A Logical Account for Linear Partial Differential Equations

Marie Kerjean
IRIF, Université Paris Diderot, France
kerjean@irif.fr

Abstract
Differential Linear Logic (DiLL), introduced by Ehrhard and Regnier, extends linear logic with a notion of linear approximation of proofs. While DiLL is classical logic, i.e., has an involutive negation, classical denotational models of it in which this notion of differentiation corresponds to the usual one, defined on any smooth function, were missing. We solve this issue by constructing a model of it based on nuclear topological vector spaces and distributions with compact support.

This interpretation sheds a new light on the rules of DiLL, as we are able to understand them as the computational principles for the resolution of Linear Partial Differential Equations. We thus introduce D-DiLL, a deterministic refinement of DiLL with a D-exponential, for which we exhibit a cut-elimination procedure, and a categorical semantics. When D is a Linear Partial Differential Operator with constant coefficients, then the D-exponential is interpreted as the space of generalised functions ψ solutions to Dψ = φ. The logical inference rules represents the computational steps for the construction of the solution φ. We recover linear logic and its differential extension DiLL as a particular case.

Keywords Differential Linear Logic, Linear Partial Differential Equations, Functional Analysis, Categorical semantics

1 Introduction
A Partial Differential Equation (PDE) is an equation Dg = f between functions f and g, where Dg is a possibly non-linear combination of partial derivatives of g, with smooth functions as coefficients. The study of PDEs through theoretical, numerical and computational methods is one of the most active areas of modern mathematics. Most research concentrates on non-linear equations such as Navier-Stokes equation. Programs are used to find approximate non-linear solutions, and applied mathematicians work at finding quick and efficient algorithms to do so.

Linear PDEs (LPDEs) are easier to solve theoretically, and when they have constant coefficients a universal method was found separately by Malgrange [25] and Ehrenpreis [5]. Examples of LPDEs with constant coefficients (LPDEcc) include fundamental examples such as the Laplacian equation or the heat equation:

\[ \sum_i \frac{\partial^2 q}{\partial x_i^2} = f \text{ or } \frac{\partial q}{\partial t} - a\left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2}\right) = f. \]

In this paper, we construct a proof syntax, with cut-elimination, with a denotational model in which formulas are interpreted as spaces of distributions and cut-elimination correspond to the resolution of LPDEs. This builds a new and strong bridge between Logic and Mathematical Physics, by extending the Proof/Function part of the Curry-Howard-Lambek correspondence to LPDEs. We understand this result as a first step towards a more general computational theory encompassing non-linear PDEs. On a more practical level, we believe D-DiLL could lead to a type system for the verification of numerical programs.

From linear to non-linear proofs and back. Linear Logic (LL) was introduced by Girard [14] as a proof-theory where a distinction is made between linear deductions of B under the hypothesis A, and non-linear ones. The former is represented by the sequent A ⊩ B, while the latter is represented by !A ⊩ B. The intuition is that a linear proof will make use of A exactly once: thus, !A is traditionally interpreted as a collection of all finite copies of A. The inference rules for the exponential connective ! of LL then represent a calculus of resources. Among these rules, the dereliction rule d allows to deduce !A ⊩ B from A ⊩ B: thus linear proofs can always be considered as non-linear ones.

DiLL was introduced by Ehrhard and Regnier [10], as a refinement of LL without its promotion rules but with dual exponential rules. It features in particular a codereletion rule d allowing to deduce from a sequent !A ⊩ B a linear approximation of it: A ⊩ B. This second sequent is considered as the differentiation of the first sequent. Both LL and DiLL are first presented under a classical form: sequents are monolateral ⊩ A⁻¹ B, and formulas are equivalent to their double linearization A⁻¹ A⁻¹. A sequent !A ⊩ B is then rewritten ⊩ ?A⁻¹, B, where ? is the “why not” modality.

Thus DiLL, and differential or quantitative λ-calculi, are traditionally understood as a logic and as calculi of approximations, as they account a syntactic variant of the Taylor Formula [31]. In this paper, we change this point of view and consider it as the basis for a calculus of Partial Differential Equations.

The equation solved by DiLL, and its generalisation. The fundamental idea behind this paper is that ψ of type A⁻¹⁻¹ is such that d(ψ) = φ, for φ of type !A.

This is true at the level of functions: a function g is linear, i.e., of type A⁻¹, if and only if there is a function g such that the differential at 0 of f corresponds to g. The previous statement extends this at the level of linear duals of spaces of functions, that is spaces of distributions.

We generalize this idea into a new connective !D, and a new codereletion rule dD:

ψ !D A is such that dD(ψ) = φ, for φ !A.

The fact that we work in a classical setting is central here, as is allows to understand d : A ⊩ !A as d : A²⁻¹⁻¹ ⊩ !A, and to generalize it as dD !D A ⊩ !A. DiLL thus corresponds to a special case where !D = Id.
We then construct a new sequent calculus D-DiLL which refines DiLL, and modelises the resolution of Linear Partial Differential Equations:

\[ \vdash \Gamma \vdash A \quad \vdash \Gamma \vdash \Delta \vdash A \]

The cut-elimination procedure of D-DiLL translates categorically into:

\[ \bar{d}_D(\bar{\epsilon}_D(\bar{w}_D, \phi)) = \phi \]

for \( \phi \vdash A \). In the syntax, this says that the solution \( \psi \) to the equation \( \bar{d}_D(\bar{\epsilon}_D(\bar{w}_D, \phi)) = \phi \) is exactly \( \bar{\epsilon}_D(\bar{w}_D, \phi) \). In the semantics of D-DiLL this is interpreted exactly as the resolution of a Linear Partial Differential Equation in the theory of distributions [19].

A classical and smooth semantics. This syntax for the resolution of LPDE comes from a semantical investigation for smooth and classical models of DiLL. Denotational semantics is the study of proofs and programs through their interpretation as denotations (functions) between spaces. In a denotational model of LL, there are spaces \( \mathcal{L}(E, F) \) of linear functions from \( E \) to \( F \), spaces \( \mathcal{C}(E, F) \) of non-linear ones, and a way to understand non-linear functions on \( E \) as linear functions on \( !E \); \( \mathcal{L}(E, F) \approx \mathcal{C}(E, F) \). In a model of DiLL, functions must also be smooth, that is able to be iteratively differentiated everywhere. We write \( \mathcal{C}^{\omega}(E, F) \) the space of all smooth functions between \( E \) and \( F \).

The first models of DiLL introduced by Ehrhard [6, 7] have a discrete basis: non-linear proofs are interpreted as power series between spaces of sequences. In order to get a better understanding of the differential nature of DiLL rules, one is bound to search for a denotational of model of DiLL where functions are interpreted as the smooth functions of differential geometry or functional analysis. But to account for linearity of functions, and for the classical setting of DiLL, one needs to interpret formulas as some topological vector spaces \( E \) which are reflexive: denoting \( E' = \mathcal{L}(E, \mathbb{R}) \), we need \( E \approx E'' \).

The requirements for relexivity to be preserved by the connectives of LL and the ones for having smooth functions work as opposite forces. More precisely:

- One needs a monoidal category of reflexive spaces, that is spaces which are isomorphic to their bidual and such that this property is preserved by tensor product and internal hom-sets. This is true for euclidean spaces, but fails in general when considering infinite dimensional spaces: it is false in particular for Banach spaces.
- One needs a cartesian closed category of smooth functions: we want \( \mathcal{C}^{\omega}(E \times F, G) \approx \mathcal{C}^{\omega}(E, \mathcal{C}^{\omega}(F, G)) \). These structures are notably scarce in analysis, but are fundamental in the semantics of LL as it accounts for the possibility to curryfy programs.

Solutions to the first point are for instance models based on spaces of sequences [6, 7], or topological vector spaces with very coarse topologies [21]. Solutions to the second point were constructed by Frölicher, Kriegl and Michor [23], leading to models of Intuitionistic DiLL [2, 22]. The attempt by Girard to interpret LL in Banach spaces fails [13], as the requirement of a norm on power series is too strong to allow a good cartesian closed category. We propose here a classical and smooth model of DiLL without promotion, while another one with a more intricate structure and interpreting promotion was recently exposed by Dabrowski and K. [3].

Computing with distributions. Distributions appears naturally in the quest for a model of LL. On the one hand, consider a model of DiLL made of \( \mathbb{E} \)-vector spaces, with spaces of linear functions \( \mathcal{L}(E, F) \), and spaces of smooth functions \( \mathcal{C}^{\omega}(E, F) \approx \mathcal{L}(E, F) \). Then as these spaces are reflexive we have necessarily:

\[ !E \approx \mathcal{C}^{\omega}(E, \mathbb{E}) \]

Thus \( !E \) is a space of linear forms acting on some space of smooth function, i.e. a space of distributions.

On the other hand, one of the major requirements in the categorical semantics of LL is the Seely’s isomorphism: \( !A !B = !(A \otimes B) \). It translates immediately into the Schwartz’s Kernel theorem [28], written here for distributions with compact support: \( \mathcal{C}^{\omega}(\mathbb{R}^n, \mathbb{R}) \otimes \mathcal{C}^{\omega}(\mathbb{R}^m, \mathbb{E}) \approx \mathcal{C}^{\omega}(\mathbb{R}^{n+m}, \mathbb{E}) \). Based on these intuitions, we find a classical semantics of DiLL in the theory of Nuclear spaces and distributions.

The language of distributions has been used for a while in Linear Logic, and this work should be seen as a way to ground this fact.

Nuclear spaces, Fréchet space, and distributions: a model of Smooth DiLL Typical examples of nuclear spaces are either euclidean spaces as \( \mathbb{R}^n \) or \( \mathbb{R}^m \), either spaces of (test) function \( \mathcal{E}(\mathbb{R}^n) = \mathcal{C}^{\omega}(\mathbb{R}^n), \mathcal{C}^{\omega}_{\text{ct}}(\mathbb{R}^n) \), or their duals, spaces of distributions \( \mathcal{E}(\mathbb{R}^n) = \mathcal{C}^{\omega}(\mathbb{R}^n)' \), \( \mathcal{D}(\mathbb{R}^n) = \mathcal{C}^{\omega}_{\text{ct}}(\mathbb{R}^n)' \). Moreover, a nuclear Fréchet space (that is a nuclear, complete and metrisable space) is reflexive, and while it is not preserved by duality, this condition is preserved by tensor product. We use the fact that Nuclear spaces which are Fréchet (i.e. complete and metrisable) form a negative interpretation for polarized MALL. When defining \( !\mathcal{E}(\mathbb{R}^n) = \mathcal{C}^{\omega}(\mathbb{R}^n)' \), the kernel theorem of distribution allows us to see ! as a monoidal functor from the category of Nuclear spaces to the category of duals of Nuclear Fréchet spaces; We translate this structure in the syntax (section 4) obtaining a polarized Smooth DiLL with a distinction between finitary and smooth formulas.

Modelizing D-DiLL by LPDEs Our definition of D-DILL is justified by the fact that for \( D \) any linear partial differential operator (LPDO) with constant coefficients, we have a model of D-DiLL.

\( !\mathcal{E}^n \) is then interpreted as the space of distribution with compact support \( \mathcal{E}(\mathbb{R}^n) \), \( D \) as a LPDO, and

\[ !D\mathcal{E}^n := (D(\mathcal{E}(\mathbb{R}^n)))' \]

Consider \( D_0 \) the operator mapping a function \( f \in \mathcal{C}^{\omega}(\mathbb{R}^n, \mathbb{R}) \) to its differential at 0, that is:

\[ D_0 := f \mapsto v \mapsto \lim_{h \to 0} \frac{f(hv) - f(0)}{h} \]

Then \( D_0(\mathcal{E}(\mathbb{R}^n)) = (\mathbb{R}^n)' \) and \( !D_0\mathbb{R}^n = (\mathbb{R}^n)' \approx \mathbb{R}^n \). The fact that we work in a classical setting, and thus with reflexive spaces, is central here, as is allows to understand the usual interpretation of \( d \) : \( v \in \mathbb{R}^n \mapsto f \mapsto D_0(f)(v) \) as operator matching \( \phi \in (\mathbb{R}^n)' \) to \( \phi \circ D_0 \in C^{\omega}(\mathbb{R}^n)' \), and to generalize it.
The coweakening \( \tilde{\psi}_D \) is then interpreted as the input of a fundamental solution \( \psi_D \), solution to \( \psi \circ D = \delta_0 \). We prove in particular that while \( \psi_D \) is not a distribution with compact support in general, it is an element of the interpretation of \( !D \). The co-contraction \( \tilde{\varepsilon}_D \) is interpreted by the convolution between a solution in \( !D \) and a distribution in \( \lambda \), producing another solution in \( !D \). Following the rules in the sequent calculus, we have, for every \( \psi \in \mathbb{R}^n \), for every \( f \in E(\mathbb{R}^n) \):

\[
\tilde{\varepsilon}_D(\psi_D, \phi)(D(f)) = \tilde{d}(\tilde{\psi}_D, \phi)(f) = \phi(f).
\]

That is, the solution \( \psi \) to \( D \psi = \phi \) is \( \varepsilon_D(\psi_D, f) \).

**Contributions**

- defines a Polarized Smooth variant of DiLL, without higher order, with a distinction between smooth and finitary formulas, and its categorical models.
- constructs a denotational model for it, based on the idempotent adjunction between Nuclear Fréchet and Nuclear DF spaces, and the construction of the exponential as a space of compact support distributions.
- defines a Polarized D-Differential Linear Logic, which refines Smooth DiLL with an indexed exponential \( !D \) whose rules represent the computation of a solution to a partial differential equation. We define a cut-elimination procedure for D-DiLL.
- shows that we have a model of Polarized D-DiLL for any LPDOcc.

**Related work**

There is a major research effort towards the understanding and the semantics of probabilistic programming [4, 12, 17]. Our work bears similarity with these, if only because we use the same language of distributions and kernels. More generally, this works takes place in a global understanding of continuous data-types and computations: machine-learning, which uses gradients to optimize the computations, is one example. The change of paradigm, allowing to go from a discrete point of-view on resource-sensitive programs to solutions of Differential Equations, relates to recent work on continuous probability distributions in probabilistic programming [9]. Notice however that models of probabilistic programming are not in general models of Differential Linear Logic.

**Organisation of the paper**

We first introduce in section 2 the rules, cut-elimination procedure and categorical semantics of DILL. Then in section 3 we give an overview of the functional analysis necessary to the paper. We barely recall any proofs, but show examples and precise references for our claims. Section 4 is quite short, as it formalizes syntactically and categorically the content of section 3 into the definition of Smooth DiLL. Section 5 defines D-DiLL, its syntax, rules, cut-elimination procedure and its categorical semantics. We also show in this section that for any \( \mathcal{D} \) LPDOcc, we have a model of D-DiLL.

---

1To avoid early confusion, we recall that for a distribution \( \psi \), \( D(\psi) \) is usually not defined as \( \psi \circ D \). See section 5.4.
There are two points of view: the first one is to refine the cut-elimination procedure follows the one of LL, and the comonad diagrams [11].

Figure 2. Cut-Elimination for the exponential group of DiLL

The cut-elimination procedure follows the one of LL for the MALL connectives, and the one for the exponential group are detailed in Ehrhard [8]. They follows the intuitions for the differentiation in euclidean spaces. We recall them semantically, through commutative diagrams in figure 2.

2.2 Categorical models of DiLL

There are two points of view: the first one is to refine the notion of Seely Model of Linear Logic with a biproduct and an interpretation for the codereliction [11], and the second one considers first models of DiLL without prom, and then extend this definition [8]. We adopt the first point of view, but make use of the numerous details and diagrams exposed by Ehrhard [8]. The following definitions are those of Fiore [11], sometimes adapted to the classical setting.

Definition 2.2. A biproduct on a category \( \mathcal{L} \) is a monoidal structure \( (\circ, I) \) together with natural transformations:

\[
\begin{align*}
I &\xrightarrow{1} \mathcal{L} \\
A &\xrightarrow{\Delta} A \circ A \\
A \circ A &\xrightarrow{\nabla} A \\
A &\xrightarrow{u} A
\end{align*}
\]

such that \((A, u, \nabla)\) is a commutative monoid and \((A, n, \Delta)\) is a commutative comonoid.

Definition 2.3. A \( \ast \)-autonomous category is a symmetric monoidal closed category \((\mathcal{L}, \otimes, I)\) with an object \( \perp \) giving an equivalence of categories \((\_ \otimes \_ )_{\mathcal{L}} : L^{op} \rightarrow L\) with the canonical map \(ev_E : E \rightarrow E''\) being a natural isomorphism.

Definition 2.4. A model of DiLL with promotion is consists of a symmetrical monoidal closed category \((\mathcal{L}, \otimes, I)\) with a \( \ast \)-autonomous structure, a biproduct structure \((\circ, \nabla)\), a co-monad \(! : \mathcal{L} \rightarrow \mathcal{L}\) which is strong monoidal from \((\mathcal{L}, \circ)\) to \((\mathcal{L}, \otimes)\), and a natural transformation \(\hat{d} : Id \rightarrow !\) satisfying strengh and comonad diagrams [11].

Remark. As shown by Fiore, from the biproduct structure follows the fact that the category \( \mathcal{L} \) is enriched over commutative monoids. This induces an additive law + on hom-sets, which is necessary to interpret the sums of proofs-trees of DiLL which stems from cut-elimination.

\[
f + g : E \xrightarrow{\lambda(f,g)} F \circ F \xrightarrow{\gamma} F.
\]

2.3 Interpreting DiLL in its categorical model.

We briefly recall how to interpret a sequent of DiLL as morphism in \( \mathcal{L} \), detailing only the action of exponential rules. The connectives \( \otimes, \forall, \oplus, \& \) are interpreted respectively by \( \otimes \) and its dual, and by the coproduct and product deduced from \( \circ \). We have \(! = 1\) by strong monoidality of !.

We write \( m_{E,F} : ! (E \circ F) \rightarrow ! E \circ ! F \) the isomorphism resulting from the monoidality of !, and \( d ! \rightarrow Id\) the co-unit of !. Then:

- from \( f : E \rightarrow F\) one construct \( f \circ d_E : !E \rightarrow F\) and from \( g : !E \rightarrow F\) one construct \( g \circ d_E : E \rightarrow F\);
- one construct \( w ! \rightarrow Id\) as \( \bar{w}_E \downarrow\bar{w} \downarrow 1 ! \) as \( \bar{w}_E \downarrow u\);
- one construct the natural transformation \( c : ! \rightarrow ! \) as \( c_A = m_{A,A}^E \circ m_{A,A}^F\) and \( c : ! \rightarrow ! \) as \( \bar{c}_A = \bar{\nabla}_A \circ m_{A,A}^{-1}\).

It should be clear then that in order to interpret the exponential rules of DiLL one requires the biproduct structure, the strong monoidality of ! and an interpretation for \( \hat{d} \) and \( d \). The co-monadic structure of ! is used only for the interpretation of the promotion rule, and enforces the definition of \( d \). We will make use of that statement in section 4 when we relax the co-monad requirement on !.

3 Topological vector spaces

In this section, we give technical accounts on some specific classes of topological vector spaces, on distribution theory and LPDOs. We refer mainly to the books by Jarchow [20] and Hörmander [19], as well as Grothendieck’ thesis [15]. We consider vector spaces on \( \mathbb{R} \).

Definition 3.1. A topological vector space (tvvs) is a vector space endowed with a topology, that is a covering class of open sets closed by infinite union and finite intersection, making the scalar multiplication and the addition continuous.

3.1 Locally convex tvvs.

A tvs is said to be Hausdorff if for any two distinct point \( x \) and \( y \) one can find two disjoint open sets containing \( x \) and \( y \) respectively. It is locally convex if every point is contained in a convex open set.

From now on we work with locally convex separated topological vector spaces and denote them by lc-tvvs. Examples of lc-tvvs includes all euclidean spaces \( \mathbb{R}^n \), normed spaces and metric spaces. For the rest of the section we consider \( E \) and \( F \) two lc-tvvs.

Notation. We will write \( E = F \) for the linear isomorphism between \( E \) and \( F \) as vector spaces, and \( E \simeq F \) for the linear homeomorphism between \( E \) and \( F \) as tvvs.

Definition 3.2. Consider \( U \subset E \) and \( x \in U \), then \( U \) is said to be a neighborhood of \( x \) if \( U \) contains an open set containing \( x \). A set \( B \subset E \) is bounded if for every \( U \) neighborhood of 0, there is \( \lambda \in \mathbb{R} \) such that \( B \subseteq \lambda U \).
Definition 3.3. For two lcvs $E$ and $F$ we consider $\mathcal{L}_b(E, F)$ the lcvs of all linear continuous functions between $E$ and $F$ and endow it with the topology of uniform convergence on bounded subsets of $E$. We write $E' = \mathcal{L}_b(E, \mathbb{R})$ for the dual of $E$.

Definition 3.4. A lcvs is reflexive if $E \simeq E''$ through the transpose of the evaluation map in $E'$:

$$\delta : \begin{cases} E & \to E'' \\ x & \mapsto \delta_x : (f \mapsto f(x)) \end{cases}$$

Typically, all euclidean spaces are reflexive, as they are isomorphic to their dual. This is also true for every Hilbert spaces, but as soon as we generalize to Banach spaces we encounter the famous counter example of $\ell_1$ and its dual $\ell_\infty$. The restriction to reflexive spaces is not preserved by tensor product nor linear hom-sets: typically, the space $L(H, H)$ is not reflexive when $H$ is a Hilbert space.\footnote{The author thanks Marc Bagnol for this clarifying example.}

Definition 3.5. Consider $E$ and $F$ two lcvs. The projective tensor product\footnote{Many topologies can be defined on the vector space resulting from the tensor product of two lcvs. The later restriction to Nuclear spaces will de facto identify all reasonable topological tensor product to the projective one.} $E \otimes_\pi F$ is the algebraic tensor product, endowed with the finest topology making the canonical bilinear map $E \times F \to E \otimes_\pi F$ continuous. Then $E \otimes_\pi F$ is a lcvs. The completion of $E \otimes_\pi F$ is called the completed projective tensor product and denoted $\hat{E} \otimes_\pi \hat{F}$.

3.1 (F)-spaces and (DF)-spaces

Definition 3.6. A Fréchet space, or $(F)$-space, is a complete and metrisable lcvs.

Recall that a lcvs is metrisable if and only if it admits a countable basis of 0-neighbourhoods. If $F$ is a metrisable space, we write $F$ its completion. Fréchet spaces are very common in analysis, but are not preserved by duality: the dual of a Fréchet space is not necessarily metrisable. In particular, the dual $\mathcal{C}_{c0}(\mathbb{R})'$ of the space of smooth scalar functions, as described in section 3.2, is not metrisable.

Definition 3.7. A $(DF)$-space is a lcvs $E$ admitting a countable basis of bounded sets $\mathcal{A} = (A_n)_n$, such that if $(U_n)_{n}$ is a sequence of closed and disked neighbourhoods of 0 whose intersection $U$ is bornorviable (i.e. absorbs all bounded subsets), then $U$ is a neighbourhood of 0.

Let us note that, by duality, the second condition is equivalent to asking every bounded subset $B$ of the strong dual $E'$ which is the union of a sequence of equicontinuous subsets to be equicontinuous. Moreover, it is costless to ask that for every $n A_n$ is be absolutely convex and $A_n + A_n \subset A_{n+1}$. We will therefore always suppose that this is the case. Although this definition may seem obscure, it is the right one for interpreting the dual and pre-dual of $(F)$-spaces.

Proposition 3.8 ([16] IV.3.1). • If $F$ is metrisable, then its strong dual $E'$ is a $(DF)$-space.

• If $E$ is a $(DF)$-space and $F$ and $(F)$-space, then $\mathcal{L}_b(E, F)$ is an $(F)$-space. In particular, $F'$ is an $(F)$-space.\footnote{A basis $\mathcal{A}$ being a collection of bounded set such that every bounded set in included into an object of $\mathcal{A}$}

Proposition 3.9 ([20] 12.4.2 and 15.6.2). The class of $(DF)$-spaces is preserved by countable inductive limits, countable direct sums, quotient and completions. The class of $(F)$-spaces is stable with the construction of products and completed projective tensor products $\hat{\otimes}_\pi$.

The following reflects the syntax of an intuitionist version of Smooth DiLL of section 4.

Example 3.10 ([20] 12.4.4). A space which is Fréchet and $(DF)$ is necessarily finite dimensional.

3.2 Distributions with compact support

We refer to the exposition of distributions by Hörmander [19] for proofs and details.

Definition 3.11. Consider $n \in \mathbb{N}$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$. The function $f$ is said to be smooth if it is differentiable at every point $x \in \mathbb{R}^n$, and if at every point its differential is smooth.

The theory of distribution is traditionally introduced by considering the space $\mathcal{D}(\mathbb{R}^n) := \mathcal{C}_{c0}^{\infty}(\mathbb{R}^n)$ of test functions, i.e. the space of scalars smooth functions on $\mathbb{R}^n$ with compact support, and define distributions as elements of its dual $\mathcal{D}(\mathbb{R}^n)'$. But because the derivelion rule d makes us consider linear continuous function as a particular case of smooth function, we work with the following:

Definition 3.12. We consider $\mathcal{E}(\mathbb{R}^n) := \mathcal{C}_{c0}^{\infty}(\mathbb{R}^n, \mathbb{R})$ the space of all scalar smooth functions on $\mathbb{R}^n$, endowed with the usual topology of uniform convergence of all differentials of order $\leq k$ on all compact subsets of $\mathbb{R}^n$, for all $k \in \mathbb{N}$. Its dual is called the space of distributions with compact support and denoted $\mathcal{E}'(\mathbb{R}^n)$.

Proposition 3.13. For any $n \in \mathbb{N}$, $\mathcal{E}(\mathbb{R}^n)$ is an $(F)$-space and $\mathcal{E}'(\mathbb{R}^n)$ is a complete $(DF)$-space.

Example 3.14. A distribution must be considered as a generalized function, and acts as such. The key idea is that, if $f \in \mathcal{C}_{c0}^{\infty}(\mathbb{R}^n)$ then on defines a compact distribution by $g \in \mathcal{C}_{c0}^{\infty}(\mathbb{R}^n) \mapsto \int f(x) g(x) dx$.

Typical examples of distributions which do not follow this pattern are the dirac distributions. For $x \in \mathbb{R}^n$ one defines the dirac at $x$ as: $\delta_x : f \in \mathcal{E}(\mathbb{R}^n) \mapsto f(x)$.

Definition 3.15. Consider $\phi \in \mathcal{E}'(\mathbb{R}^n)$ and $f \in \mathcal{E}(\mathbb{R}^n)$. Then one defines the convolution between a distribution and a functions as $\phi * f \in \mathcal{E}(\mathbb{R}^n)$ as: $\phi * f : x \mapsto \phi(y) \mapsto f(x-y)$.

This definition is extended to a convolution product between distributions. Consider $\psi \in \mathcal{E}'(\mathbb{R}^n)$. Then $\phi * \psi$ is the unique distribution in $\mathcal{E}'(\mathbb{R}^n)$ such that:

$$\forall f \in \mathcal{E}(\mathbb{R}^n), (\phi * \psi) * (f) = \phi * (\psi * f). \tag{1}$$

Although the above is not a symmetric definition, one proves easily that the convolution is commutative and associative [19].

Example 3.16. Note that $\delta_0$ defined in 3.14 acts as neutral element for the convolution law.

The central theorem of the theory of distributions is the Kernel Theorem:
Theorem 3.17 ([29] 51.6). For any $n, m \in \mathbb{N}$ we have:
\[
\mathbb{E}'(\mathbb{R}^{n+m}) \simeq \mathbb{E}'(\mathbb{R}^n) \otimes \mathbb{E}'(\mathbb{R}^m) \simeq \mathbb{L}(\mathbb{E}'(\mathbb{R}^m), \mathbb{E}(\mathbb{R}^n))
\]
This theorem is proved on the spaces of functions by showing the density of smooth functions of the kind $f \otimes g$, $f \in \mathbb{E}(\mathbb{R}^n)$, $g \in \mathbb{E}(\mathbb{R}^m)$, and then that the topology induced by $\mathbb{E}(\mathbb{R}^{n+m})$ on $\mathbb{E}(\mathbb{R}^n) \otimes \mathbb{E}(\mathbb{R}^m)$ is indeed the projective topology of the tensor product. This, and particularly the fact that $\mathbb{E} = \mathbb{E}_\pi$, is justified by the theory of Nuclear spaces, which is recalled below.

3.3 Nuclear spaces

The theory of nuclear spaces will allow us to interpret the idempotent negation of DiLL, and as the same time the theory of exponentials as distributions

Definition 3.18. An linear map $f$ between a lctvs $E$ and a Banach $X$ is said to be nuclear if there is an equicontinuous sequence $(a_n)$ in $E'$, a bounded sequence $(y_n)$ in $X$, and a sequence $(\lambda_n) \in l_1$ such that for all $x \in E$:
\[
f(x) = \sum_n \lambda_n a_n(x) y_n.
\]

Definition 3.19. Consider $E$ a lctvs. We say that $E$ is nuclear every continuous linear map of $E$ into any Banach space is nuclear.

Proposition 3.20 ([20] 21.2.3). The class of nuclear spaces is closed with respect to the formation of completion, cartesian products, countable direct sums, projective tensor products, subspaces and quotients.

An important property of nuclear spaces is that as soon as they are normed, they are finite dimensional. In other word, if a Hilbert or Banach or simply normed space is nuclear, then it is isomorphic to $\mathbb{R}^n$ for a certain $n$.

Example 3.21. Typical examples of nuclear spaces are Euclidean spaces $\mathbb{R}^n$, spaces of smooth functions $C_c^\infty(\mathbb{R}^n, \mathbb{R})$, $C^\infty(\mathbb{R}^n, \mathbb{R})$ and their duals $D'(\mathbb{R}^n)$ and $E'(\mathbb{R}^n)$.

Theorem 3.22. An $(F)$-space $F$ which is also nuclear is reflexive. As a consequence, $\mathbb{E}(\mathbb{R}^n)$ and $E'(\mathbb{R}^n)$ are reflexive.

Proof. We give a brief proof for the reader familiar with functional analysis. It is enough to prove that $F$ is semi-reflexive, that is that $F = F''$, as the equality between the topologies will follow from the metrisability of $F$. Indeed, when $F$ is metrisable $E$-equicontinuous sets and $E$-weakly bounded sets corresponds in $E'$ [20, 8.5.1]. Now we have that every bounded set of a nuclear space is precompact [27, III.7.2.2]. Thus as $F$ is nuclear and complete, its bounded sets are compact, and $F$ is endowed with the Arens-topology of uniform convergence on absolutely convex compact subsets of $F$. By the Mackey-Arens theorem, this makes $F$ semi-reflexive.

Proposition 3.23. • Consider $E$ a lctvs which is either an $(F)$-space or a $(DF)$-space. Then $E$ is nuclear if and only if $E'$ is nuclear [15, Chap II, 2.1, Thm 7].

• If $E$ is a complete $(DF)$-space and if $F$ is nuclear, then $L_\beta(E, F)$ is nuclear. If moreover $F$ is an $(F)$-space or a $(DF)$-space, then $L_\beta(E, F)$ is nuclear [15, Chapter II, 2.2, Thm 9, Cor. 3]. As a corollary, the dual of a nuclear $(DF)$-space is a nuclear $(F)$-space.

Proposition 3.24 ([15] Chapter II, 2.2, Thm 9). If $E$ and $F$ are both nuclear $(DF)$-spaces, then so is $E \otimes \pi F$.

A central result of the theory of nuclear spaces, explaining for the form of the Kernel theorem 3.17, is the following proposition. It is proved by applying the hypothesis that $E$ is reflexive and thus $E'$ is complete and barreled, and thus applying the hypothesis of Treves' book [29].

Proposition 3.25 ([29] prop. 50.5). Consider $E$ a Fréchet nuclear space, and $F$ a complete space. Then $E \otimes \pi F \simeq \mathbb{L}(E', F)$.

3.4 Linear Partial Operators

We recall the very first steps in the theory of LPDEs\footnote{We are not considering in this paper border conditions, regularity of the solutions to equations with non-constant coefficients, nor modern research subjects in the theory non-linear equations.}. For $a = (a_1, ..., a_n) \in \mathbb{N}^n$ we write $\delta a$ the linear continuous map:
\[
f \in C^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto x \in \mathbb{R}^n \mapsto \frac{\partial^{|a|} f}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}(x)
\]

Definition 3.26. Consider, for $a \in \mathbb{N}^n$ smooth functions $a_\alpha$ of $C^\infty(\mathbb{R}^n, \mathbb{R})$. Then a Linear Partial Differential Operator (LPDO) is defined as an operator $D: C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$:
\[
D = \sum_{a \in \mathbb{N}^n} a_\alpha \delta a.
\]

A LPDO with constant coefficients is a LPDO $D$ such that the $a_\alpha$ are constants.

The definition of $D$ is extended to distributions as follows:
\[
D(\phi) = f \mapsto \sum_{a} (-1)^{|a|} a_\alpha \delta a(\phi f)),
\]

so that for $g \in C^\infty(\mathbb{R}^n): D(f) \mapsto \int f g = f \mapsto \int f D(g)$.

The weak resolution of the LPDO consists then, when $\phi \in E'(\mathbb{R}^n)$, of finding $\psi$ such that, for all $f \in E(\mathbb{R}^n)$:
\[
\psi = D(f) = \phi(f).
\]

The resolution of LPDOs with constant is always possible, and particularly elegant, due to the behaviour of convolution with respect to partial differentiation:

Proposition 3.27 ([19] 4.2.5). Consider $f \in C^\infty_c(\mathbb{R}^n)$ and $\phi \in E'(\mathbb{R}^n)$. Then $\delta a \phi \ast f = f \ast (\delta \phi f)$.

Definition 3.28. A fundamental solution to equation (2) consists of a distribution $E_D \in C^\infty_c(\mathbb{R}^n)$ such that $D(E_D) = \delta_0$.

Example 3.29. Because of linear partial differential operator, we are working with distributions whose support is not necessarily compact. Indeed, the existence of a fundamental solution is not ensured when distributions must apply to any smooth function. The typical example is $D = f \in C^\infty(\mathbb{R}, \mathbb{R}) \mapsto f'$. If $f$ has compact support, one can define:
\[
E_D: f' \mapsto \int_{-\infty}^{0} f'
\]

and one has indeed $E_D (D(f)) = f(0)$. This however is not possible in generality when $f$ has no compact support.
It appears then thanks to the linearity of the convolution product and propositions 3.16 and 3.27 that:
\[ \forall \phi \in \mathcal{E}'(\mathbb{R}^n), D(\hat{E}_D \circ \phi) = \phi. \]

Again, we make a slightly different use of the fundamental solution by defining \( E_D \) such that equation 3 holds.

**Theorem 3.30** (Malgrange-Ehrenpreis [18] 3.1.1). Every LPDE admits a fundamental solution \( \hat{E}_D \), which leads to \( E_D = \hat{E}_D + b_0 \in (D(C^\infty_c(\mathbb{R}^n)))' \).

The theorem is in fact much more precise as we have information about the local growth of \( E_D \). We do not have in general that \( E_D \in \mathcal{E}'(\mathbb{R}^n) \).

However, the first and easy step of the proof consists in noticing that, if one defines \( E_D \) as the distribution \( f \mapsto \hat{E}_D(x \mapsto f(-x)) \), that is \( E_D(f) = \hat{E}_D * f(0) \), we have:
\[ \forall f \in C^\infty_c(\mathbb{R}^n), E_D(D(f)) = f(0). \]

The proof consists afterwards into the majorization of \( \hat{E}_D \) in order to extend if to \( C^\infty_c(\mathbb{R}^n) \). This is one of the arguments for the introduction of \( \mathcal{E}'(\mathbb{R}^n) \).

**Proposition 3.31.** The fundamental solution \( \hat{E}_D \) defined well and continuous on \( D(C^\infty_c(\mathbb{R}^n)) \), as it corresponds to \( D(f) \mapsto f(0) \). We have thus \( \hat{E}_D \in \mathcal{E}'(\mathbb{R}^n) \).

**4 Smooth Differential Linear Logic and its models**

In this section we introduce a Smooth Differential Linear Logic for which Nuclear spaces and distributions form a classical and smooth model. We notably show that the categorical interpretations for \( \oplus \) and \( \otimes \) correspond to the convolution and the dirac in 0 in the theory of distributions.

Let us recall the notion of polarisation in LL. In polarized linear logic [24] a distinction is made between positive connectives \( , \), \( \otimes \), \( \oplus \) whose introduction rules are non-reversible, and negative connectives \( !, \top, \& \) whose introduction rule is reversible. Negation then changes the polarity of a formula. This plays a fundamental role in proof-search [1].

**4.1 The category of Nuclear Fréchet spaces.**

Nuclear Fréchet spaces gather all the stability properties to be a (polarized) model of LL, except that we do not have an interpretation for higher-order smooth functions. Indeed if \( \mathbb{R}^n \) is interpreted as \( \mathcal{E}'(\mathbb{R}^n) \), we do not have a straightforward definition of \( \mathbb{R}^n \).

**Definition 4.1.** We write \( \text{NF} \) the category of Nuclear (F)-spaces and continuous linear maps, \( \text{NDF} \) the category of complete Nuclear (DF)-spaces and continuous linear maps, and \( \text{Eucl} \) the subcategory of both formed of euclidean spaces.

**Proposition 4.2.**
- \( \text{Eucl} \) is a model of MALL.
- \( \text{NF} \) forms a model for the negative interpretation of polarized MALL [24, 6.20], where positive formulas are thus interpreted as objects of \( \text{Ndf} \).

\footnote{Let us point out that even if \( D_b \) is not a LPDO, the equation \( D_b \phi = f \) behaves likewise. If there is of a solution to this equation it means that \( f \) is linear, and then \( D_b f = D_b(f * b_0) = f \).}
of Polarized MALL, that is an involutive , defined as an
adjunction between a category of negative smooth formulas
and a category of positive smooth formulas. Sequents would
then be interpreted as maps in the larger category of complete
levts.

Definition 4.5. A categorical model of SDiLL consists into a
model of MALL with biproduct Fin_, and a model of MALL
SmoothDiLL, such that we have a strong monoidal functor ! : 
(Fin_, ×) → (SmoothDiLL, ⊗), a forgetful functor U : Eucl → Nucl
strong monoidal in ⊗, ֒, & and two natural transformation 
\( d : ! → U \) and \( \bar{d} : ! → U \) such that \( d \circ \bar{d} = Id_{Eucl} \).

Theorem 4.6. The structure on Nuclear Spaces and Distribu-
tions defines a model of SmoothDiLL.

Proof. We interpret finitary formulas \( A \) as euclidean spaces.
Without any ambiguity, we denote also by \( A \) the interpretation
of a finitary formula into euclidean spaces. The exponential
is interpreted as \( !A = E^\prime(A) \), extended by precomposition
on functions. We briefly explain the interpretation for the rules,
which follow the intuition of [10]. We define:

\[
d : \frac{A}{!A} \quad \bar{d} : \frac{A'' \rightarrow !A}{ev_x \rightarrow (f \mapsto ev_x(D_0(f)))}
\]

This is justified by the definition of reflexivity 3.4. Then we
have indeed: \( d \circ \bar{d} = Id_{A''} \). The interpretation of \( w, c, \bar{w}, \bar{c} \)
follows from the biproduct structure on Eucl and from the
monoidality of \( ! \), as explained in 2.3.

We show that \( \bar{w}, \bar{c} \) they have a direct interpretation
which follows the intuitions of [2, 10].

Proposition 4.7. The cocontraction and cowakening defined
through the kernel theorem correspond to the convolution of
distributions and the introduction of \( \delta_0 \).

\[
\begin{align*}
\bar{c} : & \frac{!A \otimes !A}{\bar{c}} \rightarrow !A \\
\bar{w} : & \frac{\mathbb{R} \rightarrow !A}{1 \mapsto \delta_0 : (f \mapsto E(A) \mapsto f(0))}
\end{align*}
\]

Proof. As defined in section 2, \( \bar{w}_A = ![u : \{0\} \rightarrow A] \)
corresponds to \( w_A(1) = (f \in E(A) \mapsto f \circ u = f(0)) \), thus \( \bar{w} = \delta_0 \).
During the rest of the proof we use Fourier transforms and
temperates distributions, as exposed by Hornermander [19, 7.1].
The co-contraction is defined categorically as \( \bar{c} = !\forall \circ m_{!A}^A \).
In the categorical setting, addition in hom sets is
defined through the biproduct. But here the reasoning is
done backward. We know that \( \forall = \times \) is a biproduct thanks to
\( \forall : A \times A \rightarrow A(x, y) \mapsto x + y \), and thus \( !\forall : \phi \mapsto ![A \times A] \mapsto (f \in E(A) \mapsto (f \circ \psi(x, y) \mapsto f(x + y)). \)
Moreover if \( f \in E(A \times A) \) is the sequential limit of \( (f_n \otimes g_n)_n \in (E(A) \otimes E(A))^{\forall} \) (see theorem
3.17) \( m_{!A}^A(\phi \circ \psi)(f) = \lim_n(\phi(f_n) \psi(g_n)). \)
If we write by \( \hat{\psi} \) the Fourier transformation of a distribution,
we have that of \( \bar{c}(\hat{\phi}, \hat{\psi}) = \hat{\phi} \hat{\psi} \). From the details above we
deduce
\( \bar{c}(\hat{\phi}, \hat{\psi})(f) = \hat{\phi}(f) \hat{\psi}(f) \). As distributions with compact support are temperates, we
can apply the inverse Fourier transformation and thus \( \bar{c} \)
corresponds to the convolution.

\[ \]
We interpret finitary formulas $\llbracket \text{Proposition 5.2.} \rrbracket$
The rules $\bar{w}_D$, $\bar{c}_D$, $w_D$ and $c_D$ are admissible in SDiLL, when $\forall \bar{A}$ is equivalent to $A$.

Theorem 5.3. When $\bar{!}_D A \equiv A$, the proof-trees of SDiLL are sums of proof-trees of D-DiLL.

5.3 Categorical models of D-DiLL

Definition 5.4. A categorical model of D-DiLL consists in a model of MALL with biproduct Eucl, and a (polarized) model of MALL Nucl, with a strong monoidal functor $! : (\text{Eucl}, x) \rightarrow (\text{Nucl}, 0)$, a functor $! : \text{Eucl} \rightarrow \text{Nucl}$, and two natural transformations $d_D : ! \rightarrow ! D$ and $\bar{d}_D : ! \rightarrow ! D$ such that $! d_D \circ d_D = id_{\text{Eucl}}$.

Indeed, one defines the interpretations of $c_D$, $w_D$, $\bar{c}_D$, $w_D$ through the strong monoidality of $!$, the biproduct structure and $d_D$ and $\bar{d}_D$ as it is done in the proof of proposition 5.1 and in paragraph 2.3.

The cut-elimination rules of figure 7 are then easily verified. For example, we have indeed:

$\bar{c}_D(\bar{w}_D, \bar{\phi}) = d_D \circ \bar{c}_D \circ \bar{w}_D, \bar{\phi} = (d_D \circ \bar{c}_D \circ \bar{\phi}).$

5.4 A LPDE interpreted in the syntax

We show that the categories Eucl, Ndf and Nf defined in section 3.3, together with distributions of compact support and a LPDOcc $D$, form a model of D-DiLL. In this section we interpret $\mathbb{R}^n$ by the space of distributions, and not distributions with compact support.

Consider $D : \mathcal{E}(\mathbb{R}^n) \rightarrow \mathcal{E}(\mathbb{R}^n)$ a LPDOcc:

$D(f)(x) = \sum a_\alpha \partial^\alpha f(x).$

We interpret finitary formulas $A, B$ as euclidean spaces. One has indeed $1 \circ 0 = \mathbb{R}$ and $\top \circ \bot = \{0\}$. The connectives of LL are interpreted in Eucl. Nf, Ndf as in section 4.

Definition 5.5. For $A$ a finitary formula interpreted by $\mathbb{R}^n \in \text{Eucl}$, we interpret $! A$ on its dual as:

$!_{D, E} : \mathcal{E}(\mathbb{R}^n) \rightarrow \mathcal{E}(\mathbb{R}^n)$. \\

Proposition 5.6. We have that $!_{D} \mathbb{R}^n \in \text{Nf}$ and $!_{D} \mathbb{R}^n \in \text{Ndf}$.

Proof. $!_{D} \mathbb{R}^n$ is a closed subset of $\mathcal{E}(\mathbb{R}^n)$. As such, it is a nuclear $(\Gamma)$-space, see 3.20 and 3.9. Thus $!_{D} \mathbb{R}^n \in \text{Nf}$ and $!_{D} \mathbb{R}^n \in \text{Ndf}$.

From the previous proposition and proposition 3.22 it follows that $(!_{D} \mathbb{R}^n) = !_{D} \mathbb{R}^n$.

Theorem 5.7. We extend $!_{D}$ on linear maps by precomposition by $D$, and thus define a functor $!_{D} : \text{Eucl} \rightarrow \text{Nf}$. Then we have natural isomorphisms

$m_{D, A, B} : !_{D}(\mathbb{R}^{n+m}) \equiv !_{D} \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^{m}$.

Proof. This theorem encodes in particular a well used convention in LPDOs [29, chap. 52], which allows to extend $D$ defined on $\mathcal{E}(\mathbb{R}^n)$ to $\mathcal{E}(\mathbb{R}^{n+m})$. One differentiates the $n$-th variable apply to functions defined on $\mathbb{R}^{n+m}$. Our theorem is then directly deduced from the Kernel theorem 3.17.

The interpretation of $w_D$, $\bar{w}_D$, $c_D$ and $\bar{c}_D$ follows from the previous proposition and the biproduct structure:

$!_{D} : 1 \rightarrow !_{D}$ such that $!_{D} \mathbb{R}(1) = \mathbb{R}^n$. It is well defined thanks to proposition 3.31.

$c_D : !_{D} \otimes !_{D} \rightarrow !_{D}$ correspond to the convolution product (see prop. 4.7) and is well defined (prop. 3.27).

$c_D : !_{D} \rightarrow !_{D}$ corresponds to the construction of a Kernel of functions, and to the intuitions of 5.7.

$w_D : !_{D} \rightarrow 1$ corresponds to the application of a distribution to $D(x \in \mathbb{R}^n \rightarrow 1)$.

By equation 3 we have indeed the satisfaction of the diagrams of figure 7.

Definition 5.8. We interpret the dereliction $d_D : ! \rightarrow !_{D}$ as $d_{D,E}(\phi \in \mathcal{E}(\mathbb{R}^n)) \mapsto (E_{\phi} \star \phi)$ and codereliction $\bar{d}_D : ! \rightarrow !_{D}$ as $d_{D,E}(\phi \in (D(\mathcal{E}(\mathbb{R}^n)))) \mapsto (\phi \circ D) \in \mathcal{E}(\mathbb{R}^n)$.

Then one has for every $\phi \in \mathcal{E}(\mathbb{R}^n)$ and $f \in \mathcal{E}(\mathbb{R}^n)$:

$d_{D,E} \circ d_{D,E}(\phi)(f) = E_{\phi} \star (\phi(f))$

$= \phi(E_{\phi} \star f)$ by equation 1

$= \phi(f)$ by definition 3.28

Defining $d_D$ by restriction to $(D(\mathcal{E}(\mathbb{R}^n)))$, as we defined $d$ as the restriction to $E'$, would not guarantee the preceding equation. Let us notice that in the case $D = \delta_0$, we have $E_{\phi} \equiv 0$ and thus $d_{\delta_0}$ is still the restriction to $(D(\mathcal{E}(\mathbb{R}^n))) \cong (\mathbb{R}^n)'$. The preceding propositions conclude:

Theorem 5.9. For any $D$ LPDOcc, we have a polarized model of D-DiLL with Eucl, Nf, Ndf, $!_{\bot} = \mathcal{E}(\bot)$ and $!_{D, \bot} = (D(D(\bot)))'$.  

6 Conclusion

In this paper, we constructed a logical system D-DiLL accounting for the resolution of LPDEcc, generalizing DiLL. It opens several perspectives.

The generalisation to higher order is work in progress. We can easily introduce a version of D-DiLL with promotion and
no separation between finitary and smooth formulas. Cut-
elimination would be an adaptation of the cut-elimination
for DiLL with promotion [26]. Models of it should come from
smooth and classical models of Linear Logic with higher-order,
as studied recently [3].

After that one should find a deterministic classical term-
calculus, inspired by the differential λp-calculus [30], ac-
counting for D-DiLL. In a Curry-Howard-Lambek correspondence
perspective, this would correspond to the Program/Proof
bijection, while we studied here the Proof/Categories inter-
pretation. Notice that it was necessary to work with $E'(\mathbb{R}^n)$
when interpreting SDiLL, but other classes of distributions
may suit for a model of D-DiLL.

Work in progress consists in generalising D-DiLL into a sys-
tem englobing all LPDOcc. Promotion, contraction and co-
contraction lead to a BLL-like syntax, in which we would
like to give a syntactical counterpart to the construction of a
fundamental solution. Generalized to all LPDOs, this could
lead to a syntactical criterion for the resolution of LPDEs.
D-DiLL should also be generalised to account for the domain
$\Omega \subset \mathbb{R}^n$ on which LPDEs are solvable: this should be done by
introducing subtyping on finitary formulas, and could lead to
a complete semantics over Nuclear spaces. The next goal after
that should be to find a logical account for all LPDEs.
The long-term goal is of course to go towards non-linear PDEs.

Acknowledgments
The author would like to thanks Y. Dabrowski, T. Ehrhard
and T. Hirshowitz for discussions about this work. The author
was supported by the ANR Project RAPIDO, ANR-14-CE25-
0007.

References
Computing Continuous-Time Markov Chains as Transformers of
Unbounded Observables. In FOSSACS.
I. Division by a polynomial of derivation. Amer. J. Math. 76
proof-nets, models and antiderivatives. Math. Struct. in Comp.
and Stable, Measurable Functions: A Model for Probabilistic
biadditive intuitionistic linear logic. TLCA (2007).
2017. Unrestricted stone duality for Markov processes. In LICS
2017.