A Type Theory For Partial Differential Equations

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The Curry-Howard-Lambek correspondance

SCIENTIFIC CONTEXT :	Cartesian closed categories Functions $f : E \to F$	Minimal Logic Implication $E \Rightarrow F$	Simply Typed λ-calculus Programs $\lambda x^E.t^F$
		Classical Logic $((E \Rightarrow \bot) \Rightarrow \bot) \simeq E$	Simply Typed λ-calculus Programs acting on contexts : $\mu\alpha^{E\Rightarrow\bot}. < t^E \vert \alpha^{E\Rightarrow\bot} >$ [1]
	+ $*$-autonomous categories Linear functions $f \in \mathcal{L}(E,F)$ Reflexive vector spaces $E'' \simeq E$	Classical Linear Logic Linear Implication $E \multimap F$ [2] $((E \multimap \bot) \multimap \bot) \simeq E$	
	+ differentiation Smooth functions $f \in \mathcal{C}^\infty(E,F)$	Differential Linear Logic Linear approximation to non-linear proofs [3]	Differential λ-calculus
THIS WORK :	Nuclear spaces, Distributions, and Differential operators	D-DiLL Resolution of Linear Partial Differential Equations [4] (LPDE)	Conjecture : A theorical resolution of LPDE via context-program equivalence

Denotational models of Differential Linear Logic

Formulas must be interpreted by reflexive vector spaces,

$$E \simeq E''$$

and functions by smooth functions

$$f : \mathcal{C}^\infty(E,F)$$

We use the theory of topological vector spaces :

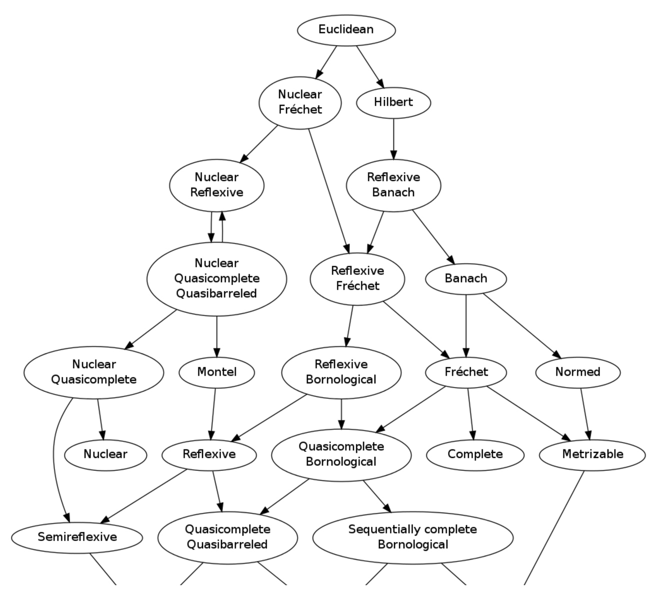


Fig. 1: A classification

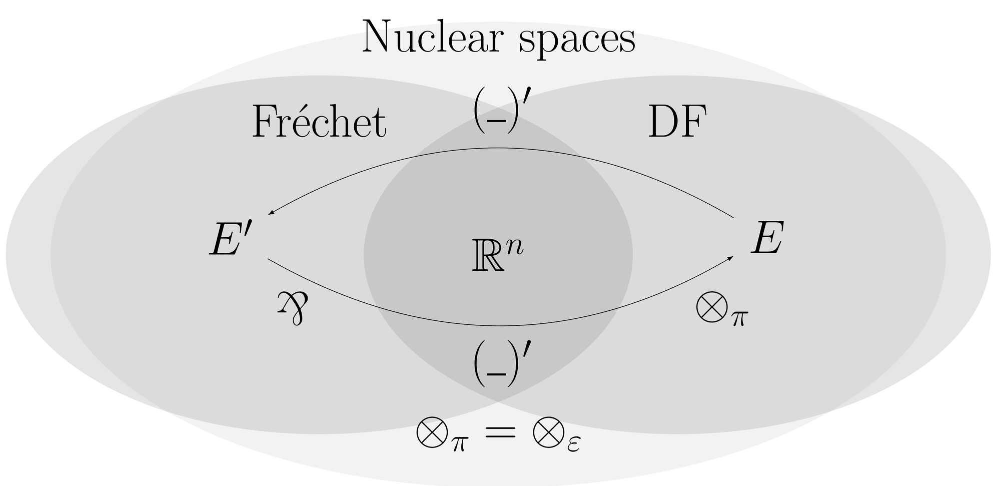


Fig. 2: A polarized model of MLL

Smooth exponentials : Distributions

A typical Nuclear Fréchet space is the space of smooth functions on $\mathbb{R}^n$:

$$\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}).$$

A typical Nuclear DF spaces is the space of distributions with compact support :

$$\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})' := \{\phi : f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \phi(f) \in \mathbb{R}\}.$$

Schwartz' Kernel Theorem :

$$\mathcal{C}^\infty(E,\mathbb{R})' \hat{\otimes} \mathcal{C}^\infty(F,\mathbb{R})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{R})'$$

$$!\mathbb{R}^n = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'.$$

Differential Operator on Distributions

$$D(g)(x) = \sum_{|\alpha| \leq n} a_\alpha(x) \frac{\partial^\alpha g}{\partial x^\alpha}.$$

$$!_D E := (D(\mathcal{C}^\infty(E))'$$

Theorems :

- If $D(g)$ is the differentiation at 0, $!_D E = E'' \simeq E$ and $DiLL =$ sums of $D - DiLL$.
- When the coefficients a_α are constant, we have a model of D-DiLL.

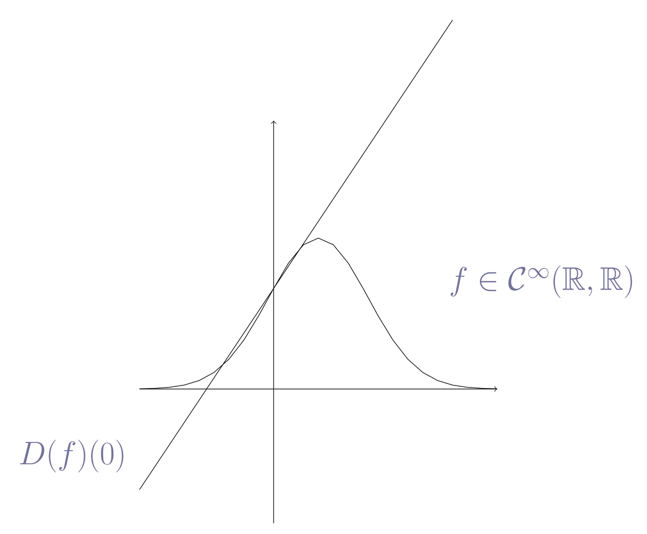
Differential Linear Logic

Linear Logic encodes non-linearity into an exponential :

$$!E \multimap F \simeq E \Rightarrow F$$

And deduces curryfication for non-linear programs from the curryfication for linear one :

$$!E \otimes !F \simeq !(E \times F).$$



Differential Linear Logic allows to compute a Linear function from a non linear one : its differentiation at 0.

$$D(E \Rightarrow F)(0) = E \multimap F$$

D-DiLL

The exponential rules of DiLL are :

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,d \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,c$$

$$\frac{\vdash \Gamma, !A, !A}{\vdash \Gamma, !A}\,\bar{c} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, !A}\,\bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\,\bar{d}$$

The exponentials rules of D-DiLL are :

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\,w_D \qquad \frac{\vdash \Gamma, ?A, ?_D A}{\vdash \Gamma, ?_D A}\,c_D \qquad \frac{\vdash \Gamma, ?_D A}{\vdash \Gamma, ?A}\,d_D$$

$$\frac{\vdash}{\vdash !_D A}\,\bar{w}_D \qquad \frac{\vdash \Gamma, !A \quad \vdash \Delta, !_D A}{\vdash \Gamma, \Delta, !_D A}\,\bar{c}_D \qquad \frac{\vdash \Gamma, !_D A}{\vdash \Gamma, !A}\,\bar{d}$$

Short term goals :

- An indexed version of D-Dill with subtyping, with an understanding at the same time of all LPDO with constant coefficients, such that $\bar{c}(!_{\partial^i})(!_{\partial^j}) = !_{\partial^i + \partial^j}$.
- It will follows from the previous system that, syntactically, $!_D \mathbb{R}^n = \bot$ if and only if, semantically, the LPDE does not have a solution.

References

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2. *Linear Logic.* Jean-Yves Girard, Theoretical Computer Science , 1987.
3. *Differential interaction nets.* Thomas Ehrhard, Laurent Regnier, Th. Comp. Sci., 2006.
4. *A Logical Account for Linear Partial Differential Equations*, K. , 2018 *preprint*