Communication between Syntax and Semantics : Linearity and Differentiation

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- **Semantics** : interpreting terms of a language (logic, calculus) by mathematical objetcs.

- We are interested in **concrete** models. Incompleteness is good for us.

- From **Syntax** to Semantics, and vice-versa.

Context

Calculus		**Logic**		**Category**
Term		Proof		Function
Type	*Curry-Howard*	Formula	*Categorical Semantics*	Space
Application		Cut		Composition
Evaluation		Normalisation		Equality

λ-calculus and CCC's

Coherent spaces and linearity

Finiteness spaces and Differentiation

λ-calculus and CCC's

λ-calculus

$$t ::= x, y | \lambda x.t | (t)u$$

Functions and applications :

$$(\lambda x.t)(u) \rightsquigarrow_\beta t\{u/x\}$$

$$(x \mapsto x + 2)(y) \rightsquigarrow y + 2$$

Typed λ-calculus

So as to ensure strong normalization, the calculus is typed.

$$t ::= x^A, y^B | (\lambda x^A.t^B) : A \Rightarrow B | (t)^{A \Rightarrow B} u^A : B$$

Curry-Howard

The type system corresponds to minimal logic.

Example of a typing rule :

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B}$$

Cartesian Closed Categories

A cartesian closed category is a category $\mathcal{C}$ with :

- **Products** : If $A, B \in \mathcal{C}$, $A \times B \in \mathcal{C}$
- **Closedeness** : $A \times B \Rightarrow C \simeq A \Rightarrow (B \Rightarrow C)$.

Example : the category of Sets and Functions.

$$((x, y) \mapsto x + y) \simeq (x \mapsto y \mapsto x + y))$$

Cartesian categories are the language of λ-calculus

Denotational semantics : interpret programs by applications, acting on data.

$$[|x_1 : A_1, ..., x_n : A_n \vdash t : B|] = A_1 \times .. \times A_n \Rightarrow B$$

$$[|x_1 : A_1, ... \vdash \lambda x_n.t : A_n \Rightarrow B|] = A_1 \times .. \times A_{n-1} \Rightarrow (A_n \Rightarrow B)$$

Soundness The interpretation is invariant under β and η reduction.

Examples of Cartesian Closed Categories

Well, the category of sets, and ...

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Not such an easy question.

$\rightsquigarrow$ Scott Domains, some special kind of partial orders.

Coherent spaces and linearity

Coherent spaces

Built by Girard in 1987, they are a simplification of Scott domains.

A coherent space E is a reflexive undirected graph. We'll make use of its set of **cliques** :

$u \subset E$ is a clique when all its element are pairwise connected.

A **stable** function from E to F is a function from the cliques of E to the cliques of F with good properties.

Stable functions

$f : Cliques(E) \Rightarrow Cliques(F)$ is **stable** if

- **Monotonicity** : the more information you have before computation, the more information you give after.
  $f(a) \subset f(b)$ when $a \subset b$.
- **Continuity** : Preserving directed unions. A finite result comes from a finite argument.
- **Stability** : The minimal finite argument is unique.

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  $f(a \cap b) = f(a) \cap f(b)$ when $a \cup b$ is a clique.

Linearity

A linear function between groups : $f(x+y) = f(x) + f(y)$

Between graphs : $F(X_1 \bigcup X_2) = F(X_1) \bigcup F(X_2)$ and $f(\emptyset) = \emptyset$

Between Coherent spaces : it's a stable and linear function.

Fundamental Observation

A stable function is just a Linear function on the set of finite cliques.

$$X \Rightarrow Y \simeq\, !X \multimap Y$$

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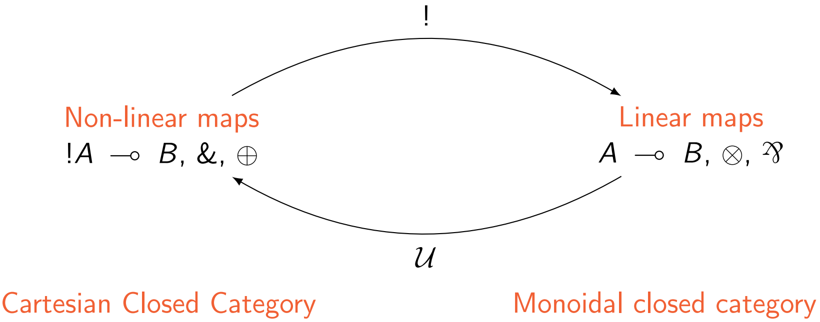
Linear Logic

$$A := 0|1|T|\bot|A \otimes B|A \parr B|A \oplus B|A \,\&\, B$$



Resource consumption

A discrete interpretation :

- A linear map is the interpretation of a proof which uses only one time its hypothesis.
- A non-linear map is the interpretation of a proof which uses a unknown but finite number of times its hypothesis.

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What about topology ? Scott domains are not even Haussdorf ..

Finiteness spaces and Differentiation

Finiteness spaces

Inspired by what happens in the relational model of Linear Logic (yet another model), Ehrhard build Finiteness spaces (2005).

A finiteness space E is a vector space, whose elements are generated by a "good" basis.

$$E = \{x = \sum_{\{a\} \in E} x_a a | x_a \in \mathbb{K}, \{a, x_a \neq 0\} \in |E|\}.$$

The "basis" $|E|$ is at most denumerable. "Good" means equals to its bi-orthogonal for a "good" orthogonality.

You endow the vector space with a linear topology : neighbourhood are sub-vector spaces.

A good orthogonality relation

In Coherent Spaces : If X is a set, $a, b \subset X$ then

$$a \perp y \textit{ iff } a \cap b \text{ contains at most one element.}$$

Then the set of cliques of X equals its bi-othogonal.

In Finiteness Spaces : If X is a set, $a, b \subset X$ then

$$a \perp y \textit{ iff } a \cap b \text{ is finite.}$$

The basis of a finiteness space verifies $|E|^{\perp\perp} = E$.

Quantitative semantics

Finiteness spaces forms a model of Linear Logic.

- **Linear maps** : Linear continous maps between vector spaces.
- **Non-linear maps** : Power series. It refines the linear vs non-linear analysis of Linear Logic.

$$P = \sum P_n$$

Vector spaces and differentiation

In analysis : The differential of a function f at a point x is the linear approximation of f at this point.

Between finiteness spaces : Functions are power series, that is $f = \sum f_n$, where f_n is n-linear.

$$f_1 \text{ is the linear approximation of } f$$

What is the meaning of differentiation on terms ? Well, differentiation is just a linear application.

Differential λ-calculus

$$\Lambda^d : S, T, U, V ::= 0 \mid s \mid s + T$$

$$\Lambda^s : s, t, u, v ::= x \mid \lambda x.s \mid sT \mid \mathrm{D}s \cdot t$$

Λ^d is the set of Differential λ-terms, Λ^s those of simple terms.

Linear and non-linear applications

We have two kind of applications, coming with two kind of reductions :

- Non-linear application $(\lambda x.s)T$ of an

$$(\lambda x.s)T \rightarrow_{\beta} s[T/x]$$

- Linear application $\mathrm{D}(\lambda x.s) \cdot t$ between two simple λ-terms.

$$\mathrm{D}(\lambda x.s) \cdot t \rightarrow_{\beta_D} \lambda x.\frac{\partial s}{\partial x} \cdot t$$

Intuition

$\frac{\partial s}{\partial x} \cdot t$ is the linear subsitution of x by t is s.

Example

$\frac{\partial (x)(x)y}{\partial x} \cdot u = (u)(x)y +$

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$\frac{\partial (x)(x)y}{\partial x} \cdot u = (u)(x)y + (Dx \cdot (u)y)(x)y$

Why ? Because :

$$f, x \mapsto f(x) \text{ is linear in } f\text{, but not in } x.$$

Conclusion

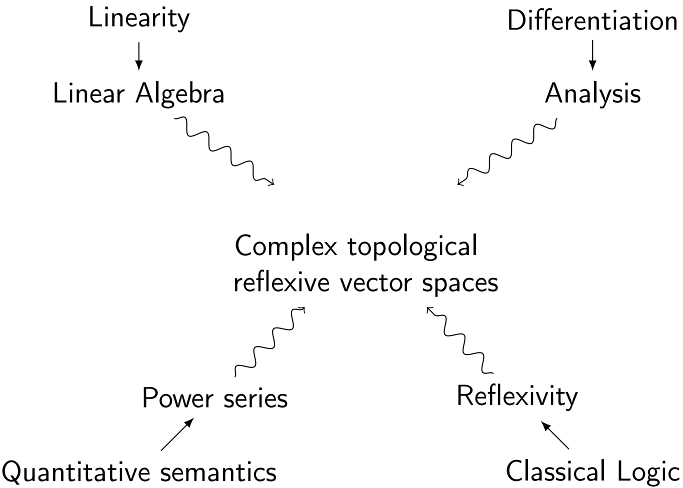
Syntax		**Semantics**
λ-calculus	Linearity	CCC's
Linear Logic	Quantitative semantics	Coherent spaces
Differential Linear Logic	Differentiation	Finiteness spaces
Differential λ-calculus		Convenient spaces Reflexive spaces

Coherent Banach Spaces

"Logic is by nature discrete ; in many situations we would like to connect its rules with analysis, i.e. with real or complex numbers."
Girard, Coherent Banach Spaces, 96

- **Objects** : $(E, E^{\perp})$ where E is a Banach space, and $E^{\perp}$ is in duality with E (a trick to interpret Classical Logic).
- **Linear maps** : Linear continuous maps between Banach Spaces.
- **Non-Linear maps** : Power series from $B_E(0,1)$ to $B_F(0,1)$. Argh !

(b-)Reflexive topological spaces



It works, thanks to the use of *bounded* sets.