A Logical Account for Linear Partial Differential Equations

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Abstract

Differential Linear Logic (DiLL), introduced by Ehrhard and Regnier, extends linear logic with a notion of linear approximation of proofs. While DiLL is classical logic, i.e. has an involutive negation, classical denotational models of it in which this notion of differentiation corresponds to the usual one, defined on any smooth function, were missing. We solve this issue by constructing a model of it based on nuclear topological vector spaces and distributions with compact support.

This interpretation sheds a new light on the rules of DiLL, as we are able to understand them as the computational principles for the resolution of Linear Partial Differential Equations. We thus introduce D-DiLL, a deterministic refinement of DiLL with a D-exponential, for which we exhibit a cut-elimination procedure, and a categorical semantics. When D is a Linear Partial Differential Operator with constant coefficients, then the D-exponential is interpreted as the space of generalised functions ψ solutions to $D\psi = \phi$. The logical inference rules represents the computational steps for the construction of the solution ϕ. We recover linear logic and its differential extension DiLL as a particular case.

Keywords Differential Linear Logic, Linear Partial Differential Equations, Functional Analysis, Categorical semantics

1 Introduction

A Partial Differential Equation (PDE) is an equation $Dg = f$ between functions f and g, where Dg is a possibly non-linear combination of partial derivatives of g, with smooth functions as coefficients. The study of PDEs through theoretical, numerical and computational methods is one of the most active areas of modern mathematics. Most research concentrates on non-linear equations such as Navier-Stokes equation. Programs are used to find approximate non-linear solutions, and applied mathematicians work at finding quick and efficient algorithms to do so.

Linear PDEs (LPDEs) are easier to solve theoretically, and when they have constant coefficients a universal method was found separately by Malgrange [Malgrange 1956] and Ehrenpreis [Ehrenpreis 1954]. Examples of LPDEs with constant coefficients (LPDEcc) include fundamental examples such as the Laplacian equation or the heat equation:

$$\sum_i \frac{\partial^2 g}{\partial x_i^2} = f \text{ or } \frac{\partial g}{\partial t} - \alpha\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) = f.$$

In this paper, we construct a proof syntax, with cut-elimination, with a denotational model in which formulas are interpreted as spaces of distributions and cut-elimination correspond to the resolution of LPDEs. This builds a new and strong bridge between Logic and Mathematical Physics, by extending the Proof/Function part of the Curry-Howard-Lambek correspondence to LPDEs. We understand this result as a first step towards a more general computational theory encompassing non-linear PDEs. On a more practical level, we believe D-DiLL could lead to a type system for the verification of numerical programs.

From linear to non-linear proofs and back. Linear Logic (LL) was introduced by Girard [Girard 1987] as a proof-theory where a distinction is made between linear deductions of B under the hypothesis A, and non-linear ones. The former is represented by the sequent $A \vdash B$, while the latter is represented by $!A \vdash B$. The intuition is that a linear proof will make use of A exactly once: thus, $!A$ is traditionally interpreted as a collection of all finite copies of A. The inference rules for the exponential connective ! of LL then represent a *calculus of resources*. Among these rules, the dereliction rule d allows to deduce $!A \vdash B$ from $A \vdash B$: thus linear proofs can always be considered as non-linear ones.

DiLL was introduced by Ehrhard and Regnier [Ehrhard and Regnier 2003], as a refinement of LL without its promotion rules but with dual exponential rules. It features in particular a *codereliction* rule $\bar{d}$ allowing to deduce from a sequent $!A \vdash B$ a *linear approximation* of it: $A \vdash B$. This second sequent is considered as the differentiation of the first sequent. Both LL and DiLL are first presented under a classical form: sequents are monolateral $\vdash A^{\perp}, B$, and formulas are equivalent to their double linear-negation $A^{\perp\perp}$. A sequent $!A \vdash B$ is then rewritten $\vdash ?A^{\perp}, B$, where ? is the "why not" modality.

Thus DiLL, and differential or quantitative λ-calculi, are traditionally understood as a logic and as calculi of *approximations*, as they account a syntactic variant of the Taylor Formula [Vaux 2017]. In this paper, we change this point of view and consider it as the basis for a calculus of Partial Differential Equations.

The equation solved by DiLL, and its generalisation. The fundamental idea behind this paper is that

ψ of type $A^{\perp\perp}$ *is such that* $\bar{d}(\psi) = \phi$*, for* ϕ *of type* $!A$.

This is true at the level of functions: a function g is linear, i.e. of type $A^{\perp}$, if and only if there is $f : ?A^{\perp}$ such that the differential at 0 of f corresponds to g. The previous statement extends this at the level of linear duals of spaces of functions, that is spaces of distributions.

We generalize this idea into a new connective $!_D$, and a new codereliction rule $\bar{d}_D$:

$\psi : !_D A$ *is such that* $\bar{d}_D(\psi) = \phi$*, for* $\phi : !A$.

The fact that we work in a classical setting is central here, as is allows to understand $\bar{d} : A \longrightarrow !A$ as $\bar{d} : A^{\perp\perp} \longrightarrow !A$, and to generalize it as $\bar{d}_D : !_D A \longrightarrow !A$. DiLL thus corresponds to a special case where $!_D = Id$.

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We then construct a new sequent calculus D-DiLL which refines DiLL, and modelises the resolution of Linear Partial Differential Equations:

$$\frac{\vdash}{\vdash !_D A}\ \bar{w}_D \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !_D A}{\vdash \Gamma, \Delta, !_D A}\ \bar{c}_D \qquad \frac{\vdash \Gamma, !_D A}{\vdash \Gamma, !A}\ \bar{d}_D$$

The cut-elimination procedure of D-DiLL translates categorically into :

$$\bar{d}_D(\bar{c}_D(\bar{w}_D, \phi)) = \phi$$

for $\phi :!A$. In the syntax, this says that the solution ψ to the equation $\bar{d}_D(\psi) = \phi$ is exactly $\bar{c}_D(\bar{w}_D, \phi)$. In the semantics of D-DiLL this is interpreted exactly as the resolution of a Linear Partial Differential Equation in the theory of distributions [Hörmander 2003].

A classical and smooth semantics. This syntax for the resolution of LPDE comes from a semantical investigation for smooth and classical models of DiLL. Denotational semantics is the study of proofs and programs through their interpretation as denotations (functions) between spaces. In a denotational model of LL, there are spaces $\mathcal{L}(E,F)$ of linear functions from E to F, spaces $\mathcal{C}(E,F)$ of non-linear ones, and a way to understand non-linear functions on E as linear functions on $!E$: $\mathcal{L}(!E,F) \simeq \mathcal{C}(E,F)$. In a model of DiLL, functions must also be smooth, that is able to be iteratively differentiated everywhere. We write $\mathcal{C}^\infty(E,F)$ the space of all smooth functions between E and F.

The first models of DiLL introduced by Ehrhard [Ehrhard 2002, 2005] have a discrete basis: non-linear proofs are interpreted as power series between spaces of sequences. In order to get a better understanding of the differential nature of DiLL rules, one is bound to search for a denotational of model of DiLL where functions are interpreted as the *smooth functions* of differential geometry or functional analysis. But to account for linearity of functions, and for the classical setting of DiLL, one needs to interpret formulas as some topological vector spaces E which are *reflexive*: denoting $E' = \mathcal{L}(E,\mathbb{R})$, we need $E \simeq E''$.

The requirements for reflexivity to be preserved by the connectives of LL and the ones for having smooth functions work as opposite forces. More precisely:

- One needs a monoidal category of reflexive spaces, that is spaces which are isomorphic to their bidual and such that this property is preserved by tensor product and internal hom-sets. This is true for euclidean spaces, but fails in general when considering infinite dimensional spaces: it is false in particular for Banach spaces.
- One needs a cartesian closed category of smooth functions: we want $\mathcal{C}^\infty(E \times F, G) \simeq \mathcal{C}^\infty(E, \mathcal{C}^\infty(F,G))$. These structures are notably scarce in analysis, but are fundamental in the semantics of LL as it accounts for the possibility to curryfy programs.

Solutions to the first point are for instance models based on spaces of sequences [Ehrhard 2002, 2005], or topological vector spaces with very coarse topologies [Kerjean 2016]. Solutions to the second point were constructed by Frölicher, Kriegl and Michor [Kriegl and Michor 1997], leading to models of Intuitionnistic DiLL [Blute et al. 2012; Kerjean and Tasson 2016]. The attempt by Girard to interpret LL in Banach spaces fails [Girard 1999], as the requirement of a norm on power series is to strong to allow a good cartesian closed category. We propose here a classical and smooth model of DiLL without promotion, while another one with a more intricate structure and interpreting promotion was recently exposed by Dabrowksi and K. [Dabrowski and Kerjean 2017].

Computing with distributions. Distributions appears naturally in the quest for a model of LL. On the one hand, consider a model of DiLL made of $\mathbb{K}$-vector spaces, with spaces of linear functions $\mathcal{L}(E,F)$, and spaces of smooth functions $\mathcal{C}^\infty(E,F) \simeq \mathcal{L}(!E,F)$. Then as these spaces are reflexive we have necessarily :

$$!E \simeq \mathcal{C}^\infty(E,\mathbb{K})' \text{ and } ?E \simeq \mathcal{C}^\infty(E,\mathbb{K})$$

Thus $!E$ is a space of linear forms acting on some space of smooth function, i.e. a space of *distributions*.

On the other hand, one of the major requirements in the categorical semantics of LL is the Seely's isomorphism: $!A \otimes !B \simeq !(A \otimes B)$. It translates immediately into the Schwartz's Kernel theorem [Schwartz 1957], written here for distributions with compact support: $\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})' \otimes \mathcal{C}^\infty(\mathbb{R}^m,\mathbb{K})' \simeq \mathcal{C}^\infty(\mathbb{R}^{n+m},\mathbb{R})$. Based on these intuitions, we find a classical semantics of DiLL in the theory of Nuclear spaces and distributions.

The language of distributions has been used for a while in Linear Logic, and this work should be seen as a way to ground this fact.

Nuclear spaces, Fréchet space, and distributions: a model of Smooth DiLL Typical examples of nuclear spaces are either euclidean spaces as $\mathbb{R}^n$ or $\mathbb{R}^m$, either spaces of (test) function $\mathcal{E}(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n)$, $\mathcal{C}^\infty_c(\mathbb{R}^n)$, or their duals, spaces of distributions $\mathcal{E}(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n)'$, $\mathcal{D}(\mathbb{R}^n) = \mathcal{C}^\infty_c(\mathbb{R}^n)'$. Moreover, a nuclear Fréchet space (that is a nuclear, complete and metrisable space) is reflexive, and while it is not preserved by duality, this condition is preserved by tensor product. We use the fact that Nuclear spaces which are Fréchet (i.e. complete and metrisable) form a negative interpretation for polarized MALL. When defining $!\mathbb{R}^n = \mathcal{E}(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n)'$, the kernel theorem of distribution allows us to see ! as a monoidal functor from the category of Nuclear spaces to the category of duals of Nucléar Fréchet spaces; We translate this structure in the syntax (section 4) obtaining a polarized Smooth DiLL with a distinction between finitary and smooth formulas.

Modelizing D-DiLL by LPDEs Our definition of D-DILL is justified by the fact that for D any linear partial differential operator (LPDO) with constant coefficients, we have a model of D-DiLL.

$!\mathbb{R}^n$ is then interpreted as the space of distribution with compact support $\mathcal{E}(\mathbb{R}^n)$, D as a LPDO, and

$$!_D\mathbb{R}^n := (D(\mathcal{E}(\mathbb{R}^n)))'.$$

Consider D_0 the operator mapping a function $f \in \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})$ to its differential at 0, that is :

$$D_0 := f \mapsto v \mapsto \lim_{h \longrightarrow 0} \frac{f(hv) - f(0)}{h}.$$

Then $D_0((\mathcal{E}(\mathbb{R}^n)) = (\mathbb{R}^n)'$ and $!_{D_0}\mathbb{R}^n \simeq (\mathbb{R}^n)'' \simeq \mathbb{R}^n$. The fact that we work in a *classical setting*, and thus with reflexive spaces, is central here, as is allows to understand the usual interpretation of $\bar{d} : v \in \mathbb{R}^n \mapsto f \mapsto D_0(f)(v)$ as operator matching $\phi \in (\mathbb{R}^n)''$ to $\phi \circ D_0 \in \mathcal{C}^\infty(\mathbb{R}^n)'$, and to generalize it.

The codereliction $\bar{d}_D : !_D \longrightarrow !$ is then postcomposition[1] by D: $\bar{d}_D(\phi) = \phi \circ D$.

The coweakening $\bar{w}_D$ is then interpreted as the input of a fundamental solution E_D, solution to $\psi \circ D = \delta_0$. We prove in particular that while E_D is not a distribution with compact support in general, it is an element of the interpretation of $!_D A$. The co-contraction $\bar{c}_D$ is interpreted by the convolution between a solution in $!_D A$ and a distribution in $!A$, producing another solution in $!_D A$. Following the rules in the sequent calculus, we have, for every $\phi \in !\mathbb{R}^n$, for every $f \in \mathcal{E}(\mathbb{R}^n)$:

$$\bar{c}_D(\bar{w}_D, \phi)(D(f)) = \bar{d}(\bar{c}_D(\bar{w}_D, \phi))(f) = \phi(f).$$

That is, *the solution* ψ *to* $D\psi = \phi$ *is* $\bar{c}_D(\bar{w}_D, f)$.

Contributions This paper:

- defines a Polarized Smooth variant of DiLL, without higher order, with a distinction between smooth and finitary formulas, and its categorical models.
- constructs a denotational model for it, based on the idempotent adjunction between Nuclear Fréchet and Nuclear DF spaces, and the construction of the exponential as a space of compact support distributions.
- defines a Polarized D-Differential Linear Logic, which refines Smooth DiLL with an indexed exponential $!_D$ whose rules represent the computation of a solution to a partial differential equation. We define a cut-elimination procedure for D-DiLL.
- shows that we have a model of Polarized D-DiLL for any LPDOcc.

Related work There is a major research effort towards the understanding and the semantics of probabilistic programming [Danos et al. 2017; Furber et al. 2017; Heunen et al. 2017]. Our work bears similarity with these, if only because we use the same language of distributions and kernels. More generally, this works takes place in a global understanding of continuous data-types and computations : machine-learning, which uses gradients to optimize the computations, is one example. The change of paradigm, allowing to go from a discrete point of-view on ressource-sensitive programs to solutions of Differential Equations, relates to recent work on continuous probabilty distributions in probabilistic programming [Ehrhard et al. 2018]. Notice however that models of probabilistic programming are not in general models of Differential Linear Logic.

Organisation of the paper We first introduce in section 2 the rules, cut-elimination procedure and categorical semantics of DiLL. Then in section 3 we give an overview of the functional analysis necessary to the paper. We barely recall any proofs, but show examples and precise references for our claims. Section 4 is quite short, as it formalizes syntactically and categorically the content of section 3 into the definition of Smooth DiLL. Section 5 defines D-DiLL, its syntax, rules, cut-elimination procedure and its categorical semantics. We also show in this section that for any D LPDOcc, we have a model of D-DiLL.

2 DiLL

In this section, we recall the rules of DiLL, the categorical structure needed to interpret them and its cut-elimination procedure. We explain the denotational intuitions behind these rules.

2.1 The formulas and proofs of DiLL

The formulas of Differential Linear Logic are constructed according to the same grammar as LL, see figure 1, with additive and multiplicative disjunction and conjunction connective. The negation of a formula A is denoted $A^\perp$ and defined as follows:

$$\begin{array}{lll} !A^\perp = ?A^\perp & (A \,\&\, B)^\perp = A^\perp \oplus B^\perp & (A \oplus B)^\perp = A^\perp \,\&\, B^\perp \\ ?A^\perp = !A^\perp & (A \parr B)^\perp = A^\perp \otimes B^\perp & (A \otimes B)^\perp = A^\perp \parr B^\perp \\ & 1^\perp = \perp \quad \perp^\perp = 1 & 0^\perp = \top \quad \top^\perp = 0 \end{array}$$

We recall the rules for the exponential connectives $\{?, !\}$ of DILL in figure 1. The other rules correspond to the usual ones for the MALL group $\{\otimes, \parr, \oplus, \times\}$ [Girard 1987].

Formulas of DiLL:

$$E, F := 0 | 1 | \top | \perp | A^\perp | A \otimes B | A \parr B | A \oplus B | A \times B | !A | ?A$$

Exponential rules of DiLL:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?E}\, w \qquad \frac{\vdash \Gamma, ?E, ?E}{\vdash \Gamma, ?E}\, c \qquad \frac{\vdash \Gamma, E}{\vdash \Gamma, ?E}\, d$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, !E}\, \bar{w} \qquad \frac{\vdash \Gamma, !E \qquad \vdash \Delta, !E}{\vdash \Gamma, \Delta, !E}\, \bar{c} \qquad \frac{\vdash \Gamma, E}{\vdash \Gamma, !E}\, \bar{d}$$

Figure 1. Syntax for the formulas and proofs of DiLL

Let us detail the denotational intuitions behind these rules. We interpret a sequent $E \vdash F$ as a linear function from E to F, and choose to denote by the same letter a formula and its interpretation, which one is which will be clear from the context.

The *codereliction* $\bar{d}$ allows then, by precomposition on a function from $!E$ to F, that is by a cut rule on a sequent $\vdash ?E^\perp, F$ to find the differential *at* 0 of f, that is a sequent $\vdash E^\perp, F$. The fact that we take here the differential at 0 must be understood as the necessity to isolate one single useful copy of E in $!E$, while the others are replaced by an empty hypothesis . In order to be able to differentiate at any point, the *cocontraction* $\bar{c}$ is introduced, which corresponds semantically to the convolution on functions:

$$f * g : x \mapsto \int f(y) g(x - y) dy.$$

This definition is then extended to the distributions of $!E$. Finally, the *coweakening* $\bar{w}$ is interpreted as $\delta_0 : f \mapsto f(0)$. It is in particular the neutral for the convolution.

Definition 2.1. Proofs of DiLL are *finite sums* of proof-trees generated by these rules. In particular, there is of an empty proof tree denoted by .

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[1] To avoid early confusion, we recall that for a distribution ψ, $D(\phi)$ is usually not defined as $\phi \circ D$. See section 5.4.

$$\begin{tikzcd}
E \arrow[dr, "\bar{d}"] \arrow[dd, "Id"'] & \\
& !E \arrow[dl, "d"] \\
E &
\end{tikzcd}$$

$$\begin{tikzcd}
E \arrow[dr, "\bar{d}"] \arrow[dd, "0"'] & \\
& !E \arrow[dl, "w"] \\
1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[dr, "c"] \arrow[dd, "\bar{d} \otimes \bar{w} + \bar{w} \otimes \bar{d}"'] & \\
& !E \arrow[dl, "\bar{d}"] \\
!E{\otimes}!E &
\end{tikzcd}$$

$$\begin{tikzcd}
!E \arrow[dr, "d"] \arrow[dd, "0"'] & \\
& E \arrow[dl, "\bar{w}"] \\
1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
!E{\otimes}!E \arrow[dr, "\bar{c}"] \arrow[dd, "d \otimes w + w \otimes d"'] & \\
& !E \arrow[dl, "d"] \\
E &
\end{tikzcd}$$

Figure 2. Cut-Elimination for the exponential group of DiLL

The cut-elimination procedure follows the one of LL for the MALL connectives, and the one for the exponential group are detailed in Ehrhard [2017]. They follows the intuitions for the differentiation in euclidean spaces. We recall them semantically, through commutative diagrams in figure 2.

2.2 Categorical models of DiLL

There are two points of view: the first one is to refine the notion of Seely Model of Linear Logic with a biproduct and an interpretation for the codereliction [Fiore 2007], and the second one considers first models of DiLL without *prom*, and then extend this definition [Ehrhard 2017]. We adopt the first point of view, but make use of the numerous details and diagrams exposed by Ehrhard [Ehrhard 2017]. The following definitions are those of Fiore [2007], sometimes adapted to the classical setting.

Definition 2.2. A biproduct on a category $\mathcal{L}$ is a monoidal structure $(\diamond, I)$ together with natural transformations:

$$\begin{tikzcd}
I \arrow[dr, "u_A"'] & & I \\
& A \arrow[ur, "n_A"'] \arrow[dr, "\triangledown_A"] & \\
A \diamond A \arrow[ur, "\triangle_A"] & & A \diamond A
\end{tikzcd}$$

such that $(A, u, \triangledown)$ is a commutative monoid and $(A, n, \triangle)$ is a commutative comonoid.

Definition 2.3. A $*$-*autonomous category* is a symmetric monoidal closed category $(\mathcal{L}, \otimes, 1)$ with an object $\perp$ giving an equivalence of categories $(\cdot)' = [\cdot, \perp]_{\mathcal{L}} : \mathcal{L}^{op} \longrightarrow \mathcal{L}$ with the canonical map $ev_E : E \longrightarrow E''$ being a natural isomorphism.

Definition 2.4. A model of DiLL with promotion is consists of a symmetrical monoidal closed category $(\mathcal{L}, \otimes, 1)$ with a *-automous structure, a biproduct structure $(\diamond, \top)$, a co-monad $! : \mathcal{L} \longrightarrow \mathcal{L}$ which is strong monoidal from $(\mathcal{L}, \diamond)$ to $(\mathcal{L}, \otimes)$, and a natural transformation $\bar{d} : Id \longrightarrow !$ satisfying strengh and comonad diagrams [Fiore 2007].

Remark. As shown by Fiore, from the biproduct structure follows the fact that the category $\mathcal{L}$ is enriched over commutative monoids. This induces an additive law $+$ on hom-sets, which is necessary to interpret the sums of proofs-trees of DiLL which stems from cut-elimination.

$$f + g : E \overset{\triangle(f,g)}{\longrightarrow} F \diamond F \overset{\triangledown}{\longrightarrow} F.$$

2.3 Interpreting DiLL in its categorical model.

We briefly recall how to interpret a sequent of DiLL as morphism in $\mathcal{L}$, detailing only the action of exponential rules. The connectives $\otimes$, $⅋$, $\oplus$, $\&$ are interpreted respectively by $\otimes$ and its dual, and by the coproduct and product deduced from $\diamond$. We have $!I = 1$ by strong monoidality of !

We write $m_{E,F} :!(E \diamond F) \longrightarrow !E \otimes !F$ the isomorphism resulting from the monoidality of !, and $d :! \longrightarrow Id$ the co-unit of !. Then:

- from $f : E \longrightarrow F$ one construct $f \circ d_E :!E \longrightarrow F$ and from $g :!E \longrightarrow F$ one construct $g \circ \bar{d}_E : E \longrightarrow F$.
- one construct $w :! \longrightarrow Id$ as $w_E = !n$ and $\bar{w} : 1 \longrightarrow !$ as $\bar{w}_E :!u$.
- one construct the natural transformation $c :! \longrightarrow !\otimes!$ as $c_A = m_{A,A} \circ !\triangle_A$ ands $\bar{c} :!\otimes! \longrightarrow !$ as $\bar{c}_A = !\triangledown_A \circ m^{-1}_{A,A}$

It should be clear then that in order to interpret the exponential rules of DiLL one requires the biproduct structure, the strong monoidality of ! and an interpretation for $\bar{d}$ and d. The co-monadic structure of ! is used only for the interpretation of the promotion rule, and enforces the definition of d. We will make use of that statement in section 4 when we relax the co-monad requirement on !.

3 Topological vector spaces

In this section, we give technical accounts on some specific classes of topological vector spaces, on distribution theory and LPDOs. We refer mainly to the books by Jarchow [Jarchow 1981] and Hörmander [Hörmander 2003], as well as Grothendieck' thesis [Grothendieck 1966]. We consider vector spaces on $\mathbb{R}$.

Definition 3.1. A *topological vector space* (tvs) is a vector space endowed with a topology, that is a covering class of open sets closed by infinite union and finite intersection, making the scalar multiplication and the addition continuous. A tvs is said to be *Hausdorff* if for any two distinct point x and y one can find two disjoint open sets containing x and y respectively. It is *locally convex* if every point is contained in a convex open set.

From now on we work with locally convex separated topological vector spaces and denote them by lctvs. Examples of lctvs includes all euclidean spaces $\mathbb{R}^n$, normed spaces and metric spaces. For the rest of the section we consider E and F two lctvs.

Notation. We will write $E = F$ for the linear isomorphism between E and F as vector spaces, and $E \simeq F$ for the linear homeomorphism between E and F as tvs.

Definition 3.2. Consider $U \subset E$ and $x \in U$, then U is said to be a neighborhood of x if U contains an open set containing x. A set $B \subset E$ is bounded if for every U neighborhood of 0, there is $\lambda \in \mathbb{R}$ such that $B \subset \lambda U$.

Definition 3.3. For two lctvs E and F we consider $\mathcal{L}_b(E,F)$ the lctvs of all *linear continuous* functions between E and F and endow it with the *topology of uniform convergence on bounded subsets* of E. We write $E' = \mathcal{L}_b(E,\mathbb{R})$ for the dual of of E.

Definition 3.4. A lctvs is *reflexive* if $E \simeq E''$ through the transpose of the evaluation map in E':

$$\delta : \begin{cases} E & \longrightarrow & E'' \\ x & \mapsto & \delta_x : (f \longrightarrow f(x)) \end{cases}$$

Typically, all euclidean spaces are reflexive, as they are isomorphic to their dual. This is also true for every Hilbert spaces, but as soon as we generalize to Banach spaces we encounter the famous counter example of ℓ_1 and its dual ℓ_∞. The restriction to reflexive spaces is not preserved by tensor product nor linear hom-sets: typically, the space $\mathcal{L}(H,H)$ is not reflexive when H is a Hilbert space [2].

Definition 3.5. Consider E and F two *lctvs*. The *projective tensor product*[3] $E \otimes_\pi F$ is the algebraic tensor product, endowed with the finest topology making the canonical bilinear map $E \times F \to E \otimes F$ continuous. Then $E \otimes_\pi F$ is a lctvs. The completion of $E \otimes_\pi F$ is called the completed projective tensor product and denoted $E \hat{\otimes}_\pi F$

3.1 (F)-spaces and (DF)-spaces

Definition 3.6. A Fréchet space, or (F)-space, is a complete and metrisable lctvs.

Recall that a lctvs is metrisable if and only if it admits a countable basis of 0-neighbourhoods. If F is a metrisable space, we write $\hat{F}$ its completion. Fréchet spaces are very common in analysis, but are not preserved by duality: the dual of a Fréchet space is not necessarily metrisable. In particular, the dual $C^\infty(\mathbb{R}^{\cdot}, R)'$ of the space of smooth scalar functions , as described in section 3.2, is not metrisable.

Definition 3.7. A (DF)-space is a lctvs E admitting a countable basis of bounded sets $\mathcal{A} = (A_n)_n$[4], and such that if $(U_n)_n$ is a sequence of closed and disked neighbourhoods of 0 whose intersection U is bornivorous (i.e. absorbs all bounded subsets), then U is a neighbourhood of 0.

Let us note that, by duality, the second condition is equivalent to asking every bounded subset B of the strong dual E' which is the union of a sequence of equicontinuous subsets to be equicontinuous. Moreover, it is costless to ask that for every n A_n is be absolutely convex and $A_n + A_n \subset A_{n+1}$. We will therefore always suppose that this is the case. Although this definition may seem obscure, it is the right one for interpreting the dual and pre-dual of (F)-spaces.

Proposition 3.8 ([Grothendieck 1973] IV.3.1)**.**

- *If F is metrisable, then its strong dual E' is a (DF)-space.*
- *If E is a (DF)-space and F and (F)-space, then $\mathcal{L}_b(E,F)$ is an (F)-space. In particular, F' is an (F)-space.*

Proposition 3.9 ([Jarchow 1981] 12.4.2 and 15.6.2)**.** *The class of (DF)-spaces is preserved by countable inductive limits, countable direct sums, quotient and completions, The class of (F)-spaces is stable with the construction of products and completed projective tensor products $\hat{\otimes}_\pi$.*

The following reflects the syntax of an intuitionist version of Smooth DiLL of section 4.

Example 3.10 ([Jarchow 1981] 12.4.4)**.** A space which is Fréchet and (DF) is necessarily finite dimensional.

3.2 Distributions with compact support

We refer to the exposition of distributions by Hörmander [Hörmander 2003] for proofs and details.

Definition 3.11. Consider $n \in \mathbb{N}$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. The function f is said to be smooth if it is differentiable at every point $x \in \mathbb{R}^n$, and if at every point its differential is smooth.

The theory of distribution is traditionally introduced by considering the space $\mathcal{D}(\mathbb{R}^n) := C_c^\infty(\mathbb{R}^n)$ of test functions, i.e. the space of scalars smooth functions on $\mathbb{R}^n$ with compact support, and define distributions as elements of its dual $\mathcal{D}(\mathbb{R}^n)'$. But because the dereliction rule d makes us consider linear continuous function as a particular case of smooth function, we work with the following:

Definition 3.12. We consider $\mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n, \mathbb{R})$ the space of all scalar smooth functions on $\mathbb{R}^n$, endowed with the usual topology of uniform convergence of all differentials of order $\leq k$on all compact subsets of $\mathbb{R}^n$, for all $k \in \mathbb{N}$. Its dual is called the space of distributions with compact support and denoted $\mathcal{E}'(\mathbb{R}^n)$.

Proposition 3.13. *For any $n \in \mathbb{N}$, $\mathcal{E}(\mathbb{R}^n)$ is an $(F)-space$ and $\mathcal{E}'(\mathbb{R}^n)$ is a complete $(DF)-space$.*

Example 3.14. A distribution must be considered as a generalized function, and acts as such. The key idea is that, if $f \in C_c^\infty(\mathbb{R}^n)$ then on defines a compact distribution by $g \in C^\infty(\mathbb{R}^n) \mapsto \int\limits_{\mathbb{R}} f(x)g(x)dx$.

Typical examples of distributions which do not follow this pattern are the dirac distributions. For $x \in \mathbb{R}^n$ one defines the dirac at x as: $\delta_x : f \in \mathcal{E}(\mathbb{R}^n) \mapsto f(x)$.

Definition 3.15. Consider $\phi \in \mathcal{E}'(\mathbb{R}^n)$ and $f \in \mathcal{E}(\mathbb{R}^n)$. Then one defines the convolution between a distribution and a functions as $\phi * f \in \mathcal{E}(\mathbb{R}^n)$ as: $\phi * f : x \mapsto \phi(y \mapsto f(x-y))$.

This definition is extended to a convolution product between distributions. Consider $\psi \in \mathcal{E}'(\mathbb{R}^n)$. Then $\phi * \psi$ is the unique distribution in $\mathcal{E}'(\mathbb{R}^n)$ such that:

$$\forall f \in \mathcal{E}(\mathbb{R}^n), (\phi * \psi) * (f) = \phi * (\psi * f). \tag{1}$$

Although the above is not a symmetric definition, one proves easily that the convolution is commutative and associative [Hörmander 2003].

Example 3.16. Note that δ_0 defined in 3.14 acts as neutral element for the convolution law.

The central theorem of the theory of distributions is the Kernel Theorem:

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[2] The author thanks Marc Bagnol for this clarifying example.

[3] Many topologies can be defined on the vector space resulting from the tensor product of two lctvs. The later restriction to Nuclear spaces will de facto identify all reasonable topological tensor product to the projective one.

[4] A basis $\mathcal{A}$ being a collection of bounded set such that every bounded set in included into an object of $\mathcal{A}$

Theorem 3.17 ([Trèves 1967] 51.6)**.** *For any $n, m \in \mathbb{N}$ we have:*

$$\mathcal{E}'(\mathbb{R}^{n+m}) \simeq \mathcal{E}'(\mathbb{R}^m)\hat{\otimes}_\pi \mathcal{E}'(\mathbb{R}^m) \simeq \mathcal{L}(\mathcal{E}'(\mathbb{R}^m), \mathcal{E}(\mathbb{R}^n))$$

This theorem is proved on the spaces of functions by showing the density of smooth functions of the kind $f \otimes g$, $f \in \mathcal{E}(\mathbb{R}^n)$, $g \in \mathcal{E}(\mathbb{R}^m)$, and then that the topology induced by $\mathcal{E}(\mathbb{R}^{n+m})$ on $\mathcal{E}(\mathbb{R}^m) \parr \mathcal{E}(\mathbb{R}^m)$ is indeed the projective topology of the tensor product. This, and particularly the fact that $\parr = \hat{\otimes}_\pi$, is justified by the theory of Nuclear spaces, which is recalled below.

3.3 Nuclear spaces

The theory of nuclear spaces will allow us to interpreted the idempotent negation of DiLL, and as the same time the theory of exponentials as distributions

Definition 3.18. An linear map f between a lctvs E and a Banach X is said to be nuclear if there is an equicontinuous sequence (a_n) in E', a bounded sequence (y_n) in X, and a sequence $(\lambda_n) \in l_1$ such that for all $x \in E$:

$$f(x) = \sum_n \lambda_n a_n(x) y_n.$$

Definition 3.19. Consider E a lctvs. We say that E is nuclear every continuous linear map of E into any Banach space is nuclear.

Proposition 3.20 ([Jarchow 1981] 21.2.3)**.** *The class of nuclear spaces is closed with respect to the formation of completion, cartesian products, countable direct sums, projective tensor products, subspaces and quotients.*

An important property of nuclear spaces is that as soon as they are normed, they are finite dimensional. In other word, if a Hilbert or Banach or simply normed space is nuclear, then it is isomorphic to $\mathbb{R}^n$ for a certain n.

Example 3.21. Typical examples of nuclear spaces are euclidean spaces $\mathbb{R}^n$, spaces of smooth functions $C^\infty_c(\mathbb{R}^n, \mathbb{R})$, $C^\infty(\mathbb{R}^n, \mathbb{R})$ and their duals $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$.

Theorem 3.22. *An (F)-space F which is also nuclear is reflexive. As a consequence, $\mathcal{E}(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ are reflexive.*

Proof. We give a brief proof for the reader familiar with functional analysis. It is enough to prove that F is semi-reflexive, that is that $F = F''$, as the equality between the topologies will follow from the metrisability of F. Indeed, when F is metrisable E-equicontinuous sets and E-weakly bounded sets corresponds in E' [Jarchow 1981, 8.5.1]. Now we have that every bounded set of a nuclear space is precompact [Schaefer 1971, III.7.2.2]. Thus as F is nuclear and complete, its bounded sets are compact, and F' is endowed with the Arens-topology of uniform convergence on absolutely convex compact subsets of F. By the Mackey-Arens theorem, this makes F semi-reflexive. □

Proposition 3.23.

- *Consider E a lctvs which is either an (F)-space or a (DF)-space. Then E is nuclear if and only if E' is nuclear [Grothendieck 1966, Chap II, 2.1, Thm 7].*
- *IF E is a complete (DF)-space and if F is nuclear, then $\mathcal{L}_b(E, F)$ is nuclear. If moreover F is an (F)-space or a (DF)-space, then $\mathcal{L}_b(E, F)'$ is nuclear [Grothendieck 1966, Chapter II, 2.2, Thm 9, Cor. 3]. As a corollary, the dual of a nuclear (DF)-space is a nuclear (F)-space.*

Proposition 3.24 ([Grothendieck 1966] Chapter II, 2.2, Thm 9)**.** *If E and F are both nuclear (DF)-spaces, then so is $E \otimes_\pi F$.*

A central result of the theory of nuclear spaces, explaining for the form of the Kernel theorem 3.17, is the following proposition. It is proved by applying the hypothesis that E is reflexive and thus E' is complete and barrelled, and thus applying the hypothesis of Treves' book [Trèves 1967].

Proposition 3.25 ([Trèves 1967] prop. 50.5)**.** *Consider E a Fréchet nuclear space, and F a complete space. Then $E\hat{\otimes}_\pi F \simeq \mathcal{L}(E', F)$.*

3.4 Linear Partial Operators

We recall the very first steps in the theory of LPDEs[5]. For $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ we write ∂^α the linear continuous map:

$$f \in C^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto x \in \mathbb{R}^n \mapsto \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}(x)$$

Definition 3.26. Consider, for $\alpha \in \mathbb{N}^n$ smooth functions $a_\alpha \in C^\infty(\mathbb{R}^n, \mathbb{R})$. Then a Linear Partial Differential Operator (LPDO) is defined as an operator $D : C^\infty_c(\mathbb{R}^n) \longrightarrow C^\infty_c(\mathbb{R}^n)$:

$$D = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial^\alpha.$$

A LPDO with constant coefficients is a LPDO D such that the a_α are constants.

The definition of D is extended to distributions as follows:

$$D(\phi) = f \mapsto \sum_\alpha (-1)^\alpha a_\alpha \partial^\alpha(\phi(f)),$$

so that for $g \in C^\infty_c(\mathbb{R}^n) : D(f \mapsto \int fg) = f \mapsto \int f D(g)$.
The *weak resolution* of the LPDE consists then, when $\phi \in \mathcal{E}'(\mathbb{R}^n)$, of finding ψ such that, for all $f \in \mathcal{E}(\mathbb{R}^n)$[6].

$$\psi \circ D(f) = \phi(f). \tag{2}$$

The resolution of LPDOs with constant is always possible, and particularly elegant, due to the behaviour of convolution with respect to partial differentiation:

Proposition 3.27 ([Hörmander 2003] 4.2.5)**.** *Consider $f \in C^\infty_c(\mathbb{R}^n)$ and $\phi \in \mathcal{E}'(\mathbb{R}^n)$. Then $\partial^\alpha \phi * f = \phi * (\partial^\alpha f)$.*

Definition 3.28. A fundamental solution to equation (2) consists of a distribution $\hat{E}_D \in C^\infty_c(\mathbb{R}^n)'$ such that $D(E_D) = \delta_0$.

Example 3.29. Because of linear partial differential operator, we are working with distributions whose support is not neccessarily compact. Indeed, the existence of a fundamental solution is not ensured when distributions must apply to any

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[5] We are not considering in this paper border conditions, regularity of the solutions to equations with non-constant coefficients, nor modern research subjects in the theory non-linear equations.

[6] This definition is specific to the paper, and necessary to be coherent DiLL. In the literature, the resolution of the equation consists in finding ψ such that $(D\psi)(f) = \phi(f)$.

smooth function. The typical example is $D = f \in C^\infty(\mathbb{R}, \mathbb{R}) \mapsto f'$. If f has compact support, one can define:

$$E_D : f' \mapsto \int_\infty^0 f'$$

and one has indeed $E_D(D(f)) = f(0)$. This however is not possible in generality when f has no compact support.

It appears then thanks to the linearity of the convolution product and propositions 3.16 and 3.27 that:

$$\forall \phi \in \mathcal{E}'(\mathbb{R}^n), D(\hat{E}_D * \phi) = \phi.$$

Again, we are make a slighlty different use of the fundamental solution by defining E_D such that equation 3 holds.

Theorem 3.30 (Malgrange-Ehrenpreis [Hörmander 1963] 3.1.1)**.** *Every LPDEcc admits a fundamental solution $\hat{E}_D$, which leads to $E_D = \hat{E_D} * \delta_0 \in (D(C^\infty_\rfloor(\mathbb{R}^n)))'$.*

The theorem is in fact much more precise as we have information about the local growth of E_D. We do not have in general that $E_D \in \mathcal{E}'(\mathbb{R}^n)$.
However, the first and easy step of the proof consists in noticing that, if one defines E_D as the distribution $f \mapsto \hat{E_D}(x \mapsto f(-x))$, that is $E_D(f) = (\hat{E_D} * f)(0)$, we have:[7]

$$\forall f \in C^\infty_c(\mathbb{R}^n), E_D(D(f)) = f(0). \tag{3}$$

The proof consists afterwards into the majoration of $\check{E}$ in order to extend if to $C^\infty_c(\mathbb{R}^n)$. This is one of the arguments for the introduction of $!_D(\mathbb{R}^n) = (D(C^\infty_c(\mathbb{R}^n))'$.

Proposition 3.31. *The fundamental solution $\check{E_D}$ defined well defined and continuous on $D(C^\infty_c(\mathbb{R}^n))$, as it corresponds to $D(f) \mapsto f(0)$. We have thus $\check{E_D} \in !_D\mathbb{R}^n$.*

4 Smooth Differential Linear Logic and its models

In this section we introduce a Smooth Differential Linear Logic for which Nuclear spaces and distributions form a classical and smooth model. We notably show that the categorical intepretations for $\bar{c}$ and bw correspond to the convolution and the dirac in 0 in the theory of distributions.
Let us recall the notion of polarisation in LL. In polarized linear logic [Laurent 2002] a distinction is made between positive connectives ?, $\otimes$, $\oplus$ whose introduction rules are non-reversible, and negative connectives !, $\parr$, & whose introduction rule is reversible. Negation then changes the polarity of a formula. This plays a fundamental role in proof-search [Andreoli 1992].

4.1 The category of Nuclear Fréchet spaces.

Nuclear Fréchet spaces gather all the stability properties to be a (polarized) model of LL, except that we do no have an interpretation for higher-order smooth functions. Indeed if $!\mathbb{R}^n$ is interpreted as $\mathcal{E}'(\mathbb{R}^n)$, we do not have a straightforward definition of $!!\mathbb{R}^n$.

Formulas : $E, F := A|N|P$
Finitary formulas: $A, B := 0|1|\top|\bot|A^\perp|A \otimes B|A \parr B|A \oplus B|A \times B.$
Negative smooth formulas: $N, M := A|?A||N \parr M|N \times M|P^\perp$
Positive smooth formulas : $P, Q := A|!A|P \otimes Q|P \oplus Q|N^\perp$

Figure 3. The syntax of Smooth DiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,d$$

$$\frac{\vdash}{\vdash !A}\,\bar{w} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\,\bar{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\,\bar{d}$$

Figure 4. Exponential Rules of SDiLL

Definition 4.1. We write Nf the category of Nuclear (F)-spaces and continuous linear maps, Ndf the category of complete Nuclear (DF)-spaces and continuous linear maps, and Eucl the subcategory of both formed of euclidean spaces.

Proposition 4.2.
- Eucl *is a model of MALL.*
- Nf *forms a model for the negative interpretation of polarized MALL [Laurent 2002, 6.20], where positive formulas are thus interpreted as objects of* Ndf.

Proof. The first point is transparent. The second point is due to the stability of Nuclear Fréchet spaces by cartesian product (interpreting &) and completed π-tensor product (interpreting $\parr$), see propositions 3.20, 3.9 and 3.25. The interpretation of the rules for $\otimes$ and $\oplus$ is possible by the fact that Nuclear Fréchet spaces are reflexive (proposition 3.22) and thus the interpretation of $A \otimes B$ is the one of $(A^\perp \parr B^\perp)^\perp$. ◻

Thus the interpretation of $\otimes$ in Ndf is $\hat{\otimes}_\pi$, and that the interpretation of $\parr$ in Nf is also $\hat{\otimes}_\pi$.

Remark. Note however that we *do not* have a compact closed category, as we are working in a polarized model of MALL with an adjunction between Nf and Ndf^{op}.

Definition 4.3. For $\mathbb{R}^n \in$ Eucl we define $!\mathbb{R}^n = \mathcal{E}'(\mathbb{R}^n)$. This is extended as a functor on Eucl by defining $!(f : \mathbb{R}^n \longrightarrow \mathbb{R}^m) : \phi \in \mathcal{E}'(\mathbb{R}^n) \mapsto \phi \circ f \in \mathcal{E}'(\mathbb{R}^m)$.

It follows from the Kernel theorem 3.17 and example 3.21 that the space of compact distribution acts as a strong monoidal functor from Eucl to Ndf:

Theorem 4.4. *The exponential* ! : Eucl $\longrightarrow$ Ndf *is a strong monoidal functor.*

4.2 Smooth Differential Linear Logic (SDiLL)

In this section, we construct a version of DILL for which Nuclear spaces and distributions are a model, by distinguishing several classes of formulas. We introduce SDiLL: its grammar, defined in figure 3, separates formulas into finitary ones and polarized smooth ones.
Its rules are those of DiLL : follows the one of MALL for the additive and multiplicative connectives, and those detailed in figure 4 for the exponential. Thus, the cut-elimination procedure is the same as the one defined originally [Ehrhard and Regnier 2003].

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[7] Let us point out that even if D_0 is *not* a LPDO, the equation $D_0 g = f$ behaves likewise. If there is of a solution to this equation it means that f is linear, and then $D_0 f = D_0(f * \delta_0) = f$.

If we forget about the polarisation of SDiLL, a model of it would be a model of DiLL where the object $!!A$ does not need to be defined. It is thus a model of DiLL where ! does not need to be an endofunctor, but just a strong monoidal functor $! : \textsc{Fin} \longrightarrow \textsc{Smooth}$ between two categories. The categories $\textsc{Fin}$ and $\textsc{Smooth}$ need to be both a model of MALL.
This distinction is necessary here to account for spaces of distributions are their dual, which cannot be understood as part of the same $*$-autonomous category. We give a categorical semantics for an unpolarized version of SDiLL. The polarized version would ask the category $\textsc{Smooth}$ below to be a model of Polarized MALL, that is an involutive , defined as an adjunction between a category of negative smooth formulas and a category of positive smooth formulas. Sequents would then be interpreted as maps in the larger category of complete lctvs.

Definition 4.5. A categorical model of SDiLL consists into a model of MALL with biproduct $\textsc{Fin}$, and a model of MALL $\textsc{Smooth}$, such that we have a strong monoidal functor $! : (\textsc{Fin}, \times) \longrightarrow (\textsc{Smooth}, \otimes)$, a forgetful functor $U : \textsc{Eucl} \longrightarrow \textsc{Nucl}$ strong monoidal in $\otimes, \parr, \&, \oplus$and two natural transformation $d : ! \longrightarrow U$ and $\bar{d} : ! \longrightarrow U$ such that $d \circ \bar{d} = Id_{\textsc{Eucl}}$.

Theorem 4.6. *The structure on Nuclear Spaces and Distributions defines a model of SmoothDiLL.*

Proof. We interpret finitary formulas A as euclidean spaces. Without any ambiguity, we denote also by A the interpretation of a finitary formula into euclidean spaces. The exponential is interpreted as $!A = \mathcal{E}'(A)$, extended by precomposition on functions. We briefly explain the interpretation for the rules, which follow the intuition of [Ehrhard and Regnier 2003]. We define:

$$d : \begin{cases} !A & \longrightarrow A'' \\ \phi & \mapsto \phi_{|A'} \end{cases} \qquad \bar{d} : \begin{cases} A'' \longrightarrow !A \\ ev_x \mapsto (f \mapsto ev_x(D_0(f))) \end{cases}$$

This is justified by the definition of reflexivity 3.4. Then we have indeed: $d \circ \bar{d} = Id_{A''}$. The interpretation of w, c, $\bar{w}$, $\bar{c}$ follows from the biproduct structure on $\textsc{Eucl}$ and from the monoidality of !, as explained in 2.3. ☐

We show that $\bar{w}$, $\bar{c}$ they have a direct interpretation which follows the intuitions of [Blute et al. 2012; Ehrhard and Regnier 2003].

Proposition 4.7. *The cocontraction and coweakening defined through the kernel theorem correspond to the convolution of distributions and the introduction of δ_0.*

$$\bar{c} : \begin{cases} !A \otimes !A & \longrightarrow !A \\ \phi \otimes \psi & \mapsto \phi * \psi \end{cases}$$

$$\bar{w} : \begin{cases} \mathbb{R} \longrightarrow !A \\ 1 \mapsto \delta_0 : (f \in \mathcal{E}(A) \mapsto f(0)) \end{cases}$$

Proof. As defined in section 2, $\bar{w}_A = !(u : \{0\} \longrightarrow A)$ corresponds to $\bar{w}_A(1) = (f \in \mathcal{E}(A) \mapsto f \circ u = f(0))$, thus $\bar{w} = \delta_0$.
During the rest of the proof we use Fourier transformations and temperates distributions, as exposed by Hörmander [Hörmander 2003, 7.1]. The co-contraction is defined categorically as $\bar{c} = !\nabla \circ m^{-1}_{A,A}$. In the categorical setting, addition in hom sets is defined through the biproduct. But here the reasoning is done backward. We know that $\oplus = \times$ is a biproduct thanks to $\nabla : A \times A \longrightarrow A; (x, y) \mapsto x + y$, and thus $!\nabla : \phi \in !(A \times A) \mapsto (f \in \mathcal{E}(A) \mapsto \phi((x, y) \mapsto f(x + y))$. Moreover if $f \in \mathcal{E}(A \times A)$ is the sequential limit of $(f_n \otimes g_n)_n \in (\mathcal{E}(A) \otimes \mathcal{E}(A))^{\mathbb{N}}$ (see theorem 3.17) $m^{-1}_{A,A}(\phi \otimes \psi)(f) = \lim_n(\phi(f_n)\psi(g_n))$.
If we write by $\hat{\phi}$ the Fourier transformation of a distribution, we have that of $\widehat{\bar{c}(\phi, \psi)} = \hat{\phi}\hat{\psi}$. From the details above we deduce
$\widehat{\bar{c}(\phi, \psi)}(f) = \hat{\phi}(f)\hat{\psi}(f)$.
As distributions with compact support are temperates, we can apply the inverse Fourier transformation and thus $\bar{c}$ corresponds to the convolution. ☐

$$\begin{gathered} E, F := A | N | M \\ A, B := 0 | 1 | \top | \bot | A \otimes B | A \parr B | A \oplus B | A \times B. \\ N, M := ?A | ?_D A || N \parr M | N \times M \\ P, Q := !A | !_D A | P \otimes Q | P \oplus Q \end{gathered}$$

Figure 5. The grammar of formulas of D-DiLL

$$\frac{\vdash}{\vdash !_D A}\ \bar{w}_D \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !_D A}{\vdash \Gamma, \Delta, !_D A}\ \bar{c}_D \qquad \frac{\vdash \Gamma, !_D A}{\vdash \Gamma, !A}\ \bar{d}_D$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\ w_D \qquad \frac{\vdash \Gamma, ?A, ?_D A}{\vdash \Gamma, ?_D A}\ c \qquad \frac{\vdash \Gamma, ?_D A}{\vdash \Gamma, ?A}\ d$$

Figure 6. Exponential rules for D-DiLL

The interpretation of the contraction c is then the construction of a Kernel of two smooth functions, while the intepretation for the weakening consists in applying a distribution to the function constant at 1. Diagrams of 2 are easily verified and follow the intuitions of [Ehrhard 2017; Ehrhard and Regnier 2003].

5 LPDEs in the Syntax

In this section we define a sequent calculus refining Smooth DiLL by introducing a connective $!_D$. We prove that the rules and cut-elimination account of $!_D$ account semantically for the resolution of LPDEs with constant coefficients. We prove that when $!_D \simeq Id$, proof trees of Smooth DiLL are sums of proof trees of D-DiLL.

5.1 The sequent calculus D-DiLL

Grammar and rules We introduce a generalisation of Smooth DiLL, where the role of $A \simeq A^{\bot\bot}$ in the exponential rules $\bar{d}$ and d is played by a new formula $!_D A$. The idea is that $A^{\bot\bot}$ represents the linear forms acting on the space of functions $f = D_0 g$ for some, and that $!_D A$ represents the type of linear forms acting on functions $f = Dg$ for some g.
The grammar of D-DiLL is defined in figure 5, and differs very little from those of SDiLL. The MALL connectives of D-DiLL follow the same rules as usual in LL or DiLL.

The cut-elimination procedure in D-DiLL The cut-elimination is described in figure 7 as commutative diagrams for their denotational interpretation. It is inspired by the one of Linear Logic and by the calculus on distributions, see section 5.4.

Remark. The differences between SDiLL and D-DiLL makes the cut-elimination procedure simpler: cuts between d and $\bar{w}$

$$\begin{tikzcd}
!_D E \arrow[rd, "\bar{d}_D"] \arrow[dd, "Id"'] & \\
& !E \arrow[ld, "d_D"] \\
!_D E &
\end{tikzcd}
\qquad
\begin{tikzcd}
!_D E \otimes !E & \\
& !_D E \arrow[lu, "c_D"'] \arrow[ld, "\bar{d}_D"] \\
!E \arrow[uu, "\bar{w}_D \otimes id"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
!_D E \otimes !E \arrow[rd, "\bar{c}_D"] \arrow[dd, "w_D \otimes id"'] & \\
& !_D E \arrow[ld, "d_D"] \\
!E &
\end{tikzcd}$$

Figure 7. Cut-Eliminitation in D-DiLL for the exponential rules

or $\bar{d}$ and w are not possible, and the cut-elimination procedure does not generates sums of proof-terms, as contraction and co-contraction are not symmetrical. The proof that the cut-elimination procedure converges to cut-free proofs is a direct adaptation of the one for DiLL.

5.2 Encoding DiLL

Proposition 5.1. *The rules $\bar{w}$, $\bar{c}$, w and c are admissible in D-DiLL.*

Proof. We write the rules here under their denotational form: $w = d_D \circ w_D$, $\bar{w} = \bar{d}_D \circ \bar{w}_D$, $c = d_D \circ c_D(\_ \otimes c_D(\_ \otimes w_D)) + d_D \circ c_D(c_D(\_ \otimes w_D) \otimes \_)$, $\bar{c} = bd \circ \bar{c}_D(\bar{c}_D(\_ \otimes \bar{w}_D) \otimes \_) + bd \circ \bar{c}_D(\_ \otimes \bar{c}_D(\_ \otimes \bar{w}_D))$. ☐

Likewise, one proves similarly the following propositions. One shows then easily that the cut-elimination procedures correspond.

Proposition 5.2. *The rules $\bar{w}_D$, $\bar{c}_D$, w_D and c_D are admissible in SDiLL, when $!_D A$ is equivalent to A.*

Theorem 5.3. *When $!_D A \simeq A$, the proof-trees of SDiLL are sums of proof-trees of D-DiLL.*

5.3 Categorical models of D-DiLL

Definition 5.4. A categorical model of D-DiLL consists in a model of MALL with biproduct EUCL, and a (polarized) model of MALL NUCL, with a strong monoidal functor $! : (\text{EUCL}, \times) \longrightarrow (\text{NUCL}, \otimes)$, a functor $!_D : \text{EUCL} \longrightarrow \text{NUCL}$, and two natural transformation $d_D : ! \longrightarrow !_D$ and $\bar{d}_D : ! \longrightarrow !_D$ such that $d_D \circ \bar{d}_D = Id_{\text{EUCL}}$.

Indeed, one defines the interpretations of c_D, w_D, $\bar{c}_D$, $\bar{w}_D$ through the strong monoidality of !, the biproduct structure and d_D and $\bar{d}_D$ as it is done in the proof of proposition 5.1 and in paragraph 2.3.
The cut-elimination rules of figure 7 are then easily verified. For example, we have indeed :
$\bar{c}_D(\bar{w}_D, \phi) = d_D \circ \bar{c}(\bar{d}_D \circ \bar{w}_D, \phi) = d_D \circ \bar{c}(\bar{w}, \phi) = d_D(\phi)$.

5.4 A LPDE interpreted in the syntax

We show that the categories EUCL, NDF and NF defined in section 3.3, together with distributions of compact support and a LPDOcc D, form a model of D-DiLL. *In this section we interpret $!\mathbb{R}^n$ by the space of distributions, and not distributions with compact support.*
Consider $D : \mathcal{E}(\mathbb{R}^n) \longrightarrow \mathcal{E}(\mathbb{R}^n)$ a LPDOcc:

$$D(f)(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial^\alpha f(x).$$

We interpret finitary formulas A, B as euclidean spaces. One has indeed $1 \simeq 0 = \mathbb{R}$ and $\top \simeq \perp = \{0\}$. The connectives of LL are interpreted in EUCL, NF and NDF as in section 4.

Definition 5.5. For A a finitary formula interpreted by $\mathbb{R}^n \in$ EUCL, we interpret $!_D A$ an its dual as:

$$!_D \mathbb{R}^n := (D(C_c^\infty(\mathbb{R}^n)))'$$
$$?_D \mathbb{R}^n = D(C_c^\infty((\mathbb{R}^n)')) = D(C_c^\infty(\mathbb{R}^n))$$

Proposition 5.6. *We have that $?_D \mathbb{R}^n \in \text{Nf}$ and $!_D \mathbb{R}^n \in \text{Ndf}$.*

Proof. $?_D \mathbb{R}^n$ is a closed subset of $\mathcal{E}(\mathbb{R}^n)$. As such, it is a nuclear (F)-space, see 3.20 and 3.9. Thus $?_D \mathbb{R}^n \in$ NF and $!_D \mathbb{R}^n \in$ NDF. ☐

From the previous proposition and proposition 3.22) it follows that $(?_D \mathbb{R}^n)' \simeq !_D \mathbb{R}^n$.

Theorem 5.7. *We extend $!_D$ on linear maps by precomposition by D, and thus define a functor $!_D : \text{Eucl} \longrightarrow \text{Nf}$. Then we have natural isomorphisms*

$$m_{D,A,B} : !_D(\mathbb{R}^{n+m}) \simeq !_D \mathbb{R}^n \hat{\otimes}_\pi !R^m.$$

Proof. This theorem encodes in particular a well used convention in LPDOs [Trèves 1967, chap. 52], which allows to extends D defined on $\mathcal{E}(\mathbb{R}^n)$ to $\mathcal{E}(\mathbb{R}^{n+m})$. One differentiate son the n-first variable apply to functions defined on $\mathbb{R}^{n+m}$. Our theorem is then directly deduced from the Kernel theorem 3.17. ☐

The interpretation of w_D, $\bar{w}_D$, c_D and $\bar{c}_D$ follows from the previous proposition and the biproduct structure :

- $\bar{w}_D : 1 \longrightarrow !_D$ is such that $\bar{w}_{D,E}(1) = E_D$. It is well defined thanks to proposition 3.31.
- $\bar{c}_D : !_D \otimes ! \longrightarrow !_D$ correspond to the convolution product (see prop. 4.7) and is well defined (prop. 3.27).
- $c_D : !_D \longrightarrow ! \longrightarrow !_D$ corresponds to the construction of a Kernel of functions, and to the intuitions of 5.7 .
- $w_D : !_D \longrightarrow 1$ corresponds to the application of a distribution to $D(x \in \mathbb{R}^n \mapsto 1)$.

By equation 3 we have indeed the satisfaction of the diagrams of figure 7.

Definition 5.8. We interpret the dereliction $d_D : ! \longrightarrow !_D$ as $d_{D,E}(\phi \in \mathcal{E}'(\mathbb{R}^n)) \mapsto (E_D * \phi)$ and codereliction $\bar{d}_D : ! \longrightarrow !_D$ as $\bar{d}_{D,E}(\phi \in (D(\mathcal{E}(\mathbb{R}^n))) \mapsto (\phi \circ D) \in \mathcal{E}'(\mathbb{R}^n)$.

Then one has for every $\phi \in \mathcal{E}'(\mathbb{R}^n)$ and $f \in \mathcal{E}(\mathbb{R}^n)$:

$$\begin{aligned} d_{D,E} \circ \bar{d}_{D,E}(\phi)(f) &= E_D * (\phi(D(f)) \\ &= \phi(E_D * D(f)) \text{ by equation 1} \\ &= \phi(f) \text{ by definition 3.28} \end{aligned}$$

Defining d_D by restriction to $(D(\mathcal{E}(\mathbb{R}^n)))$, as we defined d as the restriction to E', would not guarantee the preceding equation. Let us notice that in the case $D = D_0$, we have $E_D = \delta_0$ and thus d_{D_0} is still the restriction to $(D(\mathcal{E}(\mathbb{R}^n))) \simeq (\mathbb{R}^n)'$. The preceding propositions conclude:

Theorem 5.9. *For any D LPDOcc, we have a polarized model of D-DiLL with* Eucl*,* Nf*,* Ndf*,* $!(\_) = \mathcal{E}'(\_)$ *and* $!_D(\_) = (D(\mathcal{D}(\_)))'$*.*

6 Conclusion

In this paper, we constructed a logical system D-DiLL accounting for the resolution of LPDEcc, generalizing DiLL. It opens several perspectives.

The generalisation to higher order is work in progress. We can easily introduce a version of D-DiLL with promotion and no separation between finitary and smooth formulas. Cut-elimination would be an adaptation of the cut-elimination for DiLL with promotion [Pagani 2009]. Models of it should come from smooth and classical models of Linear Logic with higher-order, as studied recently [Dabrowski and Kerjean 2017].

After that one should find a deterministic classical term-calculus, inspired by the differential $\lambda\mu$-calculus [Vaux 2007], accounting for D-DiLL. In a Curry-Howard-Lambek correspondence perspective, this would correspond to the Program/Proof bijection, while we studied here the Proof/Categories interpretation. Notice that it was necessary to work with $\mathcal{E}'(\mathbb{R}^n)$ when interpreting SDiLL, but other classes of distributions may suits for a model of D-DiLL.

Work in progress consists in generalising D-DiLL into a system englobing all LPDOcc. Promotion, contraction and co-contraction lead to a BLL-like syntax, in which we would like to give a syntactical counterpart to the construction of a fundamental solution. Generalized to all LPDOs, this could lead to a syntactical criterion for the resolution of LPDEs.

D-DiLL should also be generalised to account for the domain $\Omega \subset \mathbb{R}^n$ on which LPDEs are solvable: this should be done by introducing subtyping on finitary formulas, and could lead to a complete semantics over Nuclear spaces. The next goal after that should be to find a logical acount for all LPDEs. The long-term goal is of course to go towards non-linear PDEs.

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References

J.-M. Andreoli. 1992. Logic programming with focusing proofs in linear logic. (1992).

R. Blute, T. Ehrhard, and C. Tasson. 2012. A convenient differential category. *Cah. Topol. Géom. Différ. Catég.* (2012).

Y. Dabrowski and M. Kerjean. 2017. Models of Linear Logic based on the Schwartz epsilon product. (2017). arXiv:1712.07344

V. Danos, T. Heindel, I. Garnier, and J. G. Simonsen. 2017. Computing Continuous-Time Markov Chains as Transformers of Unbounded Observables. In *FOSSACS*.

Leon Ehrenpreis. 1954. Solution of some problems of division. I. Division by a polynomial of derivation. *Amer. J. Math.* 76 (1954), 883–903.

T. Ehrhard. 2002. On Köthe sequence spaces and linear logic. *Mathematical Structures in Computer Science* (2002).

T. Ehrhard. 2005. Finiteness spaces. *Mathematical Structures in Computer Science* 15, 4 (2005).

T. Ehrhard. 2017. An introduction to differential linear logic: proof-nets, models and antiderivatives. *Math.Struct. in Comp. Sci.* (2017).

T. Ehrhard, M. Pagani, and C. Tasson. 2018. Measurable Cones and Stable, Measurable Functions: A Model for Probabilistic Higher-Order Programming. POPL (2018).

T. Ehrhard and L. Regnier. 2003. The differential lambda-calculus. *Theoretical Computer Science* 309, 1-3 (2003).

M. Fiore. 2007. Differential structure in models of multiplicative biadditive intuitionistic linear logic. *TLCA* (2007).

R. Furber, D. Kozen, K. Larsen, R. Mardare, and P. Panangaden. 2017. Unrestricted stone duality for Markov processes. In *LICS 2017*.

J.-Y. Girard. 1999. Coherent Banach Spaces: A Continuous Denotational Semantics. *Theor. Comput. Sci.* (1999).

Jean-Yves Girard. 1987. Linear logic. *Theoret. Comput. Sci.* 50, 1 (1987).

A. Grothendieck. 1966. Produits tensoriels topologiques et espaces nucléaires. *Memoirs of the AMS* 16 (1966).

A. Grothendieck. 1973. *Topological vector spaces.* Gordon and Breach Science Publishers. Traducteur O. Chaljub.

C. Heunen, O. Kammar, S. Staton, and H. Yang. 2017. A convenient category for higher-order probability theory. In *LICS 2017*.

L. Hörmander. 1963. *Linear partial differential operators.* Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg.

L. Hörmander. 2003. *The analysis of linear partial differential operators. I.* Springer-Verlag, Berlin.

H. Jarchow. 1981. *Locally convex spaces.* B. G. Teubner, Stuttgart. Mathematische Leitfäden.

M. Kerjean. 2016. Weak topologies for Linear Logic. *Logical Methods in Computer Science* 12, 1 (2016).

M. Kerjean and C. Tasson. 2016. Mackey-complete spaces and power series. *MSCS* (2016).

A. Kriegl and P. W. Michor. 1997. *The convenient setting of global analysis.* AMS.

O. Laurent. 2002. *Etude de la polarisation en logique.* Thèse de Doctorat. Université Aix-Marseille II.

B. Malgrange. 1956. Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution. *Ann. Inst. Fourier, Grenoble* 6 (1956).

M. Pagani. 2009. The Cut-Elimination Thereom for Differential Nets with Boxes. In *TLCA 2009*.

H.H Schaefer. 1971. *Topological vector spaces.* Vol. GTM 3. Springer-Verlag.

L. Schwartz. 1957. Théorie des distributions à valeurs vectorielles. I. *Ann. Inst. Fourier, Grenoble* 7 (1957).

F. Trèves. 1967. *Topological vector spaces, distributions and kernels.* Academic Press, New York-London.

Lionel Vaux. 2007. The differential lambdamu-calculus. *Theor. Comput. Sci.* 379, 1-2 (2007), 166–209.

L. Vaux. 2017. Taylor Expansion, lambda-Reduction and Normalization. In *26th EACSL CSL 2017 (LIPIcs)*, Vol. 82.