Complete spaces and Differential Linear Logic

Marie Kerjean

June 2013

marie.kerjean@ens-lyon.fr

Complete spaces and LL

- Complete topological vector spaces and power series between them are a **continuous and quantitative** model of Linear Logic.
- Bounded sets are fundamental. They allow us to do scalar testing almost everywhere.
- It's a continuous denotational semantics, generalizing Coherent Banach spaces.

Motivations

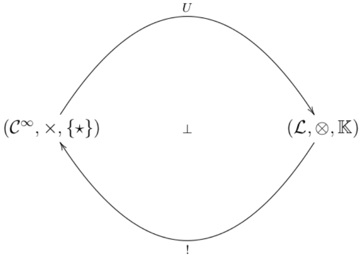
- Quantitative semantics : we want to decompose a program as a sum of finite-ressource consuming programs. e.g. : Finiteness spaces
- Finiteness spaces are based on the relational model and therefore are too dependant of their basis.
- Continuous semantics : we want to construct an easy topological model of Linear Logic, where a discrete basis is not needed anymore. e.g. : Convenient spaces

A Linear Non-linear adjunction

We want a monoidal closed category on the side of linear functions.

Non Linear Functions

- Coherent spaces : stable
- Finiteness spaces : power series
- Convenient spaces : New-smooth



Linear Functions

We will have a cartesian category on the side of the non linear functions.

Continuous semantics : topological vector spaces

The question of having a monoidal cartesian closed topological category is an old and difficult question.

$$\mathcal{C}^\infty(E \times F, G) = \mathcal{C}^\infty(E, \mathcal{C}^\infty(F, G))$$

- ~~(Topological spaces, continuous maps)~~ .
- ~~(Smooth manifolfs, smooth maps)~~

Frölicher, Kriegl, Michor find the solution by looking into what happens in Banach spaces : they created Convenient spaces.

Quantitative semantics : paying attention to ressources

- A linear program is a program using only once its argument.
- A n-linear program is a program using n times its argument.
- A program uses a finite number of times its argument x :

$$P(x) = \sum_{n=0}^{\infty} P_n(x)$$

In a quantitative model of Linear Logic, functions carry this idea. They have a Taylor developement converging at least somewhere.

$$f(x) = \sum_{n=0}^{\infty} \frac{d^n f(0)(x^n)}{k!}$$

Three spaces of functions : Looking for the Taylor formula

By defining a slightly different notion of smoothness we have three models of Differential linear logic :

- The category of complete vector spaces, smooth functions and smooth linear maps.
- The category of complete complex vector spaces, holomorphic functions and smooth linear maps.
- The category of complete complex vector spaces, power series, and smooth linear maps.

The last one is a generalisation of Coherent Banach spaces, and resolves the problem in these.

Complete vector spaces and Spaces of Linear Functions

Tools for differentiation.

Locally convex vector spaces

E vector space over $\mathbb{K}$.

A subset C is convex when $\forall \lambda \in \mathbb{K}, \forall x, y \in C, \lambda x + (1 - \lambda)y \in C$.
A subset C is absolutely convex when
$\forall \lambda, \mu, |\lambda| + |\mu| \leq 1, \lambda x + \mu y \in C$.

A topology (a collection open sets) on E is needed. We want :

- A linear topology (addition and multiplication by a scalar are continuous).
- a basis of absolutely convex open sets.

Completeness in tvs

We want to differentiate (exponentials and power series), we need completeness.

$$\begin{gathered}\text{Complete}\\ \Downarrow \\ \text{Complete for bounded Cauchy net}\\ \Downarrow \\ \text{Mackey-complete (net or sequences)}\end{gathered}$$

Scalar testing in Mackey-complete spaces

A real curve in E is smooth *iff* it is scalarly smooth into $\mathbb{R}$.

Completeness in topological vector spaces

Let E be a lcs, with a basis of 0-neighbourhood $\mathcal{O}$.

Cauchy sequence

A net $(x_\gamma)_{\gamma\in\Gamma}$ is a Cauchy net iff
$\forall O \in \mathcal{O}, \exists \gamma_O \in \Gamma, \alpha, \beta \geq \gamma_0, x_\alpha - x_\beta \in O$.

Mackey-cauchy sequence

A net $(x_\gamma)_{\gamma\in\Gamma}$ is a Mackey-Cauchy net iff
$\exists B, \exists (\lambda_{\alpha,\beta}) \to 0, x_\alpha - x_\beta \in \lambda_{\alpha,\beta} B$.

A complete space is a space where every Cauchy-net converges.

Bounded sets in lcs

Definition

A set B in a locally convex topological vector space E is told to be bounded if for every 0-neighbourhood $\mathcal{U}$ there is $\lambda \in \mathbb{K}$ such that $B \subseteq \lambda\mathcal{U}$.

Bounded sets are dramatically simpler to work with :

Scalar testing for bounded sets

A set B is bounded iff for all $l \in E^\star$, $l(B)$ is bounded in $\mathbb{K}$.

$E^\star$ is the space of linear continuous forms on E.

Bornological

A linear map $l : E \to F$ is bornological when $\forall B \subseteq E$, B bounded, $l(B)$ is bounded in F.

$\mathcal{L}(E, F)$ is the space of all linear maps between E and F, with the topology of uniform convergence on bounded subsets of E.

Completeness of $\mathcal{L}(E, F)$

When F is complete, $\mathcal{L}(E, F)$ is complete.

We will denote E' the space $\mathcal{L}(E, \mathbb{K})$ of linear bornological maps from E to $\mathbb{K}$.

Complete spaces and Linear Bornological functions

We define $E \otimes F$ as the completed of the algebraic tensor product between E and F. At the beginning, the tensor product is endowed the finest topology making $h : E \times F \to E \otimes F$ bornological.

$\mathbb{K}$ is the unit for $\otimes$. And it is a complete topological vector space !

Theorem

The Category of Complete spaces and Linear Bornological functions, endowed with $\otimes$, is a monoidal closed category.

A new definition for smoothness

Objects : Complete vector spaces.

Functions : New-smooth maps between them.

New Smoothness

Smooothness for curves

A curve $c : \mathbb{R} \to E$ is smooth when it is infinitely many times differentiable.

Smooothness for maps

A function $F : E \to F$ is (new-)smooth *iff* it sends smooth curves to smooth curves.

Boman's theorem, extended by Frölicher

A smooth map between Banach spaces is smooth *iff* it is new-smooth.

Cartesian closedeness

The category of locally convex spaces and new-smooth maps is cartesian closed.

$$\mathcal{C}^\infty(E \times F, G) \simeq \mathcal{C}^\infty(E, \mathcal{C}^\infty(F, G))$$

- We can define on $\mathcal{C}^\infty(E, F)$ a vector topology such that when E and F are complete, so is $\mathcal{C}^\infty(E, F)$.
- The product of two complete spaces is complete.

Boundedness and smoothness

Linear, bornological and smooth

A linear functions between topological vector spaces is new-smooth *iff* bornological.

The proof boils down to a version of the mean-value theorem :

If c is a continuous differentiable curve in E, if A is closed and convex and $\forall t, c'(t) \in A$, then $c(b) - c(a) \in A$ for all $a, b \in \mathbb{R}$.

Complex analysis : Holomorphic functions

Objects : Complete complex vector spaces.

Functions : Holomorphic maps between them.

Towards Taylor Formula

A function $f : \mathbb{C} \to \mathbb{C}$ is said to be holomorphic if it is complex differentiable.

$$f(z) = \lim_{w \to 0, w \in \mathbb{C}} \frac{f(Z + w) - f(z)}{w}$$

A holomorphic function is :

- Infinitely many times differentiable.
- Analytic : around each point $z_0 \in \mathbb{C}$, $f(z + z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n$.
- It verifies the Cauchy formula : $\frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{f(\lambda)}{\lambda^{n+1}} d\lambda$.
- It verifies the Cauchy inequalities : $|\frac{f^{(n)}(0)}{n!}| \leq |\frac{sup\{f(\lambda||\lambda|=r\}}{r^n}|$ for all sufficiently small r.

Holomorphic curve and functions in lcs

Curves and functions

A curve $c : \mathbb{D} \rightarrow E$ is holomorphic when it is complex differentiable.
A function $f : E \rightarrow F$ is a holomorphic mapping when it maps a holomorphic curve to a holomorphic curve

And it works ! Since an holomorphic function is smooth,

Cartesian closedeness

$$\mathcal{H}(E \times F, G) \simeq \mathcal{H}(E, \mathcal{H}(F, G))$$

The Taylor formula

Every holomorphic map verifies a Taylor formula on a finitary open, around each point in its codomain.

$\forall x \in E$, there is a set U containing 0, and for ever $y \in U$:

$$f(x+y) = \sum_{n=0}^{\infty} \frac{df^n(x)(y^n)}{n!}$$

The Cauchy Formula

Working with scalars

Let E be a complete space. Then $c : \mathbb{D} \to E$ is complex differentiable *iff* it is scalarly complex differentiable

The Cauchy Formula still holds for functions : since E is complete, we can integrate.

$$\frac{(f \circ c)^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{f \circ c(\lambda)}{\lambda^{n+1}} d\lambda$$

Cauchy Inequality

If U is a set such that $f(rB) \subseteq U$, then $\frac{d^n f(0)}{n!}(B) \subseteq \frac{\bar{U}}{r^n}$.

Power series between lcs

Objects : Complete vector spaces.

Functions : Power series between them.

From Power series in $\mathbb{C}$

In $\mathbb{C}$, a power serie is a converging sum

$$\sum_{n\in\mathbb{N}} a_n z^n$$

This sum converges uniformly in a certain bounded set.

For $x \in F$, x^n doesn't exists. But :

- $a_n = \frac{1}{n!}(z \mapsto \sum_{n\in\mathbb{N}} a_n z^n)^{(n)}(0)$
- $z \mapsto a_n z^n = \frac{1}{n!} d^n(z \mapsto \sum_{n\in\mathbb{N}} a_n z^n)(0)(z, ..., z)$

... to power series in topological vector spaces

In $\mathcal{C}^\infty(E,F)$, a power serie is a sum of smooth n-homogeneous maps.

$$f(x) = \sum_{n\in\mathbb{N}} f_n(x)$$
$$f_n \in \mathcal{H}_n^\infty(E,F) \simeq \mathcal{L}_{sym}^n(E,\ldots,E;F)$$

For all $w \in \mathbb{C}$, we have $f_n(wx) = w^n f_n(x)$. Equivalently, $f_n(x) = \tilde{f}_n(x,\ldots,x)$, f_n being n-linear and smooth.

$S(E, F)$

More than pointwise convergence

$S(E, F)$ is the space of the power series from E to F, where the $\sum f_n$ *converge uniformly on bounded sets of* E.

We give to $S(E, F)$ the topology of uniform convergence on bounded subsets of E.

$$\text{0-neighbourhoods} : \mathcal{U}_{B,U} = \{f | f(B) \subseteq U\}$$

where B is a bounded set in E and U is a neighbourhood of 0 in F.

The first step to cartesian closedeness : completeness of $S(E, F)$

Bornological, smooth and Holomorphic

A power serie in $S(E, F)$ is bornological, and
$S(E, F) \subset \mathcal{H}(E, F) \subset \mathcal{C}^\infty(E, F)$.

$S(E, F)$ is complete

$S(E, F)$ is a complete space when F is a complete space.

In fact, $S(E, F)$ is the completed of the space of polynomials.

The second step : switching terms in sums

If $f = \sum f_n \in S(E, S(F,G))$, then

$$\forall x \in E, f_n(x) = \sum f_{n,x,m} \in S(F,G)$$

.

We want $f(x)(y) = \sum_p \tilde{f}_p(x,y)$, where $\tilde{f}_p$ is p-homogeneous.

$$\sum_p \sum_{n+m=p} f_{n,x,m}(y) = f(x)(y) = \sum_n \sum_m f_{n,x,m}$$

.

Scalar testing for Power series

A serie $f(x) = \sum f_n(x)$ converges in F *iff* it converges weakly in F.

$S(E \times F, G) \simeq S(E, S(F, G))$

Fubini Theorem in $\mathbb{C}$

Let $(a_{n,m})_{n,m\in\mathbb{N}}$ be a serie such that $\sum_n \sum_m |a_{n,m}|$ is convergent.

Then $\sum_m \sum_n a_{n,m} = \sum_n \sum_m a_{n,m} = \sum_p \sum_{n+m=p} a_{n,m}$

Then we have pointwise convergence of the permuted sums.

The Cauchy formula allows us to have uniform convergence on each bounded subset.

Cauchy Inequality

If B_0 is a closed bounded set such that $f(2B) \subseteq U$, then $f_n(B) \subseteq \frac{B_0}{2^n}$.

The exponential and the Taylor formula

We need :

- A comonad !
- Verifying the Seely isomorphism.
- Making the power series as the maps of the co-Kleisli category.
- Reflexive spaces...

$$\text{Wanted} : S(E,F) \simeq \mathcal{L}(!E,F).$$

If E were *reflexive* in our category

$$E = \mathcal{L}(\mathcal{L}(E,\mathbb{C}),\mathbb{C}) = E'' = (E^{\perp})^{\perp}$$

then $!E \simeq (!E)'' \simeq (S(E,\mathbb{C}))'$.

Since we **don't have** reflexivity , we are going to form $!E$ by embedding it into $(S(E,\mathbb{C}))'$. The classical way to do so is through the evaluation application.

$$\delta : E \to S(E,\mathbb{C}))'$$

$$\delta_x = ev_x : f \mapsto f(x)$$

The exponential

δ is well defined

δ_x is linear and smooth, so $\delta_x \in \mathcal{L}(S(E,\mathbb{C}),\mathbb{C})$.

$!E$ is the completion of $< \delta(E) >$ in $\mathcal{L}(S(E,\mathbb{C}),\mathbb{C})$.

- It inherits the topology of $\mathcal{L}(S(E,\mathbb{C}),\mathbb{C})$.
- ! is an endofunctor in the category of complete spaces and linear bornological maps.
- ! is a co-monad with natural transformations $\rho : ! \to !!$ $\rho(\delta_x) = \delta_{\delta_x}$ ans $\epsilon : ! \to 1$ $\epsilon(\delta_x) = x$.

δ is a power serie

Forall $x \in E$, define $x^n \in !E$ with :

- $x^0 : f \mapsto f(0)$
- $x^1 = \lim_{t \to 0} \frac{\delta_{tx} - \delta_0}{t} : f \mapsto df(0)(x)$,
- $x^n = \bigtriangledown(\frac{\delta_{tx} - \delta_0}{t} \otimes x^{n-1}) : f \mapsto d^n f(0)(x^{\otimes n})$.

x^n extracts the n^{th} derivative in 0, applied to $x^{\otimes n}$, from a power serie.

$$\delta_x = \sum_n x^n \text{ in } S(E, \mathbb{C})'.$$

$$\delta = \sum_n (x \mapsto x^n) \in S(E, S(E, \mathbb{C})')$$

The co-Kleisli category

We want $S(E,F) \simeq \mathcal{L}(!E,F)$

- If $f \in S(E,F)$, define $\hat{f} : !E \multimap F$ with $\hat{f}(\delta_x) = f(x)$. $\hat{f}$ is linear and bornological .
- If $g \in \mathcal{L}(!E,F)$, define $\breve{g} : E \rightarrow F$ with $\breve{g} = g \circ \delta$. Since δ is a power serie, and g is linear, $\breve{g}$ is a power serie.

This is an adjunction, with co-unit $\epsilon : ! \rightarrow 1$ $\epsilon(\delta_x) = x$, and unit δ.

The Seely isomorphism

Let us show that

$$!E \otimes !F \simeq !(E \times F)$$

$!(E \times F)$ verifies the universal property of the tensor product :

- $\delta \in S(E \times F, !(E \times F))$, so $\tilde{\delta} \in \mathcal{L}(!E \otimes !F, !(E \times F))$
- Let f be a smooth bilinear map from $!E \times !F$ to G. Then

$$\begin{aligned} f &\in \mathcal{L}(!E, \mathcal{L}(!F, G)) \\ &\simeq S(E, S(F, G)) \\ &\simeq S(E \times F, G) \\ &\simeq \mathcal{L}(!(E \times F), G) \end{aligned}$$

.

- So $\tilde{f} \in \mathcal{L}(!(E \times F), G)$ is unique, and $f = \tilde{f} \circ \tilde{\delta}$.

Synthesis

- Complete spaces, Linear bornological maps and Power series form a model of Linear Logic, missing reflexivity.
- Our non-linear maps verify the Taylor formula everywhere.
- We have a differential operator $x^1 \in !E$: a model of DiLL is within reach.

A generalisation of Coherent Banach spaces

The Norm problem resolved.

Norm and coherence

- A continuous version of Coherent spaces.
- Complete to mimick the infinite cliques.
- Normed to do simple.

Cliques are bounded sets.

Example : the difference between the two additives & and $\oplus$.

Coherent Banach spaces

... are reflexive because the dual is specified : A space is a triplet $(E, E^{\perp}, < ., . >)$.

Linear maps between coherent Banach spaces are linear bornological maps.

Non-Linear maps from E to F are power series defined on the unit ball of E. They do not compose.

$(?E)^{\perp}$ is the set of analytical maps from the unit ball in E to $\mathbb{C}$.

Coherent Banach spaces

We can introduce the same kind of reflexivity on our spaces.

A full subcategory

Coherent Banach spaces and Linear maps form a full subcategory of our complete spaces and smooth maps.

Bornology solve the problem of the norm

We don't need to multiply our maps by scalar to make them compose.

Not the end

Reflexive space

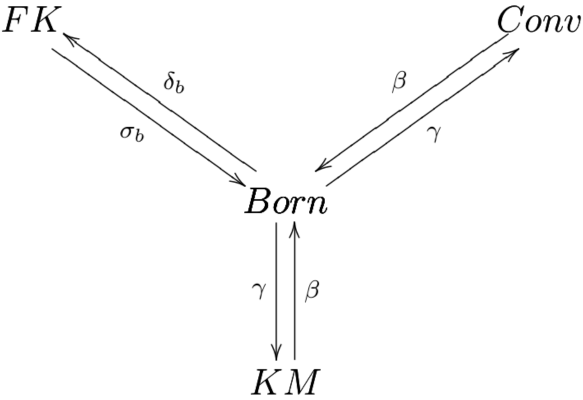
- We want a category with an "intern" reflexivity.
- $\mathcal{L}(\mathcal{L}(E, \mathbb{C}), \mathbb{C}) \simeq E$ in our category *iff* their bornologized topology is complete, and their dual is reflexive

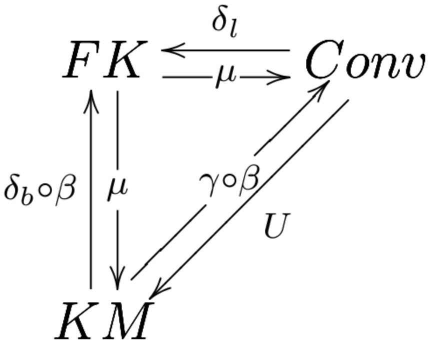
Fixpoints

- We would like a fixpoint operator.

References

1. Blute, Richard ; Ehrhard, Thomas ; Tasson, Christine *A convenient differential category.*
2. Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis*
3. Frölicher, Alfred ; Kriegl, Andreas Linear spaces and differentiation theory.
4. Bochnak, Jacek ; Siciak, Józef Analytic functions in topological vector spaces.
5. Girard, Jean-Yves. Coherent Banach spaces : a continuous denotational semantics.


FK
$Conv$
$Born$
KM
δ_b
σ_b
β
γ
γ
β


δ_l
FK
μ
$Conv$
$\delta_b \circ \beta$
μ
$\gamma \circ \beta$
U
KM