A formal, classical proof of the Hahn-Banach theorem

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The state of the art in formalized classical analysis is probably the corpora available for the HOL-Light and Isabelle/HOL proof assistants \cite{4}. This topic has been much less investigated using proof assistants based on dependent type theory like Coq or Agda. By contrast, the latter make possible the investigation of formalized constructive analysis \cite{8}. In this abstract, we present and discuss a formalization of the Hahn-Banach theorem \cite{9, 2}, developed using the Coq proof assistant and based on the Mathematical Components libraries \cite{1}, extended with some axioms. The Hahn-Banach Theorem is a cornerstone of functional and convex analysis \cite{10, 7}. Here is the so-called analytical form of the theorem:

**Theorem 1. Hahn-Banach.** Consider $V$ a real vector space, $F$ a sub-vector space of $V$. Consider $p$ a convex scalar function on $V$ and $f$ a scalar linear map on $F$. There is a linear scalar map $g$ defined on $V$, majored by $p$ on $V$ and extending $f$.

The convex map $p$ is usually instantiated as a semi-norm in locally convex topological spaces, or as a norm in normed spaces, so as to allow the extension of continuous linear scalar maps. The so-called geometrical form of the Hahn-Banach theorem is a corollary of the latter analytical form. It establishes, for any affine subspace $L$ of a given topological vector space, the existence of a separating affine hyperplane containing $L$ and disjoint from $C$, an arbitrary non-empty open convex set.

The proof of Theorem 1 goes as follows. By elementary arithmetical computations, and taking benefit of the convexity of $p$, one shows that a linear partial scalar function bounded by $p$ can always be extended to a real-line not included in its domain. When the co-dimension of $F$ is finite, a recurrence on the co-dimension concludes the proof. When this co-dimension is not known, the existence of a maximal extension is established using a form of choice axiom. For instance, Rudin \cite[3.3.2]{10} considers the collection of pairs $(G, g)$, where $G$ is a vector space containing $F$ and $g$ a scalar linear map defined on $G$, extending $f$ and bounded by $p$, and the partial order:

$$(G_1, g_1) \leq (G_2, g_2) \iff G_1 \subseteq G_2 \text{ and } g_1 \leq g_2 \text{ on } G_1.$$ 

This collection is non-empty, as it contains the pair $(F, f)$, and $\leq$ is a partial order. Rudin then uses Zorn’s lemma to conclude.

Up to our best knowledge, there exists few formal proofs of this result in a proof assistant based on some form of type theory. An entry in the Journal of Formalized Mathematics\textsuperscript{1} describes its verification using the Mizar proof assistant, which is based on a typed, first-order presentation of set theory. We are also aware of a proof in the Isabelle/HOL proof assistant \cite{3}, using the Isar language. Cederquist, Coquand and Negri \cite{6} have given a constructive version of the Hahn-Banach theorem, in a point-free formulation based on formal topology, which seems to have been formalized \cite{5} in the Alf proof assistant, based on Martin-Löf Type Theory (we were not able to retrieve the code and the paper seems unpublished).

In this submission, we propose to discuss the formalization of the analytic form of the Hahn-Banach theorem in the Coq proof assistant, using the Mathematical Components libraries, extended with some axioms. The corresponding Coq file is available at the following location:

\textsuperscript{1}http://mizar.uwb.edu.pl/JFM/Vol5/hahnban.html
The standard library of the Coq proof assistant includes possible extensions which make classical principles, like choice or excluded middle, available to the user. But the resulting, dependently typed, formal language remains quite different in essence from the one used in Isabelle/HOL, HOL-Light or Mizar. The main point of this submission is to discuss the appropriate formalization choices pertaining to this classical, dependently typed setting.

In this formalization work, we make use of the formal definition of vector spaces available in the Mathematical Components libraries, and we consider a vector space $V$ on a real field $\mathbb{R}$ with a supremum operator. The main issue in this proof is to address the gradual extension of the initial linear application and the possible pitfalls related to partiality. For instance, the explicit handling of sub-vector spaces as done in the aforementioned proof by Rudin [10] can prove quite laborious.

Instead, we consider linear applications defined on the entire space $V$, and extend gradually the locus where the application is bounded by $p$. Moreover, instead of considering an order relation on subsets of $V$, we reason on subsets of $V \times \mathbb{R}$, namely on type $V \to \mathbb{R} \to \text{Prop}$. This type is used to represent the graph of the applications successively considered. We define an appropriate partial order on this type, so as to carry successfully the construction of the successive extensions.

This talk will include a discussion on the axioms added to CIC for the purpose of the proof, namely functional extentionality, propositional extentionality, propositional irrelevance and a choice axiom in the sort $\text{Prop}$. We will also discuss the relation with the constructive approach in localic spaces [6].

References