, the resolution would consists in finding $\psi$ such that $(D\psi)(f) = \phi(f)$.}.

$$\psi \circ D(f) = \phi(f).$$  \hspace{1cm} (2)

The resolution of LPDOs with constant is always possible, and particularly elegant, due to the behaviour of convolution with respect to partial differentiation:

**Proposition 3.26 ([17], 4.2.5).** Consider $f \in \mathcal{E}(\mathbb{R}^n)$ and $\phi \in \mathcal{E}'(\mathbb{R}^n)$. Then $\partial^a \phi \ast f = \phi \ast (\partial^a f)$.

**Definition 3.27.** A fundamental solution to equation (2) consists in finding $D_{\phi} \ast f = \phi \ast (Df)$. We introduce now SDiLL: its grammar, defined in figure 3, separates formulas into finitary ones and polarized smooth ones. In this section we introduce a Smooth Differential Linear Logic (SDiLL). We write $\mathcal{L}: \varphi \vdash \psi$ if the proof-search $\varphi, \psi$ is a derivation.

**Proposition 3.29.** The fundamental solution defined above is continuous on $D(\mathcal{E}(\mathbb{R}^n))$, as it corresponds to $D(f) \mapsto f(0)$. We have thus $E_D \in 1D^{\mathbb{R}^n}$.

4 Smooth Differential Linear Logic and its models

In this section we introduce a Smooth Differential Linear Logic for which Nuclear spaces and distributions form a classical and smooth model.

Let us recall the notion of polarisation in LL. In polarized linear logic [22] a distinction is made between positive connectives $\land, \lor, \lor$ whose introduction rules are non-reversible, and negative connectives $\land, \lor, \land$ whose introduction rule is reversible. Negation then changes the polarity of a formula. This plays a fundamental role in proof-search [1].

4.1 The categorical structure of Nuclear and Fréchet spaces.

Nuclear Fréchet spaces gather all the stability properties to be a (polarized) model of LL, except that we do no have an interpretation for higher-order smooth functions.

**Figure 3.** The syntax of Smooth DILL

Indeed if $\mathbb{R}^n$ is interpreted as $\mathcal{E}'(\mathbb{R}^n)$, we do not have a straightforward definition of $\mathbb{R}^n$.

**Definition 4.1.** We write $\mathcal{N}f$ the category of Nuclear (F)-spaces and continuous linear maps, $\mathcal{N}df$ the category of complete Nuclear (DF)-spaces and continuous linear maps, and $\mathcal{E}uc$ the subcategory of both of them with properties of euclidean spaces.

**Proposition 4.2.**

- Eucl is a model of MALL.
- $\mathcal{N}f$ forms a model for the negative interpretation of polarized MALL [22, 6.20], where positive formulas are thus interpreted as objects of $\mathcal{N}df$.

**Proof.** The first point is transparent. The second point is due to the stability of Nuclear Fréchet spaces by cartesian product (interpreting $\otimes$) and completed $\pi$-tensor product (interpreting $\otimes$), see propositions 3.19, 3.9 and 3.24. The interpretation of the rules for $\otimes$ and $\otimes$ is possible by the fact that Nuclear Fréchet spaces are reflexive (proposition 3.21) and thus the interpretation of $A \otimes B$ is the one of $(A^2 \otimes B^2)^\perp$.

Thanks to propositions 3.9, 3.22 and 3.24 we have that the interpretation of $\otimes$ in $\mathcal{N}df$ is $\otimes$, and that the interpretation of $\otimes$ in $\mathcal{N}f$ is also $\otimes$.

**Remark 2.** Note however that we do not have a compact closed category, as we are working in a polarized model of MALL with an unquantification between $\mathcal{N}f$ and $\mathcal{N}df$. For $\mathbb{R}^n \in \mathcal{E}uc$ we define $\mathbb{R}^n = \mathcal{E}'(\mathbb{R}^n)$. This is extended as a functor on $\mathcal{E}uc$ by defining $f : (\mathbb{R}^n \rightarrow \mathbb{R}^m) : \phi \in \mathcal{E}'(\mathbb{R}^m) \mapsto \phi \circ f \in \mathcal{E}'(\mathbb{R}^m)$.

It follows from the Kernel theorem 3.16 and example 3.20 that the space of compact distribution acts as a strong monoidal functor from $\mathcal{E}uc$ to $\mathcal{N}df$.

**Theorem 4.4.** The exponential $!A : \mathcal{E}uc \rightarrow \mathcal{N}df$ is a strong monoidal functor.

4.2 Smooth Differential Linear Logic (SDiLL)

We introduce now SDiLL: its grammar, defined in figure 3, separates formulas into finitary ones and polarized smooth ones.

Its rules are those of DiLL: follows the one of MALL for the additive and multiplicative connectives, and those detailed in figure 4 for the exponential. Thus, the cut-elimination procedure is the same as the one defined originally [9].

If we forget about the polarisation of SDiLL, a model of it would be a model of DiLL where the object $!A$
does not need to be defined. It is thus a model of DiLL where ! does not need to be an endofunctor, but just a strong monoidal functor ! : FIN \longrightarrow \text{SMOOTH}

between two categories. The categories FIN and SMOOTH need to be both a model of MALL.

We give a categorical semantics for an unpolarized version of SDiLL. The polarized version would ask the category SMOOTH below to be a model of Polarized MALL, that is an involutive, defined as an adjunction between a category of negative smooth formulas and a category of positive smooth formulas.

\textbf{Definition 4.5.} A categorical model of SDiLL consists into a model of MALL with biproduct FIN, and a model of MALL SMOOTH, such that we have a strong monoidal functor ! : (FIN, X) \longrightarrow (SMOOTH, \otimes), a forgetful functor \( U : \text{EUCL} \longrightarrow \text{NUCL} \) strong monoidal in \( \otimes, \land, \& \) and two natural transformation \( d! : ! \longrightarrow U \) and \( \dd : ! \longrightarrow U \) such that \( \dd \circ d = Id_{\text{EUCL}} \).

\textbf{Theorem 4.6.} The structure on Nuclear Spaces and Distributions defines a model of SmoothDiLL.

\textbf{Proof.} We interpret finitary formulas A as euclidean spaces. Without any ambiguity, we denote also by A the interpretation of a finitary formula into euclidean spaces. The exponential is interpreted as \( !A = E'(A), \) extended by precomposition on functions. We briefly explain the interpretation for the rules, which follow the intuition of [9]. We define:

\[ d : \begin{cases} !A & \longrightarrow A' \\
\phi & \mapsto \phi|_{A'} \\
v_{x} & \mapsto (f \mapsto ev_{x}(D_{0}(f))) \end{cases} \quad \dd : \begin{cases} A'' & \longrightarrow !A \\
\phi \otimes \phi' & \mapsto \phi \cap \phi' \\
v_{x} & \mapsto (f \mapsto ev_{x}(D_{0}(f))) \end{cases} \]

This is justified by the definition of reflexivity 3.4. Then we have indeed: \( \dd \circ d = Id_{A'} \). The interpretation of \( w, c, \dd \) follows from the biproduct structure on EUCL and from the monoidality of !, as explained in 2.3.

We show that \( \dd, \dd \) they have a direct interpretation which follows the intutions of [2, 9].

\textbf{Proposition 4.7.} The cocontraction and coweakening defined through the kernel theorem correspond to the convolution of distributions and the introduction of \( \delta_{0} \).

\[ \dd : \begin{cases} A' & \longrightarrow !A \\
\phi & \mapsto \phi|_{A} \\
v_{x} & \mapsto (f \mapsto ev_{x}(D_{0}(f))) \end{cases} \]

\[ \dd : \begin{cases} A & \longrightarrow !A \\
\phi \otimes \phi' & \mapsto \phi \cap \phi' \\
v_{x} & \mapsto (f \mapsto ev_{x}(D_{0}(f))) \end{cases} \]

\textbf{During the rest of the proof we use Fourier transformations and temperates distributions, as exposed by Hörmander in [17, 7.1]. The co-contraction is defined categorically as \( \delta = !\nabla \circ m^{-1}_{A,A} \). In the categorical setting, addition in hom sets is defined through the biproduct. But here the reasoning is done backward. We know that \( \otimes = x \) is a biproduct thanks to \( \nabla : A \times A \longrightarrow A; (x, y) \mapsto x+y \), and thus \( !\nabla : \phi \in !A \times A \mapsto (f \in \mathcal{E}(A) \mapsto \hat{\phi}(x, y) \mapsto f(x+y)) \). Moreover if \( f \in \mathcal{E}(A \times A) \) is the sequential limit of \( \left( f_{k} \otimes g_{n} \right)_{n} \in (\mathcal{E}(A) \otimes \mathcal{A}(A))^{\mathbb{R}} \) (see theorem 3.16) \( m_{\Lambda,A}^{-1} \left( \phi \otimes \psi \right)(f) = \lim_{n}(\phi(f_{n})\psi(g_{n})). \)

If we write by \( \hat{\psi} \) the Fourier transformation of a distribution, we have that \( \dd(\hat{\phi} \otimes \hat{\psi}) = \hat{\phi} \circ \hat{\psi}. \) From the details above we deduce \( \dd(\hat{\phi} \otimes \hat{\psi})(f) = m_{\Lambda,A}^{-1} \left( \phi \otimes \psi \right)(x+y) = m_{\Lambda,A}^{-1} \left( \phi \otimes \psi \right)(f) \).

As distributions with compact support are temperates, we can apply the inverse Fourier transformation and thus \( \dd \) corresponds to the convolution. \( \square \)

The interpretation of the contraction \( c \) is then the construction of a Kernel of two smooth functions, while the interpolation for the weakening consists in applying a distribution to the function constant at 1. Diagrams of 2 are easily verified and follow the intutions of [7, 9].

\textbf{5} \ LPDEs in the Syntax

In this section we define a sequent calculus refining Smooth DiLL by introducing a connective \( !_{D} \). We prove that the rules and cut-elimination account of \( !_{D} \) account semantically for the resolution of LPDEs with constant coefficients. We prove that when \( !_{D} \approx Id \), proof trees of Smooth DiLL are sums of proof trees of \( D \sim DlL \).

\textbf{5.1} \ The sequent calculus D-DiLL

\textbf{Grammar and rules} We introduce a generalisation of Smooth DiLL, where the role of A ≡ A\perp\perp in the exponential rules d and \( \dd \) is played by a new formula \( !_{D}A \). The idea is that \( A\perp\perp \) represents the linear forms acting on the space of functions f = \( Dg \) for some, and that \( !_{D}A \) represents the type of linear forms acting on the type of function f = \( Dg \) for some g.

The grammar of D-DiLL is defined in figure 5, and differs very little from those of SDiLL. The MALL connectives of D-DiLL follow the same rules as usual in LL or DiLL.

\textbf{The cut-elimination procedure in D-DiLL} The cut-elimination is described in figure 7 as commutative diagrams for their denotational interpretation, is inspired by the one of Linear Logic and by the calculus on distributions, see section 5.4.
Remark 3. Notice that the differences between SDiLL and D-DiLL makes the cut-elimination procedure much more simpler: cuts between $d$ and $\bar{w}$ or $d$ and $w$ are not possible, and the cut-elimination procedure does not generates sums of proof-terms, as contraction and co-contraction are not symmetrical. The proof that the cut-elimination procedure converges to cut-free proofs is a direct adaptation of the one for DiLL.

5.2 Encoding DiLL

Proposition 5.1. The rules $\bar{w}, \bar{c}, w$ and $c$ are admissible in D-DiLL.

Proof. We write the rules here under their denotational form: $w = d \circ w_d, \bar{w} = \bar{d} \circ \bar{w}_d, c = c \circ c_d \circ (c_d \& \bar{w}_d) + d \circ c_d(c_d \& \bar{w}_d) \& (c_d \& \bar{w}_d) + b \circ \bar{c}_d(c_d \& \bar{w}_d) \& (c_d \& \bar{w}_d))$. Likewise, one proves similarly that:

Proposition 5.2. The rules $w_d, \bar{c}_d, w_d$ and $c_d$ are admissible in SDiLL, when $!_D A$ is equivalent to $A$.

Theorem 5.3. When $!_D A = A$, the proof-trees of SDiLL are sums of proof-trees of D-DiLL.

One shows then easily that the cut-elimination procedures correspond.

5.3 Categorical models of D-DiLL

Definition 5.4. A categorical model of D-DiLL consists in a model of MALL with biproduct Eucl, and a (polarized) model of MALL Nucl, with a strong monoidal functor $! : (\text{Eucl}, x) \rightarrow (\text{Nucl}, \&)$, a functor $!_D : \text{Eucl} \rightarrow \text{Nucl}$, and two natural transformations $d_D : ! !_D$ and $\bar{d}_D : ! !_D$ such that $d_d \circ \bar{d}_D = 1_d_{\text{Eucl}}$.

Indeed, one defines the interpretations of $c_d, w_d, \bar{c}_d, \bar{w}_d$ through the strong monoidality of $!$, the biproduct structure and $d_D$ and $\bar{d}_D$ as it is done in the proof of proposition 5.1 and in paragraph 2.3.

The cut-elimination rules of figure 7 are then easily verified. For example, we have indeed $\bar{c}_d(w_d, \phi) = d_d \circ \bar{c}(d_D \circ \bar{w}_d, \phi)$.

5.4 A LPDE interpreted in the syntax

In this section we show that the categories Eucl, Nucl and Nf defined in section 3.3, together with distributions of compact support and a LPDOcc $D$, form a model of D-DiLL.

Consider $D : E(\mathbb{R}) \rightarrow E(\mathbb{R})$ a LPDOcc:

$$D(f)(x) = \sum_{a \in \mathbb{N}^*} a_a \partial^a f(x).$$

We interpret finitary formulas $A,B$ as euclidean spaces. One has indeed $1 \simeq 0 = \mathbb{R}$ and $\top \simeq 1 = \{0\}$. The connectives of LL are interpreted in Eucl, Nucl and Nf as in section 4.

Definition 5.5. For $A$ a finitary formula interpreted by $\mathbb{R}^n \in \text{Eucl}$, we interpret $!_D A$ as its dual as:

$$!_D \mathbb{R}^n := (D(C^{\infty}(\mathbb{R}^n)))$$

$$?_D \mathbb{R}^n = D(C^{\infty}(\mathbb{R}^n))^\dagger = D(C^{\infty}(\mathbb{R}^n))$$

Proposition 5.6. We have that $?_D \mathbb{R}^n \in \text{Nucl}$ and $!_D \mathbb{R}^n \in \text{Nf}$.

Proof. $?_D \mathbb{R}^n$ is a closed subset of $?_D \mathbb{R}^n$. As such, it is a nuclear (F)-space, see 3.19 and 3.9. Thus $?_D \mathbb{R}^n \in \text{Nucl}$ and $!_D \mathbb{R}^n \in \text{Nf}$. From the previous proposition and the reflexivity of nuclear spaces (prop. 3.21) it follows that $(?_D \mathbb{R}^n)^\dagger = !_D \mathbb{R}^n$.

Theorem 5.7. We extend $!_D$ on linear maps by precomposition by $D$, and thus define a functor $!_D : \text{Eucl} \rightarrow \text{Nucl}$. Then we have natural isomorphisms

$$m_{D,A,B} : !_D(\mathbb{R}^{n+m}) \simeq !_D \mathbb{R}^n \hat{\otimes} !_D \mathbb{R}^m.$$ 

Proof. This theorem encodes in particular a well used convention in the theory of LPDEs, which allows to extend $D$ defined on $E(\mathbb{R}^n)$ to $E(\mathbb{R}^{n+m})$: we differentiate on the $n$-first variable apply to functions defined on $\mathbb{R}^{n+m}$. Our theorem is then directly deduced from the Kernel theorem 3.16, as $D(E(\mathbb{R}^{n+m})) \simeq D(E(\mathbb{R}^n)) \otimes E(\mathbb{R}^m)$. The interpretation of $w_d, \bar{w}_d, c_d$ and $\bar{c}_d$ follows from the previous proposition, by the same step as those detailed in section 2. In particular, we have:

- $\bar{w}_D : 1 ! !_D$ such that $\bar{w}_D, 1 = E_D$. It is well defined thanks to proposition 3.29.
- $\bar{c} : !_D \otimes ! !_D$ correspond to the convolution product (see prop. 4.7) and is well defined thanks to proposition 3.26.

By the definition 3.27 of the fundamental solution we have indeed the satisfaction of the diagrams of figure 7.

Definition 5.8. We interpret the dereliction $d_D : ! !_D$ as $d_{D,E}(\phi \in E'(\mathbb{R}^n)) \mapsto (E_D(\phi))$ and codereliction $\bar{d}_D : ! !_D$ as $\bar{d}_{D,E}(\phi \in D(E(\mathbb{R}^n))) \mapsto (\phi, D(\phi)) \in E'(\mathbb{R}^n)$.

Then one has for every $\phi \in E'(\mathbb{R}^n)$ and $f \in E(\mathbb{R}^n)$:

$$d_{D,E}(\phi)(f) = E_D(\phi(D(f)))$$

$$= \phi(D(f)) by equation 1$$

$$= \phi(f) by definition 3.27.$$ 

Defining $d_D$ by restriction to $(D(E(\mathbb{R}^n)))$, as we defined $d$ as the restriction to $E'$, would not guarantee the preceding equation. Let us notice that in the case $D = D_0$, we have $E_D = \delta_0$ and thus $d_{D_0}$ is still the restriction to $(D(E(\mathbb{R}^n))) \simeq (\mathbb{R}^n)$. The preceding propositions conclude:

Theorem 5.9. For any $D$ LPDOcc, we have a polarized model of D-DiLL with Eucl, Nucl, Nf, $! = E'$ and $!_D = (D(E))'$. 

}\end{figure}
6 Conclusion

In this paper, we constructed a logic system D-DiLL accounting for the resolution of LPDEec, generalizing DiLL. It opens several perspectives. The first work to tackle should be to find a deterministic classical term-calculus, inspired by the differential $\lambda\mu$-calculus [27], accounting for D-DiLL. In a Curry-Howard-Lambek correspondence perspective, this would correspond the Program/Proof bijection, while we studied here a Proof/Categories interpretation. Secondly, work in progress consists in generalising D-DiLL into a system engulfing all LPDOcc. Promotion, contraction and co-contraction lead to a BLL-like syntax, in which we would like to give a syntactical counterpart to the construction of a fundamental solution. Generalized to all LPDO, this should lead to a syntactical criterion for the resolution of LPDEs.

Of course, one would like to allow for higher order, and in particular an interpretation of differential equations at higher order. Notice that it was necessary to work with $\mathcal{C}(\mathbb{R}^n)$ when interpreting S-DiLL, but other classes of distributions may suit for a model of D-DiLL. We can easily introduce a version of D-DiLL with promotion and no separation between finitary and smooth formulas. Cut-elimination would be an adaptation of the cut-elimination for DiLL with promotion [29]. Models of it should come from smooth and classical models of Linear Logic with higher-order, as studied recently [3]. D-DiLL should also be generalised to understand equations whose solutions are defined on subsets $\Omega \subset \mathbb{R}^n$, which form the majority of LPDEs: this should be done by introducing subtyping on finitary formulas, and could lead to a complete semantics over Nuclear spaces. The next goal after that should be to find a logical account for all LPDEs. The long-term goal is of course to go towards non-linear PDEs.

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References


Appendix
Formulas $E, F = 0 | 1 | \top | \bot | E \otimes F | E \multimap F | E \oplus F | E \times F | ?E | !E$

MALL group:

1. $\vdash A, E \perp$ (axiom)
   $\vdash \Gamma, E \vdash E, \Delta$ (cut)
   $\vdash \Gamma, \Delta$ mix

2. $\vdash \Gamma$ (1)
   $\vdash \Gamma, \bot$ (⊥)
   $\vdash \Gamma, E, F$ (∥)
   $\vdash \Gamma, \Delta, E \otimes F$ ($\otimes$)

3. $\vdash \Gamma, \top$ (⊤)
   $\vdash \Gamma, E \vdash \Gamma, F, \&$
   $\vdash \Gamma, E \vdash \Gamma, F \oplus F$ ($\oplus_L$)
   $\vdash \Gamma, F \vdash \Gamma, F \oplus F$ ($\oplus_R$)

Exponential group:

4. $\vdash \Gamma, ?E \vdash w$
5. $\vdash \Gamma, \top$ (⊤)
6. $\vdash \Gamma, ?E \vdash \Gamma, !E$
7. $\vdash \Gamma, E \vdash \Gamma, !E$

$\vdash \Gamma, ?E \vdash w$
$\vdash \Gamma, \bot$ (⊥)
$\vdash \Gamma, E, F$ (∥)
$\vdash \Gamma, \Delta, E \otimes F$ ($\otimes$)

$\vdash \Gamma, \top$ (⊤)
$\vdash \Gamma, E \vdash \Gamma, F, \&$
$\vdash \Gamma, E \vdash \Gamma, F \oplus F$ ($\oplus_L$)
$\vdash \Gamma, F \vdash \Gamma, F \oplus F$ ($\oplus_R$)

Figure 8. Finitary Differential Linear Logic

Figure 9. MALL rules for SDiLL and D-DiLL

$\vdash \Gamma, !A \vdash w$
$\vdash \Gamma, ?A \vdash !A$

$\vdash \Gamma, !A \vdash w$
$\vdash \Gamma, ?A \vdash !A$

$\vdash \Gamma, !A \vdash w$
$\vdash \Gamma, ?A \vdash !A$

$\vdash \Gamma, !A \vdash w$
$\vdash \Gamma, ?A \vdash !A$

$\vdash \Gamma, !A \vdash w$
$\vdash \Gamma, ?A \vdash !A$

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$\vdash \Gamma, !A \vdash w$
$\vdash \Gamma, ?A \vdash !A$

Figure 10. Admissible rules of SDiLL in D-DiLL

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Figure 11. Cut-elimination for the exponential rules of SDiLL
Figure 12. Cut-elimination for the exponential rules of D-DiLL