A Logical Account for Linear Partial Differential Equations

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Abstract

Differential Linear Logic (DiLL), introduced by Ehrhard and Regnier, extends linear logic with a notion of linear approximation of proofs. While DiLL is classical logic, i.e. has an involutive negation, classical denotational models of it in which this notion of differentiation corresponds to the usual one, defined on any smooth function, were missing. We solve this issue by constructing a model of it based on nuclear topological vector spaces and distributions with compact support. This interpretation sheds a new light on the rules of DiLL, or to allow others to do so, for Government purposes only.

This interpretation sheds a new light on the rules of DiLL, as we are able to understand them as the computational principles for the resolution of Linear Partial Differential Equations. We thus introduce D-DiLL, a deterministic refinement of DiLL with a D-exponential, for which we exhibit a cut-elimination procedure, and a categorical semantics. When D is a Linear Partial Differential Operator with constant coefficients, then the D-exponential is interpreted as the space of generalised functions ψ solutions to Dψ = φ. The logical inference rules represent the computational steps for the construction of the solution φ. We recover linear logic and its differential extension DiLL as a particular case.

Keywords
Differential Linear Logic, Linear Partial Differential Equations, Functional Analysis, Categorical semantics

1 Introduction

A Partial Differential Equation (PDE) is an equation Dg = f between functions f and g, where Dg is a possibly non-linear combination of partial derivatives of g, with smooth functions as coefficients. The study of PDEs through theoretical, numerical and computational methods is one of the most active areas of modern mathematics. Most research concentrates on non-linear equations such as Navier-Stokes equation. Approximations are used to find approximate non-linear solutions, and applied mathematicians work at finding quick and efficient algorithms to do so.

Linear PDEs (LPDEs) are easier to solve theoretically, and when they have constant coefficients a universal method was found separately by Malgrange [25] and Ehrenpreis [5]. Examples of LPDEs with constant coefficients (LPDEcc) include fundamental examples such as the Laplacian equation or the heat equation:

\[ \sum_i \frac{\partial^2 g}{\partial x_i^2} = f \text{ or } \frac{\partial g}{\partial t} - \alpha \left( \frac{\partial^2 g}{\partial x_1^2} + \frac{\partial^2 g}{\partial x_2^2} + \frac{\partial^2 g}{\partial x_3^2} \right) = f. \]

In this paper, we construct a proof syntax, with cut-elimination, with a denotational model in which formulas are interpreted as spaces of distributions and cut-elimination correspond to the resolution of LPDEs. This builds a new and strong bridge between Logic and Mathematical Physics, by extending the Proof/Function part of the Curry-Howard-Lambek correspondence to LPDEs. We understand this result as a first step towards a more general computational theory encompassing non-linear PDEs. On a more practical level, we believe D-DiLL could lead to a type system for the verification of numerical programs.

From linear to non-linear proofs and back. Linear Logic (LL) was introduced by Girard [14] as a proof-theory where a distinction is made between linear deductions of B under the hypothesis A, and non-linear ones. The former is represented by the sequent A ⊢ ⊥ and the latter is represented by |A ⊢ B. The intuition is that a linear proof will make use of A exactly once: thus, |A is traditionally interpreted as a collection of all finite copies of A. The inference rules for the exponential connective ! of LL then represent a calculus of resources. Among these rules, the dereliction rule d allows to deduce |A ⊢ B from A ⊢ B: thus linear proofs can always be considered as non-linear ones.

DiLL was introduced by Ehrhard and Regnier [10], as a refinement of LL without its promotion rules but with dual exponential rules. It features in particular a codereliction rule d allowing to deduce from a sequent !A ⊢ B a linear approximation of it: A ⊢ B. This second sequent is considered as the differentiation of the first sequent. Both LL and DiLL are first presented under a classical form: sequents are monolateral ⊢ B, while the latter is represented by !A ⊢ B. The intuition is that a linear proof will make use of A exactly once: thus, !A is traditionally interpreted as a collection of all finite copies of A. The inference rules for the exponential connective ! of LL then represent a calculus of resources. Among these rules, the dereliction rule d allows to deduce |A ⊢ B from A ⊢ B: thus linear proofs can always be considered as non-linear ones.

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The equation solved by DiLL, and its generalisation. The fundamental idea behind this paper is that ψ of type A⊥⊥ is such that d(ψ) = φ, for φ of type !A.

This is true at the level of functions: a function g is linear, i.e. of type A⊥, if and only if there is f :?A⊥ such that the differential at 0 of f corresponds to g. The previous statement extends this at the level of linear duals of spaces of functions, that is spaces of distributions. We generalize this idea into a new connective |D, and a new codereliction rule dD.
\( \psi : !_DA \) is such that \( d_D(\psi) = \phi \), for \( \psi : !A \).

The fact that we work in a classical setting is central here, as is allows to understand \( d : A \rightarrow ^*!A \) as \( d : A'^{+1} \rightarrow ^*!A \), and to generalize it as \( d_D : !_DA \rightarrow ^*!DA \). DILL thus corresponds to a special case where \( !_D = Id \).

We then construct a new sequent calculus D-DiLL which refines DiLL, and models the resolution of Linear Partial Differential Equations:

\[
\begin{align*}
\vdash !_DA & \quad \vdash \Gamma, !_DA & \quad \vdash \Gamma, !_DA & \quad \vdash \Gamma, !DA \\
x_D & \quad \varepsilon_D & \quad \varepsilon_D & \quad \varepsilon_D \\
\end{align*}
\]

The cut-elimination procedure of D-DiLL translates categorically into :

\[
\vdash_d \varepsilon_D(x_D, \phi) = \phi
\]

for \( \phi : !A \). In the syntax, this says that the solution \( \psi \) to the equation \( d_D(\psi) = \phi \) is exactly \( \varepsilon_D(x_D, \phi) \). In the semantics of D-DiLL this is interpreted exactly as the resolution of a Linear Partial Differential Equation in the theory of distributions [19].

**A classical and smooth semantics.** This syntax for the resolution of LPDE comes from a semantic investigation for smooth and classical models of DiLL. Denotational semantics is the study of proofs and programs through their interpretation as denotations (functions) between spaces. In a denotational model of LL, there are spaces \( \mathcal{L}(E,F) \) of linear functions from \( E \) to \( F \), spaces \( C(E,F) \) of non-linear ones, and a way to understand non-linear functions on \( E \) as linear functions on \( !E = \mathcal{L}(E,F) \). In a model of DiLL, functions must also be smooth, that is able to be iteratively differentiated everywhere. We write \( C^\omega(E,F) \) the space of all smooth functions between \( E \) and \( F \).

The first models of DiLL introduced by Ehrhard [6, 7] have a discrete basis: non-linear proofs are interpreted as power series between spaces of sequences. In order to get a better understanding of the differential nature of DiLL rules, one is bound to search for a denotational model of DiLL where functions are interpreted as the smooth functions of differential geometry or functional analysis. But to account for linearity of functions, and for the classical setting of DiLL, one needs to interpret formulas as some topological vector spaces \( E \) which are reflexive: denoting \( E' = \mathcal{L}(E, \mathbb{R}) \), we need \( E \cong E'' \). The requirements for reflexivity to be preserved by the connectives of LL and the ones for having smooth functions work as opposite forces. More precisely:

- One needs a monoidal category of reflexive spaces, that is spaces which are isomorphic to their dual and such that this property is preserved by tensor product and internal hom-sets. This is true for euclidean spaces, but fails in general when considering infinite dimensional spaces: it is false in particular for Banach spaces.
- One needs a cartesian closed category of smooth functions: we want \( C^\omega(E \times F, G) \cong C^\omega(E, C^\omega(F,G)) \). These structures are notably scarce in analysis, but are fundamental in the semantics of LL as it accounts for the possibility to curry programs.

Solutions to the first point are for instance models based on spaces of sequences [6, 7], or topological vector spaces with very coarse topologies [21]. Solutions to the second point are constructed by Fröhlicher, Kriegl and Michor [23], leading to models of Intuitionnistic DiLL [2, 22]. The attempt by Girard to interpret LL in Banach spaces fails [13], as the requirement of a norm on power series is to strong to allow a good cartesian closed category. We propose here a classical and smooth model of DiLL without promotion, while another one with a more intricate structure and interpreting promotion was recently exposed by Babrowski and K. [3].

**Computing with distributions.** Distributions appears naturally in the quest for a model of LL. On the one hand, consider a model of DiLL made of \( \mathbb{K} \)-vector spaces, with spaces of linear functions \( \mathcal{L}(E,F) \), and spaces of smooth functions \( C^\omega(E,F) \). Then as these spaces are reflexive we have necessarily :

\[
\begin{align*}
1_E & \cong C^\omega(\mathbb{E}(\mathbb{E}(\mathbb{K}))') \quad \text{and} \quad ? E \cong C^\omega(\mathbb{E}(\mathbb{E}(\mathbb{K})))
\end{align*}
\]

Thus \( !E \) is a space of linear forms acting on some space of smooth function, i.e. a space of distributions.

On the other hand, one of the major requirements in the categorical semantics of LL is the Seely’s isomorphism: \( !A \otimes !B \cong ! (A \otimes B) \). It translates immediately into the Schwartz’s Kernel theorem [28], written here for distributions with compact support: \( C^\omega(\mathbb{K}^m, \mathbb{K}) \otimes C^\omega(\mathbb{K}^m, \mathbb{K})' \cong C^\omega(\mathbb{K}^{2m}, \mathbb{K}) \). Based on these intuitions, we find a classical semantics of DiLL in the theory of Nuclear spaces and distributions. The language of distributions has been used for a while in Linear Logic, and this work should be seen as a way to ground this fact.

**Nuclear spaces, Fréchet space, and distributions: a model of Smooth DiLL** Typical examples of nuclear spaces are either euclidean spaces as \( \mathbb{R}^n \) or \( \mathbb{R}^n \), either spaces of (test) function \( \mathcal{E}(\mathbb{R}^n) = C^\omega(\mathbb{R}^n) \), or their duals, spaces of distributions \( \mathcal{E}(\mathbb{R}^n) = C^\omega(\mathbb{R}^n) \)' , \( D(\mathbb{R}^n) = C^\omega_c(\mathbb{R}^n) \)' . Moreover, a nuclear Fréchet space (that is a nuclear, complete and metrisable space) is reflexive, and while it is not preserved by duality, this condition is preserved by tensor product. We use the fact that Nuclear spaces which are Fréchet (i.e. complete and metrisable) form a negative interpretation for polarized MALL. When defining \( \mathbb{R}^n = \mathcal{E}(\mathbb{R}^n) = C^\omega(\mathbb{R}^n) \)’, the kernel theorem of distribution allows us to see as a monoidal functor from the category of Nuclear spaces to the category of duals of Nuclear Fréchet spaces; We translate this structure in the syntax (section 4) obtaining a polarized Smooth DiLL with a distinction between finitary and smooth formulas.

**Modelizing D-DiLL by LPDEs** Our definition of D-DiLL is justified by the fact that for \( D \) any linear partial differential operator (LPDO) with constant coefficients, we have a model of D-DiLL.

\( \mathbb{R}^n \) is then interpreted as the space of distribution with compact support \( \mathcal{E}(\mathbb{R}^n) \), \( D \) as a LPDO, and

\[
1_{\mathbb{R}^n} := (D(\mathcal{E}(\mathbb{R}^n)))'.
\]

Consider \( D_0 \) the operator mapping a function \( f \in C^\omega(\mathbb{R}^n, \mathbb{R}) \) to its differential at 0, that is :

\[
D_0 := f \mapsto v \mapsto \lim_{h \to 0} \frac{f(hv) - f(0)}{h}
\]

Then \( D_0(\mathcal{E}(\mathbb{R}^n)) = (\mathbb{R}^n)' \) and \( 1_{\mathbb{R}^n} \cong (\mathbb{R}^n)' \cong \mathbb{R}^n \). The fact that we work in a classical setting, and thus with reflexive
spaces, is central here, as is allows to understand the usual interpretation of \(d : v \in \mathbb{R}^n \rightarrow f \mapsto D(f)(v)\) as operator matching \(\phi \in (\mathbb{R}^n)'\) to \(\phi \circ D \in C^{\infty}(\mathbb{R}^n)',\) and to generalize it. The codereliction \(d_D(\phi) = \phi \circ D\).

The coweakening \(\tilde{w}_D\) is then interpreted as the input of a fundamental solution \(E_D\), solution to \(\psi \circ D = \delta_0\). We prove in particular that while \(E_D\) is not a distribution with compact support in general, it is an element of the interpretation of \(\psi_D A\).

The co-contraction \(\tilde{c}_D\) is interpreted by the convolution between a solution in \(\psi_D A\) and a distribution in \(\psi A\), producing another solution in \(\psi_D A\). Following the rules in the sequent calculus, we have, for every \(\phi \in \mathbb{R}^n\), for every \(f \in \mathcal{E}(\mathbb{R}^n)\):

\[
\tilde{c}_D(\tilde{w}_D, \phi)(D(f)) = \tilde{d}(\tilde{c}_D(\tilde{w}_D, \phi))(f) = \phi(f).
\]

That is, the solution \(\psi\) to \(D\psi = \phi\) is \(\tilde{c}_D(\tilde{w}_D, f)\).

**Contributions**

This paper:

- defines a Polarized Smooth variant of DiLL, without higher order, with a distinction between smooth and finitary formulas, and its categorical models.
- constructs a denotational model for it, based on the idempotent adjunction between Nuclear Fréchet and Nuclear DF spaces, and the construction of the exponential as a space of compact support distributions.
- defines a Polarized D-Differential Linear Logic, which refines Smooth DiLL with an indexed exponential \(\psi_D\) whose rules represent the computation of a solution to a partial differential equation. We define a cut-elimination procedure for D-DiLL.
- shows that we have a model of Polarized D-DiLL for any LPDOcc.

**Related work**

There is a major research effort towards the understanding and the semantics of probabilistic programming \([4, 12, 17]\). Our work bears similarity with these, if only because we use the same language of distributions and kernels. More generally, this work takes place in a global understanding of continuous data-types and computations: machine-learning, which uses gradients to optimize the computations, is one example. The change of paradigm, allowing to go from a discrete point of-view on resource-sensitive programs to solutions of Differential Equations, relates to recent work on continuous probability distributions in probabilistic programming \([9]\). Notice however that models of probabilistic programming are not in general models of Differential Linear Logic.

**Organisation of the paper**

We first introduce in section 2 the rules, cut-elimination procedure and categorical semantics of DiLL. Then in section 3 we give an overview of the functional analysis necessary to the paper. We barely recall any proofs, but show examples and precise references for our claims. Section 4 is quite short, as it formalizes syntactically and categorically the content of section 3 into the definition of Smooth DiLL. Section 5 defines D-DiLL, its syntax, rules, cut-elimination procedure and its categorical semantics. 

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1To avoid early confusion, we recall that for a distribution \(\psi\), \(D(\psi)\) is usually not defined as \(\psi \circ D\). See section 5.4.

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Definition 2.1. Proofs of DiLL are finite sums of proof-trees generated by these rules. In particular, there is of an empty proof tree denoted by .

The cut-elimination procedure follows the one of LLICS '18, July 9–12, 2018, Oxford, United Kingdom autonomous structure, a biproduct structure (Definition 2.4. such that (⋄, A) is a monoidal structure of DiLL without promotion, and then extend this definition [8]. We adopt the first point of view, but make use of the numerous details and diagrams exposed by Ehrhard [8]. The following definitions are those of Fiore [11], sometimes adapted to the classical setting.

Definition 2.2. A biproduct on a category L is a monoidal structure (⋄, I) together with natural transformations:

\[ A \otimes \bar{A} \to \lambda A \]

such that \((A, u, \triangleright\triangleright)\) is a commutative monoid and \((A, n, \triangle)\) is a commutative comonoid.

Definition 2.3. A *-autonomous category is a symmetric monoidal closed category \((L, \otimes, 1)\) with an object \(\bot\) giving an equivalence of categories \((\cdot, \bot) : L^{op} \to L\) with the canonical map \(ev_L : E \to E''\) being a natural isomorphism.

Definition 2.4. A model of DiLL with promotion is consists of a symmetrical monoidal closed category \((L, \otimes, 1)\) with a *-autonomous structure, a biproduct structure \((\triangleright\triangleright, \triangleright\triangleright\triangleright)\), a co-monad \(! : L \to L\) which is strong monoidal from \((L, \otimes)\) and a natural transformation \(\bar{d} : 1d \to !d\) satisfying strength and comonad diagrams [11].

Remark. As shown by Fiore, from the biproduct structure follows the fact that the category \(L\) is enriched over commutative monoids. This induces an additive law + on hom-sets, which is necessary to interpret the sums of proofs-trees of DiLL which stems from cut-elimination.

\[ f + g : E \to F \circ F \to F. \]

2.3 Interpreting DiLL in its categorical model.

We briefly recall how to interpret a sequent of DiLL as morphism in \(L\), detailing only the action of exponential rules. The connectives \(\otimes, \triangleright\triangleright, \triangleright\triangleright\triangleright, \&\) are interpreted respectively by \(\otimes\) and its dual, and by the coproduct and product deduced from \(\otimes\). We have \(!1 = 1\) by strong monoidality of !. We write \(m_{E,F} : !(E \circ F) \to !E \otimes !F\) the isomorphism resulting from the monoidality of !, and \(d !d \to 1d\) the co-unit of !. Then:

- from \(f : E \to F\) one construct \(f \circ d_E : !E \to F\) and from \(g : !E \to F\) one construct \(g \circ d_E : E \to F\).
- one construct \(w !d \to 1d\) as \(wE = !u\) and \(w !d\) as \(wE = !u\).
- one construct the natural transformation \(c : !0 \to !0\) as \(c_A = m_{A,A} !\Delta_A\) and \(c : !0 \to !0\) as \(c_A = !\nabla_A \circ m_{A,A}\).

It should be clear then that in order to interpret the exponential rules of DiLL one requires the biproduct structure, the strong monoidality of ! and an interpretation for \(d\) and \(d\). The co-monadic structure of ! is used only for the interpretation of the promotion rule, and enforces the definition of \(d\). We will make use of that statement in section 4 when we relax the co-monad requirement on !.

3 Topological vector spaces

In this section, we give technical accounts on some specific classes of topological vector spaces, on distribution theory and LPDOs. We refer mainly to the works by Jarchow [20] and Hörmander [19], as well as Grothendieck’ thesis [15]. We consider vector spaces on \(\mathbb{R}\).

Definition 3.1. A topological vector space (tvs) is a vector space endowed with a topology, that is a covering class of open sets closed by infinite union and finite intersection, making the scalar multiplication and the addition continuous. A tvs is said to be Hausdorff if for any two distinct point \(x\) and \(y\) one can find two disjoint open sets containing \(x\) and \(y\) respectively. It is locally convex if every point is contained in a convex open set.

From now on we work with locally convex separated topological vector spaces and denote them by lcts. Examples of lcts includes all euclidean spaces \(\mathbb{R}^n\), normed spaces and metric spaces. For the rest of the section we consider \(E\) and \(F\) two lcts.

Notation. We will write \(E = F\) for the linear isomorphism between \(E\) and \(F\) as vector spaces, and \(E \simeq F\) for the linear homeomorphism between \(E\) and \(F\) as tvs.

Definition 3.2. Consider \(U \subset E\) and \(x \in U\), then \(U\) is said to be a neighborhood of \(x\) if \(U\) contains an open set containing
A set $B \subseteq E$ is bounded if for every $U$ neighborhood of $0$, there is $\lambda \in \mathbb{R}$ such that $B \subseteq \lambda U$.

**Definition 3.3.** For two lctvs $E$ and $F$ we consider $L_b(E, F)$ the lctvs of all linear continuous functions between $E$ and $F$ and endow it with the topology of uniform convergence on bounded subsets of $E$. We write $E' = L_b(E, \mathbb{R})$ for the dual of $E$.

**Definition 3.4.** A lctvs is reflexive if $E \cong E''$ through the transpose of the evaluation map in $E'$:

$$\delta : \begin{cases} E & \to E'' \\ x & \mapsto \delta_x : (f \mapsto f(x)) \end{cases}$$

Typically, all euclidean spaces are reflexive, as they are isomorphic to their dual. This is also true for all Hilbert spaces, but as soon as we generalize to Banach spaces we encounter the famous counterexample of $\ell_1$ and its dual $\ell_\infty$. The restriction to reflexive spaces is not preserved by tensor product nor linear hom-set: typically, the space $L(H, H)$ is not reflexive when $H$ is a Hilbert space.

**Definition 3.5.** Consider $E$ and $F$ two lctvs. The projective tensor product\(^3\) $E \otimes \pi F$ is the algebraic tensor product, endowed with the finest topology making the canonical bilinear map $E \times F \to E \otimes F$ continuous. Then $E \otimes \pi F$ is a lctvs. The completion of $E \otimes \pi F$ is called the completed projective tensor product and denoted $E_{\hat{\otimes}} F$.

### 3.1 (F)-spaces and (DF)-spaces

**Definition 3.6.** A Fréchet space, or (F)-space, is a complete and metrisable lctvs.

Recall that a lctvs is metrisable if and only if it admits a countable basis of $0$-neighbourhoods. If $F$ is a metrisable space, we write $\hat{F}$ its completion. Fréchet spaces are very common in analysis, but are not preserved by duality: the dual of a Fréchet space is not necessarily metrisable. In particular, the dual $C^\infty(\mathbb{R}, R)'$ of the space of smooth scalar functions, as described in section 3.2, is not metrisable.

**Definition 3.7.** A (DF)-space is a lctvs $E$ admitting a countable basis of bounded sets $\mathcal{A} = (A_\lambda)_{\lambda}$, and such that if $(U_n)_n$ is a sequence of closed and sliced neighbourhoods of $0$ whose intersection $U$ is bornivorous (i.e. absorbs all bounded subsets), then $U$ is a neighbourhood of $0$.

Let us note that, by duality, the second condition is equivalent to asking every bounded subset $B$ of the strong dual $E'$ which is the union of a sequence of equicontinuous subsets to be equicontinuous. Moreover, it is costless to ask that for every $n A_\lambda$ be absolutely convex and $A_n + A_n \subseteq A_{n+1}$. We will therefore always suppose that this is the case. Although this definition may seem obscure, it is the right one for interpreting the dual and pre-dual of (F)-spaces.

**Proposition 3.8.** ([16] IV.3.1). \(\bullet\) If $E$ is metrisable, then its strong dual $E'$ is a (DF)-space.

\(^2\)The author thanks Marc Bagnol for this clarifying example.

\(^3\)Many topologies can be defined on the vector space resulting from the tensor product of two lctvs. The later restriction to Nuclear spaces will de facto identify all reasonable topological tensor product to the projective one.

\(^4\)A basis $\mathcal{A}$ being a collection of bounded set such that every bounded set in included into an object of $\mathcal{A}$.

- If $E$ is a (DF)-space and $F$ and (F)-space, then $L_b(E, F)$ is an (F)-space. In particular, $F$ is an (F)-space.

**Proposition 3.9** ([20] 12.4.2 and 15.6.2). The class of (DF)-spaces is preserved by countable inductive limits, countable direct sums, quotient and completions. The class of (F)-spaces is stable with the construction of products and completed projective tensor products $\otimes _\pi$.

The following reflects the syntax of an intuitionist version of Smooth DiLL of section 4.

**Example 3.10** ([20] 12.4.4). A space which is Fréchet and (DF) is necessarily finite dimensional.

### 3.2 Distributions with compact support

We refer to the exposition of distributions by Hörmander [19] for proofs and details.

**Definition 3.11.** Consider $n \in \mathbb{N}$ and $f : \mathbb{R}^n \to \mathbb{R}$. The function $f$ is said to be smooth if it is differentiable at every point $x \in \mathbb{R}^n$, and if at every point its differential is smooth.

The theory of distribution is traditionally introduced by considering the space $D(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ of test functions, i.e. the space of scalars smooth functions on $\mathbb{R}^n$ with compact support, and define distributions as elements of its dual $D'(\mathbb{R}^n)$. But because the delification rule $d$ makes us consider linear functions as a particular case of smooth function, we work with the following:

**Definition 3.12.** We consider $E(\mathbb{R}^n) = C^\infty(\mathbb{R}^n, \mathbb{R})$ the space of all scalar smooth functions on $\mathbb{R}^n$, endowed with the usual topology of uniform convergence of all differentials of order $\leq$ on all compact subsets of $\mathbb{R}^n$, for all $k \in \mathbb{N}$. Its dual is called the space of distributions with compact support and denoted $E'(\mathbb{R}^n)$.

**Proposition 3.13.** For any $n \in \mathbb{N}$, $E(\mathbb{R}^n)$ is an (F)−space and $E'(\mathbb{R}^n)$ is a complete (DF)−space.

**Example 3.14.** A distribution must be considered as a generalized function, and acts as such. The key idea is that, if $f \in C^\infty(\mathbb{R}^n)$ then one defines a compact distribution by $g \in C^\infty(\mathbb{R}^n) \mapsto \int f(x)g(x)dx$. Typical examples of distributions which do not follow this pattern are the dirac distributions. For $x \in \mathbb{R}^n$ one defines the dirac at $x$ as: $\delta_x : f \mapsto f(x)$.

**Definition 3.15.** Consider $\phi \in E'(\mathbb{R}^n)$ and $f \in E(\mathbb{R}^n)$. Then one defines the convolution between a distribution and a function as $\phi \ast f \in E(\mathbb{R}^n)$ as: $\phi \ast f : x \mapsto \phi(y) \mapsto f(x - y)$. This definition is extended to a convolution product between distributions. Consider $\psi \in E'(\mathbb{R}^n)$. Then $\phi \ast \psi$ is the unique distribution in $E'(\mathbb{R}^n)$ such that:

$$\forall f \in E(\mathbb{R}^n), (\phi \ast \psi) \ast f = \phi \ast (\psi \ast f).$$

Although the above is not a symmetric definition, one proves easily that the convolution is commutative and associative [19].

**Example 3.16.** Note that $\delta_0$ defined in 3.14 acts as neutral element for the convolution law.
The central theorem of the theory of distributions is the Kernel Theorem:

**Theorem 3.17** ([29] 51.6). For any \( n, m \in \mathbb{N} \) we have:

\[
E'(\mathbb{R}^{n+m}) = E'(\mathbb{R}^m) \otimes \mathcal{E}'(\mathbb{R}^n) 
\]

This theorem is proved on the spaces of functions by showing the density of smooth functions of the kind \( f \otimes g, f \in \mathcal{E}(\mathbb{R}^m), g \in \mathcal{E}(\mathbb{R}^m) \), and then that the topology induced by \( E(\mathbb{R}^{n+m}) \) on \( \mathcal{E}(\mathbb{R}^m) \) is indeed the projective topology of the tensor product. This, and particularly the fact that \( \mathcal{E}(\mathbb{R}^m) \) is nuclear, is justified by the theory of Nuclear spaces, which is recalled below.

### 3.3 Nuclear spaces

The theory of nuclear spaces will allow us to interpreted the idempotent negation of DiLL, and as the same time the theory of exponentials as distributions

**Definition 3.18.** An linear map \( f \) between a lcts \( E \) and a Banach X is said to be nuclear if there is an equicontinuous sequence \((a_n)\) in \( E'\), a bounded sequence \((y_n)\) in \( X \), and a sequence \((\lambda_n)\) in \( \mathbb{R} \) such that for all \( x \in E \):

\[
f(x) = \sum_n \lambda_n a_n(x) y_n.
\]

**Definition 3.19.** Consider \( E \) a lcts. We say that \( E \) is nuclear every continuous linear map of \( E \) into any Banach space is nuclear.

**Proposition 3.20** ([20] 21.2.3). The class of nuclear spaces is closed with respect to the formation of completion, cartesian products, countable direct sums, projective tensor products, subspaces and quotients.

An important property of nuclear spaces is that as soon as they are normed, they are finite dimensional. In other word, if a Hilbert or Banach or simply normed space is nuclear, then it is isomorphic to \( \mathbb{R}^n \) for a certain \( n \).

**Example 3.21.** Typical examples of nuclear spaces are euclidean spaces \( \mathbb{R}^n \), spaces of smooth functions \( C_c^\infty(\mathbb{R}^n, \mathbb{R}) \), \( C^\infty(\mathbb{R}^n, \mathbb{R}) \) and their duals \( D'(\mathbb{R}^n) \) and \( E'(\mathbb{R}^n) \).

**Theorem 3.22.** An \( (F) \)-space \( F \) which is also nuclear is reflexive. As a consequence, \( \mathcal{E}(\mathbb{R}^n) \) and \( \mathcal{E}'(\mathbb{R}^n) \) are reflexive.

**Proof.** We give a brief proof for the reader familiar with functional analysis. It is enough to prove that \( F \) is semi-reflexive, that is that \( F = F'' \), as the equality between the topologies will follow from the metrisability of \( F \). Indeed, when \( F \) is metrisable E-equicontinuous sets and E-weakly bounded sets corresponds in \( E' \) [20, 8.5.1]. Now we have that every bounded set of a nuclear space is precompact [27, III.7.2.2]. Thus as \( F \) is nuclear and complete, its bounded sets are compact, and \( F' \) is endowed with the Arens-topology of uniform convergence on absolutely convex compact subsets of \( F \). By the Mackey-Arens theorem, this makes \( F \) semi-reflexive.

**Proposition 3.23.** • Consider \( E \) a lcts which is either an \( (F) \)-space or a \( (DF) \)-space. Then \( E \) is nuclear if and only if \( E' \) is nuclear [15, Chap II, 2.1, Thm 7].

• If \( E \) is a complete \( (DF) \)-space and if \( F \) is nuclear, then \( \mathcal{L}_b(E,F) \) is nuclear. If moreover \( F \) is an \( (F) \)-space or a \( (DF) \)-space, then \( \mathcal{L}_b(E,F') \) is nuclear [15, Chapter II, 2.2, Thm 9, Cor. 3]. As a corollary, the dual of a nuclear \( (DF) \)-space is a nuclear \( (F) \)-space.

**Proposition 3.24** ([15] Chapter II, 2.2, Thm 9). If \( E \) and \( F \) are both nuclear \( (DF) \)-spaces, then so is \( E \otimes \mathcal{F} \).

A central result of the theory of nuclear spaces, explaining the form of the Kernel theorem 3.17, is the following proposition. It is proved by applying the hypothesis that \( E \) is reflexive and thus \( E' \) is complete and barrelled, and thus applying the hypothesis of Treves’ book [29].

**Proposition 3.25** ([29] prop. 50.5). Consider \( E \) a Fréchet nuclear space, and \( F \) a complete space. Then \( E \otimes \mathcal{F} \cong \mathcal{L}(E',F) \).

### 3.4 Linear Partial Operators

We recall the very first steps in the theory of LPDEs\(^5\). For \( \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n \) we write \( \partial^\alpha \) the linear continuous map:

\[
f \in C^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto x \in \mathbb{R}^n \mapsto \frac{\partial^{\alpha}|f|}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_n}}(x)
\]

**Definition 3.26.** Consider, for \( \alpha \in \mathbb{N}^n \) smooth functions \( a_\alpha \in C^\infty(\mathbb{R}^n, \mathbb{R}) \). Then a Linear Partial Differential Operator (LPDO) is defined as an operator \( D : \mathcal{E}(\mathbb{R}^n) \rightarrow \mathcal{E}(\mathbb{R}^n) \):

\[
D = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial^\alpha.
\]

A LPDO with constant coefficients is a LPDO \( D \) such that the \( a_\alpha \) are constants.

The definition of \( D \) is extended to \( E'(\mathbb{R}^n) \) as follows:

\[
D(\phi) = f \mapsto \sum_{\alpha} (-1)^n a_\alpha \partial^\alpha(\phi(f)),
\]

so that for \( g \in C^\infty(\mathbb{R}^n) \): \( D(f) = g \Leftrightarrow f = \int f \, D(g) \).

The weak resolution of the LPDE consists then, when \( \phi \in E'(\mathbb{R}^n) \), of finding \( \psi \) such that, for all \( f \in \mathcal{E}(\mathbb{R}^n)^6 \):

\[
\psi = D(f) = \phi(f).
\]

The resolution of LPDOs with constant is always possible, and particularly elegant, due to the behaviour of convolution with respect to partial differentiation:

**Proposition 3.27** ([19] 4.2.5). Consider \( f \in \mathcal{E}(\mathbb{R}^n) \) and \( \phi \in E'(\mathbb{R}^n) \). Then \( \partial^\alpha \phi \ast f = \phi \ast (\partial^\alpha f) \).

**Definition 3.28.** A fundamental solution to equation (2) consists of a distribution \( \hat{E}_D \) such that \( D(\hat{E}_D) = \delta_0 \).

I appears then thanks to the linearity of the convolution product and propositions 3.16 and 3.27 that:

\[
\forall \phi \in E'(\mathbb{R}^n), D(\hat{E}_D \ast \phi) = \phi.
\]

Again, we make a slightly different use of the fundamental solution by defining \( E_D \) such that equation 3 holds.

\(^5\)We are not considering in this paper border conditions, regularity of the solutions to equations with non-constant coefficients, nor modern research subjects in the theory non-linear equations.

\(^6\)This definition is specific to the paper, and necessary to be coherent DiLL. In the literature, the resolution of the equation consists in finding \( \psi \) such that \( D(\psi)(f) = \phi(f) \).
Theorem 3.29 (Malgrange-Ehrenpreis [18] 3.1.1). Every LPDEcc admits a fundamental solution $E_D$, which leads to $E_D = D_0 + e \in (D(E^m(\mathbb{R}^q)))'$. The theorem is in fact much more precise as we have information about the local growth of $E_D$. We do not have in general that $E_D \in E'(\mathbb{R}^n)$.

However, the first and easy step of the proof consists in noticing that, if one defines $E_D$ as the distribution $f \mapsto E_D(x \mapsto f(-x))$, that is $E_D(f) = (E_D * f)(0)$, we have:

$$\forall f \in E(\mathbb{R}^n), E_D(D(f)) = f(0). \quad (3)$$

The proof consists afterwards into the majoration of $E_D$ in order to extend if to $C_c^\infty(\mathbb{R}^n)$. This is one of the arguments for the introduction of $F_D(\mathbb{R}^n) = (D(C^\infty(\mathbb{R}^n)))'$. Proposition 3.30. The fundamental solution $E_D$ defined above is continuous on $D(E(\mathbb{R}^n))$, as it corresponds to $D(f) \mapsto f(0)$. We have thus $E_D \in D(\mathbb{R}^n)$.

4 Smooth Differential Linear Logic and its models

In this section we introduce a Smooth Differential Linear Logic for which Nuclear spaces and distributions form a classical and smooth model. We notably show that the categorical interpretations for $\wedge$ and $\otimes$ correspond to the convolution and the dirac in 0 in the theory of distributions.

Let us recall the notion of polarisation in LL. In polarized linear logic [24] a distinction is made between positive connectives $\Rightarrow, \otimes, \oplus$ whose introduction rules are non-reversible, and negative connectives $\setminus, \exists$, whose introduction rule is reversible. Negation then changes the polarity of a formula. This plays a fundamental role in proof-search [1].

4.1 The category of Nuclear Fréchet spaces.

Nuclear Fréchet spaces gather all the stability properties to be a (polarized) model of LL, except that we do no have an interpretation for higher-order smooth functions. Indeed if $\mathbb{R}^n$ is interpreted as $E'(\mathbb{R}^n)$, we do not have a straightforward definition of $\mathbb{R}^n$.

Definition 4.1. We write NF the category of Nuclear (F)-spaces and continuous linear maps, NDF the category of complete Nuclear (DF)-spaces and continuous linear maps, and EUCL the subcategory of both formed of euclidean spaces.

Proposition 4.2. \begin{itemize} 
\item Eucl is a model of MALL.
\item NF forms a model for the negative interpretation of polarized MALL [24, 6.20], where positive formulas are thus interpreted as objects of Ndf.
\end{itemize}

Proof. The first point is transparent. The second point is due to the stability of Nuclear Fréchet spaces by cartesian product (interpreting $\wedge$) and completed $\pi$-tensor product (interpreting $\setminus$), see propositions 3.20, 3.9 and 3.25. The interpretation of the rules for $\otimes$ and $\oplus$ is possible by the fact that Nuclear Fréchet spaces are reflexive (proposition 3.22) and thus the interpretation of $A \otimes B$ is the one of $(A^\perp \otimes B^\perp)^\perp$. $\Box$

\footnote{Let us point out that even if $D_0$ is not a LPDO, the equation $D_0 g = f$ behaves likewise. If there is of a solution to this equation it means that $f$ is linear, and then $D_0 f = D_0(f * \delta_0) = f$.}

Thus the interpretation of $\otimes$ in NDF is $\otimes$, and that the interpretation of $\exists$ in NF is also $\otimes$.

Remark. Note however that we do not have a compact closed category, as we are working in a polarized model of MALL with an adjunction between NF and NDF$^{op}$.

Definition 4.3. For $\mathbb{R}^n \in EUCL$ we define $\mathbb{R}^n = E'(\mathbb{R}^n)$. This is extended as a functor on EUCL by defining $(f : \mathbb{R}^n \rightarrow \mathbb{R}^m) : \phi \mapsto \phi \circ f \in E'(\mathbb{R}^m)$.

It follows from the Kernel theorem 3.17 and example 3.21 that the space of compact distribution acts as a strong monoidal functor from EUCL to NDF:

Theorem 4.4. The exponential $! : EUCL \rightarrow NDF$ is a strong monoidal functor.

4.2 Smooth Differential Linear Logic (SDiLL)

In this section, we construct a version of DILL for which Nuclear spaces and distributions are a model, by distinguishing several classes of formulas. We introduce SDiLL: its grammar, defined in figure 3, separates formulas into finitary ones and polarized smooth ones.

Its rules are those of DiLL : follows the one of MALL for the additive and multiplicative connectives, and those detailed in figure 4 for the exponential. Thus, the cut-elimination procedure is the same as the one defined originally [10].

If we forget about the polarisation of SDiLL, a model of it would be a model of DILL where the object $A$ does not need to be defined. It is thus a model of DILL where $!$ does not need to be an endofunctor, but just a strong monoidal functor $! : Fin \rightarrow Smooth$ between two categories. The categories Fin and Smooth need to be both a model of MALL.

This distinction is necessary here to account for spaces of distributions are their dual, which cannot be understood as part of the same $+$-autonomous category. We give a categorical semantics for an unpolarized version of SDiLL. The polarized version would ask the category Smooth below to be a model of Polarized MALL, that is an involutive, defined as an adjunction between a category of negative smooth formulas and a category of positive smooth formulas. Sequents would then be interpreted as maps in the larger category of complete lctvs.

Definition 4.5. A categorical model of SDiLL consists into a model of MALL with biproduct Fin, and a model of MALL...
The interpretation of the contraction \( c \) is then the construction of a Kernel of two smooth functions, while the interpretation for the weakening consists in applying a distribution to the function constant at 1. Diagrams of 2 are easily verified and follow the intuitions of [8, 10].
We interpret finitary formulas Eucl, and a LPDOcc, with distributions of compact support
and a LPDOcc $D$, form a model of D-DiLL.

Consider $D : E(\mathbb{R}^n) \to E(\mathbb{R}^n)$ a LPDOcc:

$$D(f)(x) = \sum_{\alpha \in \mathbb{N}} a_{\alpha} \varphi^\alpha f(x).$$

We interpret finitary formulas $A, B$ as euclidean spaces. One has indeed $1 \times 0 = \mathbb{R}$ and $T = \mathbb{N} = \{0\}$. The connectives of LL are interpreted in Eucl, Nf and Nf as in section 4.

**Definition 5.5.** For $A$ a finitary formula interpreted by $\mathbb{R}^n \in$ Eucl, we interpret $!A$ an its dual as:

$$!_D^\mathbb{R}^n := (D(C_\infty(\mathbb{R}^n)))'$$

$$\cdot_D^\mathbb{R}^n = D(C_\infty(\mathbb{R}^n))' = D(C_\infty(\mathbb{R}^n))$$

**Proposition 5.6.** We have that $\cdot_D^\mathbb{R}^n \in$ Nf and $!_D^\mathbb{R}^n \in$ Nf.

**Proof.** This theorem encodes in particular a well used convention in LPDOs [29, chap. 52], which allows to extends $D$ defined on $E(\mathbb{R}^n)$ to $E(\mathbb{R}^{n+m})$. One differentiate son the $n$-first variable apply to functions defined on $\mathbb{R}^{n+m}$. Our theorem is then directly deduced from the Kernel theorem 3.17.

The interpretation of $w_D, \tilde{w}_D, c_D$ and $\tilde{c}_D$ follows from the previous proposition and the biproduct structure:

- $\tilde{w}_D : 1 \to !_D$ is such that $w_D E(1) = E_D$. It is well defined thanks to proposition 3.30.
- $c_D : !_D \otimes !_D \to !_D$ correspond to the convolution product (see prop. 4.7) and is well defined (prop. 3.27).
- $c_D : !_D \to !_D$ corresponds to the construction of a Kernel of functions, and to the intuitions of 5.7.
- $w_D : !_D \to 1$ corresponds to the application of a distribution to $D(x \in \mathbb{R}^n \mapsto 1)$.

By equation 3 we have indeed the satisfaction of the diagrams of figure 7.

**Definition 5.8.** We interpret the dereliction $d_D : !_D \to !_D$ as $d_D : E_D(\mathbb{R}^n) = (E_D(\mathbb{R}^n))'$, and codereliction $d_D : !_D \to !_D = (D(E_D(\mathbb{R}^n)))'$.

Then one has for every $\phi \in E'(\mathbb{R}^n)$ and $f \in E(\mathbb{R}^n)$:

$$d_D(\phi(f)) = \phi(D(f))$$

$$\cdot_D^\mathbb{R}^n \cdot_D^\mathbb{R}^n \cdot_D^\mathbb{R}^n.$$
After that one should find a deterministic classical term-calculus, inspired by the differential λµ-calculus [30], accounting for D-DiLL. In a Curry-Howard-Lambek correspondence perspective, this would correspond to the Proof/Program bijection, while we studied here the Proof/Categories interpretation. Notice that it was necessary to work with \( E'(\mathbb{R}^n) \) when interpreting SDiLL, but other classes of distributions may suit for a model of D-DiLL.

Work in progress consists in generalising D-DiLL into a system englobing all LPDOcc. Promotion, contraction and co-contraction lead to a BLL-like syntax, in which we would like to give a syntactical counterpart to the construction of a fundamental solution. Generalized to all LPDOs, this could lead to a syntactical criterion for the resolution of LPDEs. D-DiLL should also be generalised to account for the domain \( \Omega \subset \mathbb{R}^n \) on which LPDEs are solvable: this should be done by introducing subtyping on finitary formulas, and could lead to a complete semantics over Nuclear spaces. The next goal after that should be to find a logical account for all LPDEs. The long-term goal is of course to go towards non-linear PDEs.

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