Soutenance de thèse

19 Octobre 2018

Espaces reflexifs de fontions lisses : une approche logique des equations linéaires aux dérivées partielles

Marie Kerjean

IRIF, Université Paris Diderot

THE STATUS OF NEGATION

- The **negation** $(\neg A) \Leftrightarrow (A \Rightarrow \bot)$

 : A

 : $\neg A$

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- **Classical logic** $\neg\neg A \Leftrightarrow A$.

  $\rightsquigarrow$ It is false that there is no chocolate is this house $\Leftrightarrow$ There is chocolate.

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- **Intuitionnistic logic** $\neg\neg A \not\Rightarrow A$

$\rightsquigarrow$ This does not tell me where is that chocolate !

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 $: A$  $: \neg A$

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- **Linear Logic** : Negation is *involutive* but *linear* : $A^{\perp} = A \multimap \bot$.

player : A	opponent : $A^{\perp}$
	

Interaction : A can act on $A^{\perp}$ as $A^{\perp}$ acts on A.

Linear Logic, Girard, TCS (1987)

$\rightsquigarrow$ A logic for linear algebra.

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What's a proof ?

Proving formulas

$$\frac{C \vdash A \qquad C \vdash B}{C \vdash A \wedge B}$$

$$A \vdash B \equiv \vdash A \Rightarrow B$$

Eliminating Hypothesis via the Cut-rule

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}\ \text{cut}$$

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Proofs are about more than distinguishing what's true from what is not.

p, a path towards chocolate :

CURRY-HOWARD-LAMBEK

Programs		**Logic**		**Categories**
Term	$\leftrightsquigarrow$	Proof	$\leftrightsquigarrow$	Morphisms
$\lambda x^A.t^B$		$\dfrac{\vdots}{A \vdash B}$		$f : A \to B$
Type		Formulas		Objects
Execution		Cut - elimination		Equality

Denotational semantics focuses its interests on the input output of a program

Curry-Howard-Lambek

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Curry-Howard-Lambek

Programs		Logic		Concrete categories
Term	⇝	Proof	**Context** ⇝	Smooth or Linear maps
$\lambda x^A.t^B$		$\dfrac{\vdots}{A \vdash B}$		$f : A \to B$
Type		Formulas		Topological vector spaces
Execution		Cut - elimination		Equality

Denotational semantics focuses its interests on the input output of a program

Linear logic

Usual Implication

Linear Logic

$$A \Rightarrow B = \; !A \multimap B$$

A proof is linear when it uses only once its hypothesis A.

Linear logic

Usual implication

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Exponential

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Linear logic

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A linear proof is in particular non-linear.

$$\frac{A \vdash B \text{ is linear}}{!A \vdash B \text{ is non-linear}}\ \text{dereliction}$$

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POLARIZATION

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \times B} \times \qquad\qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \otimes$$

Polarization *distinguishes between positive and negative connectives.*

Categorical Models of Linear Logic

An adjunction

$$A \Rightarrow B \simeq \,!A \multimap B$$

$$\mathscr{C}^\infty(E,F) \simeq \mathscr{L}(!E,F)$$

A linear proof is in particular non-linear.

$$\frac{A \vdash \Gamma}{!A \vdash \Gamma} \text{ dereliction}$$

$$\mathscr{L}(E,F) \hookrightarrow \mathscr{C}^\infty(E,F)$$

A strong monoidal exponential

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \times B} \times \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \otimes$$

$$!A \otimes !B \simeq \,!(A \times B)$$

Seely's isomorphism

DIFFERENTIAL LINEAR LOGIC

$$\frac{\vdash \Gamma, A^{\perp}}{\vdash \Gamma, ?A^{\perp}}\ d$$

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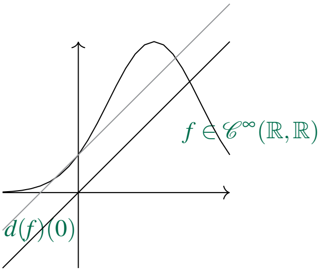
From a non-linear proof we can extract a linear proof



Differential interaction nets, Ehrhard and Regnier, TCS (2006)

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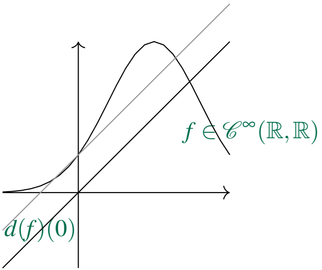

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CUT-ELIMINATION:

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$$\rightsquigarrow \qquad \frac{\vdash \Gamma, v : A \qquad \vdash \Delta, \ell : A^{\perp}}{\vdash \Gamma, \Delta, D_0(\ell)(x) = \ell(x) : \mathbb{R} = \perp}\ \text{cut}$$

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PROBLEMATIC: SMOOTH AND REFLEXIVE SPACES.

- Historical models of DILL are *spaces of sequences* : $E \subset \mathbb{K}^{\mathbb{N}}$.

Ehrhard, *Köthe spaces*, 2002. *Finiteness spaces*, 2005.

- Differentiation and analysis calls for bases-free, analytical spaces.

SMOOTHNESS

Spaces : E is a **locally convex** and **Haussdorf** topological vector space.
Functions: $f \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is infinitely and everywhere differentiable.

- DILL calls for vectorial models of **reflexive** :

$$A^\perp := A \multimap \perp \qquad (A \multimap \perp) \multimap \perp \equiv A.$$

$$E' := \mathscr{L}(E, \mathbb{R}) \qquad \mathscr{L}(\mathscr{L}(E,\mathbb{R}),\mathbb{R}) \simeq E.$$

$$\phi \in E'' \rightsquigarrow \exists x \in E, \phi = (ev_x : \ell \in E' \mapsto \ell(x)).$$

Duality is a test of functions : ℓ acts on x as ev_x acts on ℓ.

CHALLENGES

The two requirements works as opposite forces .

× A cartesian closed category with smooth functions.

$\rightsquigarrow$ Completeness, and a dual topology fine enough.

× Interpreting $(E^{\perp})^{\perp} \simeq E$.

$\rightsquigarrow$ Reflexivity, and a dual topology coarse enough.

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Convenient differential category Blute, Ehrhard Tasson Cah. Geom. Diff. (2010)

Mackey-complete spaces and Power series, K. and Tasson, MSCS 2016.

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Weak topologies for Linear Logic, K. LMCS 2015.

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- *A model of LL with Schwartz' epsilon product*, Dabrowski and K., *Preprint*.
- *A logical account for PDEs*, K., *LICS18*
- *A positive interpretation of DiLL*, K. PhD Thesis, Chapter 6.

Polarization

From mathematics to proof-theory and back.

Differential Equations

Polarization

From mathematics to proof-theory and back.

Differential Equations

Outline of this talk :

- Introduction
- Distributions and PDEs : a focus on *functions*

  ⇝ A Syntax for Distributions and LPDEs
- Mackey and weak topologies : a focus on *spaces*.

  ⇝ Smooth, classical, and polarized models of DILL
- Conclusion

The exponential as a space of distributions: from differentiation to differential operators

PhD thesis, Chapters 7 and 9

PROOFS AS DISTRIBUTIONS

- Distributions with compact support are elements of $\mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\phi_f : g \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int fg.$$

Théorie des distributions, Schwartz, 1947.

Proofs as Distributions

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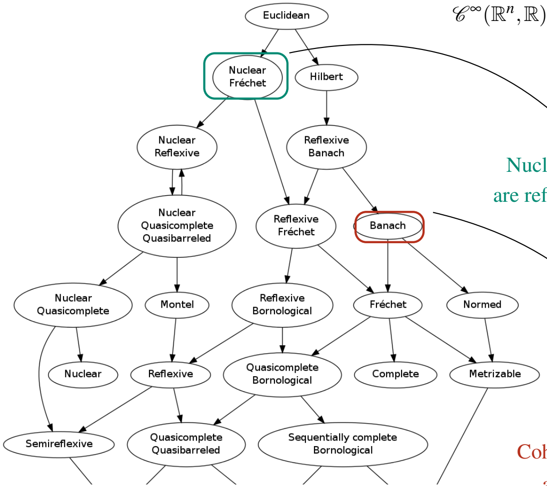
- Seely's isomorphism corresponds to Kernel theorems:

$$!A \otimes !B \simeq !(A \times B)$$
$$\mathscr{C}^\infty(E,\mathbb{R})' \tilde{\otimes} \mathscr{C}^\infty(F,\mathbb{R})' \simeq \mathscr{C}^\infty(E \times F,\mathbb{R})'$$

A quest for reflexivity

$\mathscr{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$ is not finite dimensional

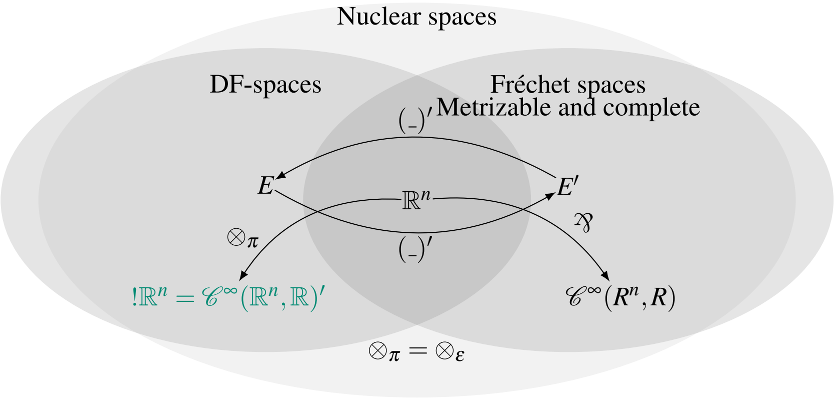


Nuclear Fréchet spaces
are reflexive and complete

Coherent Banach spaces, Girard 2004,
a *norm* is too restrictive

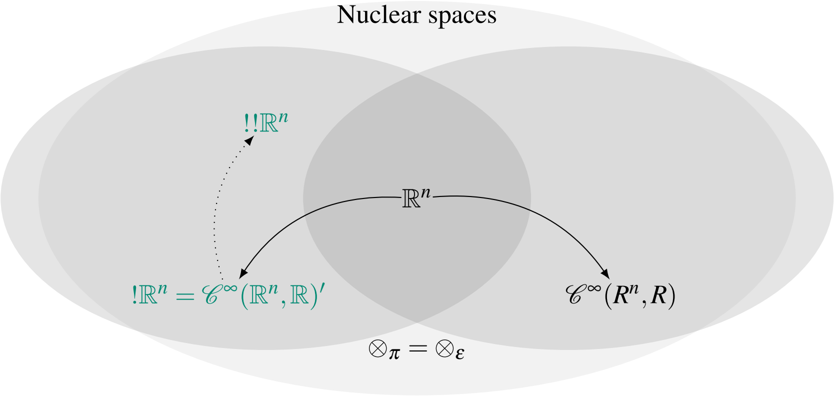
Let us take the other way around, through Nuclear Fréchet spaces.

A Smooth classical Differential Linear Logic with Distributions

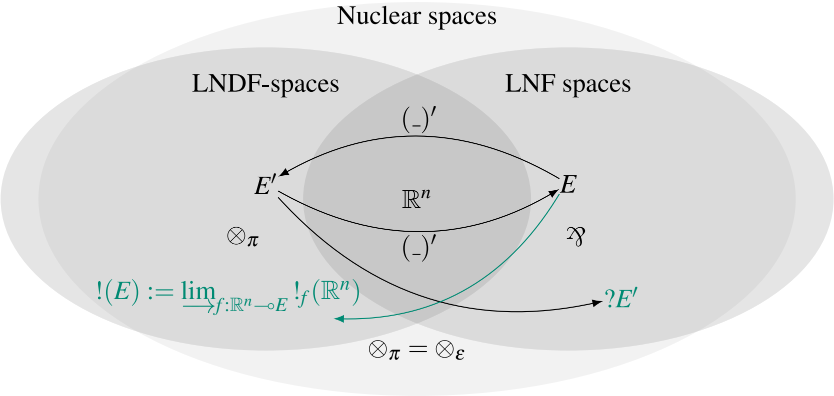


A Logical Account for LPDEs, K. LICS 2018.

Higher-order



HIGHER-ORDER



DEFINITIONS

$$?E' := \varprojlim_{f:\mathbb{R}^n \multimap E} ?_f(\mathbb{R}^n) \qquad !E := \varinjlim_{f:\mathbb{R}^n \multimap E} !_f(\mathbb{R}^n)$$

Collaboration with J.S. Lemay.

DERELICTION AND CO-DERELICTION IN A SMOOTH CLASSICAL MODEL

$$\bar{d} : \begin{cases} E \to \mathscr{C}^\infty(E, \mathbb{R})', \\ x \mapsto (f \mapsto D_0 f(x)) \end{cases}$$

$$d : \begin{cases} !E \to E \\ \psi \mapsto \psi_{|E'} \end{cases}$$

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When $E \simeq E''$

$$\bar{d} : \begin{cases} E'' \to \mathscr{C}^\infty(E,\mathbb{R})', \\ \phi = ev_x \mapsto \phi \circ D_0 = (f \mapsto ev_x(D_0(f)) \end{cases} \qquad d : \begin{cases} !E \to E'' \\ \psi \mapsto \psi_{|E'} \end{cases}$$

As $\mathscr{L}(E,\mathbb{R}) = D_0(\mathscr{C}^\infty(E,\mathbb{R}))$:

$$\bar{d} : \begin{cases} (D_0(\mathscr{C}^\infty(E,\mathbb{R})))' \to \mathscr{C}^\infty(E,\mathbb{R})', \\ \phi \mapsto \phi \circ D_0 \end{cases} d : \begin{cases} \mathscr{C}^\infty(E,\mathbb{R})' \to (D_0(\mathscr{C}^\infty(E,\mathbb{R}))' \\ \psi \mapsto \psi_{|D_0(\mathscr{C}^\infty(E,\mathbb{R})} \end{cases}$$

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Reflexivity allows to unveil a more general behaviour :

$$\bar{d}_D: \begin{cases} (D(\mathscr{C}^\infty(E,\mathbb{R}))' \to \mathscr{C}^\infty(E,\mathbb{R})' \\ \phi \mapsto \phi \circ D \end{cases} \quad d_D: \begin{cases} \mathscr{C}^\infty(E,\mathbb{R})' \to (D(\mathscr{C}^\infty(E,\mathbb{R}))' \\ \psi \mapsto \psi_{|D(\mathscr{C}^\infty(E,\mathbb{R}))} \end{cases}$$

ANOTHER EXPONENTIAL IS POSSIBLE

$$!_D E := D^{-1}((\mathscr{C}^\infty(E,\mathbb{R})') \subset (\mathscr{C}_c^\infty(E,\mathbb{R}))'$$

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- $!_{D_0} E := E'' \simeq E$.
- $!_{Id} E := !E = \mathscr{C}^\infty(E,\mathbb{R})'$.

Linear Partial Differential Equations with constant coefficient

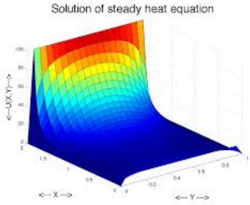
Consider D a LPDO with constant coefficients:

$$D = \sum_{\alpha, |\alpha| \leq n} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}.$$

The heat equation in $\mathbb{R}^2$

$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$

$u(x,y,0) = f(x,y)$



Theorem (Malgrange 1956)

For any D LPDOcc, there is $E_D \in \mathscr{C}_c^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})'$ such that :

$$D(E_D) = \delta_0$$

and thus : output $D(E_D * \phi) = \phi$ input

The same cut-elimination

$$\bar{d}_D : \begin{cases} !_D E \to \mathscr{C}^\infty(E,\mathbb{R})' \\ \phi \mapsto D\phi \end{cases} \qquad d_D : \begin{cases} \mathscr{C}^\infty(E,\mathbb{R})' \to !_D E \\ \psi \mapsto \psi \end{cases}$$

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !_D A}{\vdash \Gamma, D\phi : !A}\,\bar{d}_D \qquad \dfrac{\vdash \Delta, g : ?_D A^\perp}{\vdash \Delta, g : ?A^\perp}\,d_D}{\vdash \Gamma, \Delta, D(\phi)(E_D * g) : \mathbb{R} = \perp}\,\text{cut} \rightsquigarrow$$

$$\dfrac{\vdash \Gamma, \phi : !_D A \qquad \vdash \Delta, g : ?_D A}{\vdash \Gamma, \Delta, D(\phi(g)) \neq \phi(g) : \mathbb{R} = \perp}\,\text{cut}$$

THE SAME CUT-ELIMINATION

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D-DILL

DILL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, !A}\, \bar{w} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\, \bar{c} \qquad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x)!A}\, \bar{d}$$

$D - DiLL$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \int D : ?_D A}\, w_D \qquad \frac{\vdash \Gamma, f : ?A, g : ?_D A}{\vdash \Gamma, f.g : ?_D A}\, c \qquad \frac{\vdash \Gamma, f : ?_D A}{\vdash \Gamma, f * E_D : ?A}\, d_D$$

$$\frac{\vdash}{\vdash E_D : !_D A}\, \bar{w}_D \qquad \frac{\text{param.} \vdash \Gamma, \phi : !A \qquad \text{sol.} \vdash \Delta, \psi : !_D A}{\vdash \Gamma, \Delta, \phi * \psi : !_D A}\, \bar{c}_D \qquad \frac{\vdash \Gamma, \psi : !_D A}{\vdash \Gamma, D\psi : !A}\, \bar{d}_D$$

A deterministic cut-elimination.

A Logical Account for LPDEs, K. LICS 2018.

D-DILL

DILL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ d$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, !A}\ \bar{w} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\ \bar{c} \qquad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x)!A}\ \bar{d}$$

$D - DiLL$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \int D : ?_D A}\ w_D \qquad \frac{\vdash \Gamma, f : ?A, g : ?_D A}{\vdash \Gamma, f.g : ?_D A}\ c \qquad \frac{\vdash \Gamma, f : ?_D A}{\vdash \Gamma, f * E_D : ?A}\ d_D$$

$$\frac{\vdash}{\vdash E_D : !_D A}\ \bar{w}_D \qquad \frac{\text{input} \vdash \Gamma, \phi : !A \qquad \text{output} \vdash \Delta, \psi : !_D A}{\vdash \Gamma, \Delta, \phi * \psi : !_D A}\ \bar{c}_D \qquad \frac{\vdash \Gamma, \psi : !_D A}{\vdash \Gamma, D\psi : !A}\ \bar{d}_D$$

A deterministic cut-elimination.

A Logical Account for LPDEs, K. LICS 2018.

Smooth and classical models of DILL via polarization

PhD thesis, Chapters 4, 5, 6.

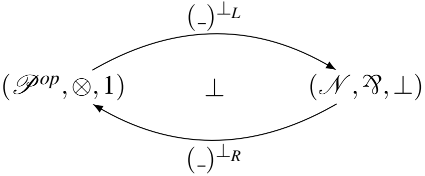
Polarized Linear Logic

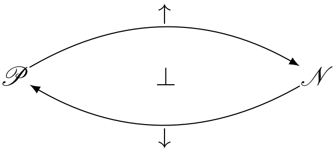
Syntax

Negative Formulas: $N, M := a \mid ?P \mid N \parr M \mid \bot \mid N \,\&\, M \mid \top \mid$

Positive Formulas: $P, Q := a^\perp \mid !N \mid P \otimes Q \mid 0 \mid P \oplus Q \mid 1$

Semantics for polarized MLL : Mellies Chiralities

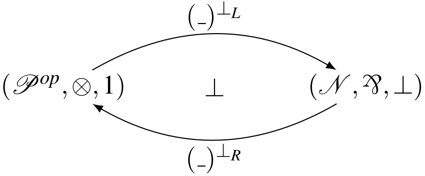


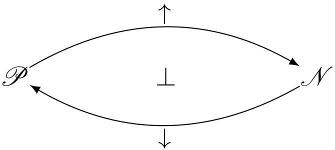


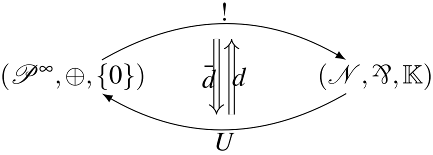
We do not ask for an equivalence but : $N^{\perp_R \perp_L} \simeq N$.

POLARIZED LINEAR LOGIC

Semantics for polarized DiLL

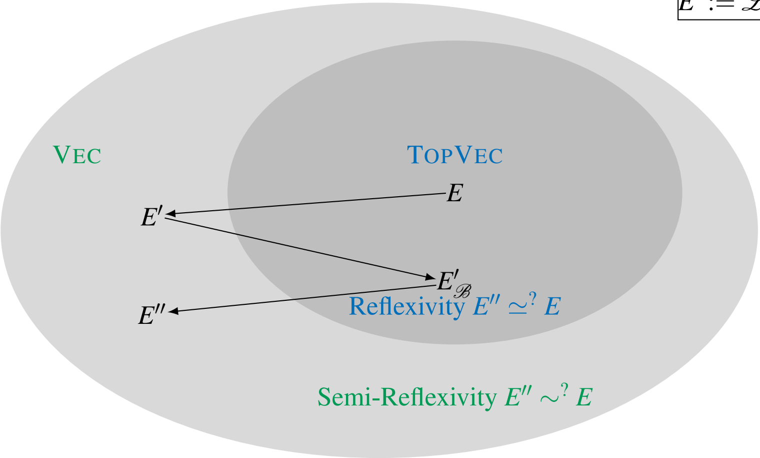






Duality in topological vector spaces

$E' := \mathscr{L}(E, \mathbb{R})$



MLL in TopVect

It's a mess.

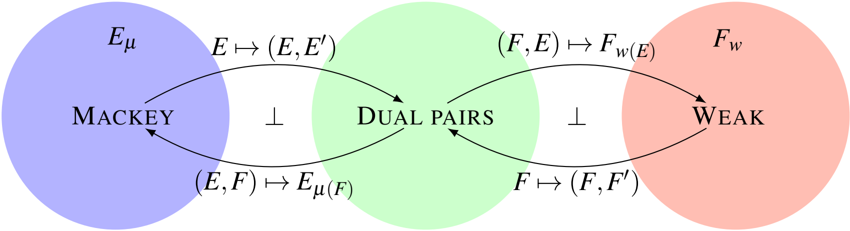
Duality is not an orthogonality in general :

- It depends of the topology E'_β, E'_c, E'_w, E'_μ on the dual.
- It is typically *not* preserved by $\otimes$.
- It is in the canonical case not an orthogonality E'_β is not reflexive.

Monoidal closedness does not extends easily to the topological case :

- Many possible topologies on $\otimes$: $\otimes_\beta$, $\otimes_\pi$, $\otimes_\varepsilon$.
- $\mathscr{L}_{\mathscr{B}}(E \otimes_{\mathscr{B}} F, G) \simeq \mathscr{L}_{\mathscr{B}}(E, \mathscr{L}_{\mathscr{B}}(F, G))$
  $\Leftrightarrow$ "Grothendieck problème des topologies".

THE MACKEY-ARENS THEOREM, BY BARR

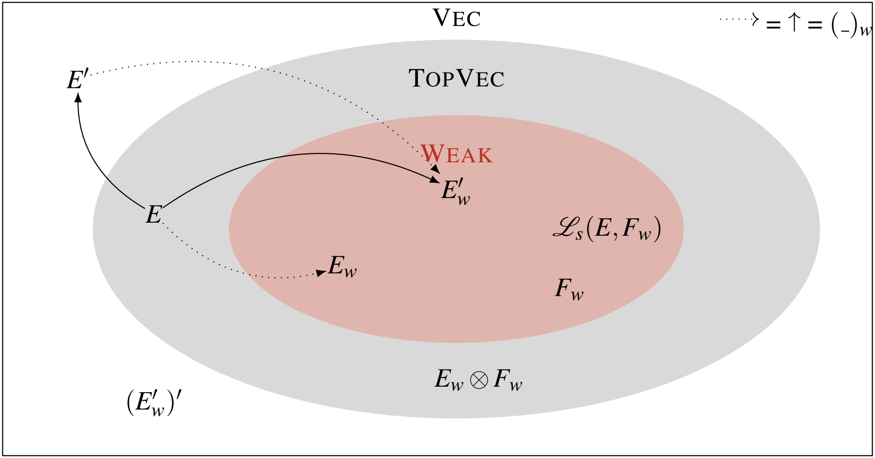


$$\mathscr{L}(E_\mu, F) = \mathscr{L}(E, F_w)$$

On $*$-autonomous categories of topological vector spaces, M. Barr Cahiers Topologie Géom. Différentielle Catég., 2000.

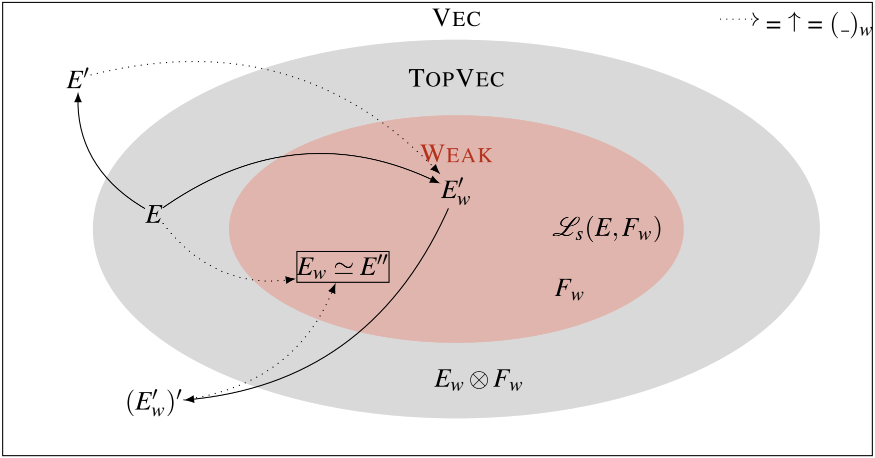
On convex topological vector spaces, G. Mackey, Trans. Amer. Math. Soc., 1946.

Weak spaces, a negative interpretation of DiLL



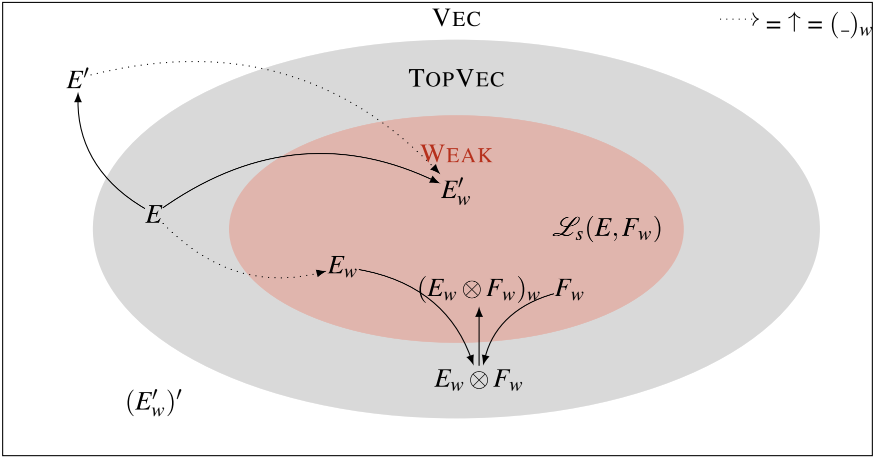
Weak spaces, a negative interpretation of DiLL

Weak spaces are reflexive, when the dual is the weak dual.



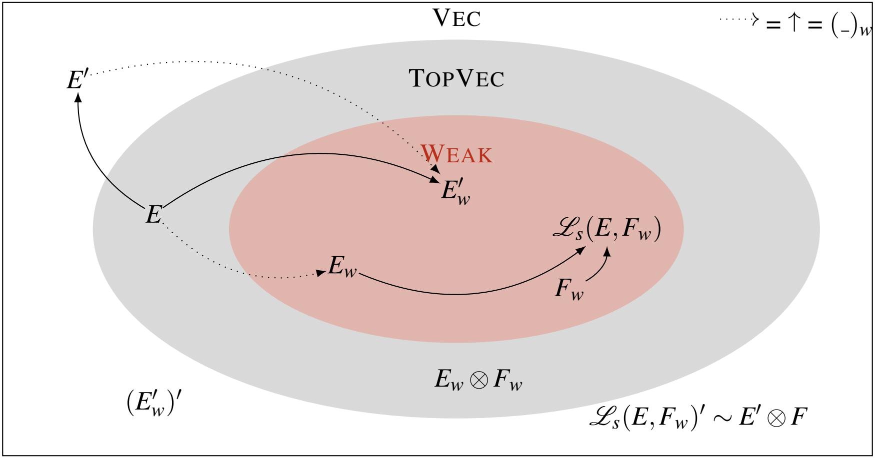
WEAK SPACES, A NEGATIVE INTERPRETATION OF DILL

The tensor product needs to undergo a shift to be a weak space.



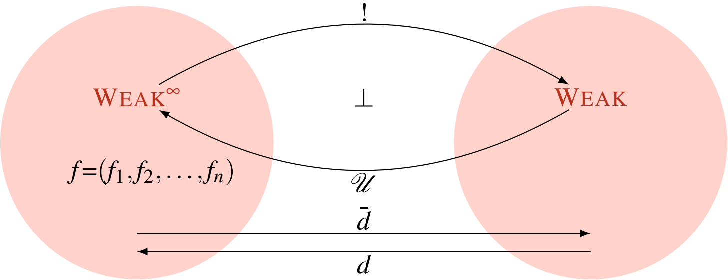
Weak spaces, a negative interpretation of DiLL

The ⅋ is internally endowed with its weak topology.



Weak spaces, a quantitative interpretation of DiLL

Without completeness, no smooth exponential is possible.



$$\mathscr{L}(!E, \mathbb{K}) \simeq \prod_n \mathbb{H}^n(E, \mathbb{K})$$

Weak topologies for Linear Logic, K. LMCS 2015.

MODELS BASED ON $⅋=\varepsilon$, AN UNPOLARIZED SETTING

Models of Linear Logic based on the Schwartz' ε product Y. Dabrowski and K.

A FIRST MODEL OF DILL : K-REF

- $((E'_c)'_c)'_c \simeq E'_c$.
- Associativity of ε through a minimal completeness condition.

SMOOTH FUNCTIONS WITH PARAMETERS IN $\mathscr{C} \subset$ K-REF

$\mathscr{C}^{\infty}_{\mathscr{C}}(E,F) :=$
$\{f : E \to F, \forall X \in \mathscr{C}, \forall c \in \mathscr{C}^{\infty}_{co}(X,E) \Rightarrow f \circ c \in \mathscr{C}^{\infty}_{co}(X,F)\}$

A NEW INDUCED TOPOLOGY

Dereliction : $E \hookrightarrow \mathscr{C}^{\infty}_{\mathscr{C}}(E'_{\mu}, \mathbb{R})$,
induces a new topology $\mathscr{S}_{\mathscr{C}}(E)$ on E.

Models based on $\parr = \varepsilon$, an unpolarized setting

Models of Linear Logic based on the Schwartz' ε product Y. Dabrowski and K.

Then when E is Mackey-complete :

$\mathscr{C}$	$\mathscr{S}_{\mathscr{C}}(E)$
Fin	The Schwartzification of E
Ban	The Nuclearification of E
$\{0\}$	The weak topology on E

Smooth and classical models of LL

- k-complete spaces with Arens reflexivity.
- Schwartz Mackey-complete spaces with Mackey reflexivity.
- Nuclear Mackey-complete spaces with Mackey reflexivity.

MODELS BASED ON $⅋=\varepsilon$, AN UNPOLARIZED SETTING

Models of Linear Logic based on the Schwartz' ε product Y. Dabrowski and K.

SMOOTH AND CLASSICAL MODELS OF LL

- k-complete spaces with Arens reflexivity.
- Schwartz Mackey-complete spaces with Mackey reflexivity.
- Nuclear Mackey-complete spaces with Mackey reflexivity.

DIFFERENTIATION: ONE DRAWBACK

Differentials are bounded in general, and we need an ad-hoc definition to have continuous differentials.

ONE GENERAL QUESTION

What's the different role of completeness and $\mathscr{S}_{\mathscr{C}}$?

KEEP CALM AND POLARIZE EVERYTHING

Convenient differential category Blute, Ehrhard Tasson Cah. Geom. Diff. (2010)

$\rightsquigarrow$ Mackey-complete and bornological spaces : a smooth model of IDILL.

γ = topology of bornivorous subsets

BORNVEC $\perp$ TOPVEC

β = bornology of the subsets absobed by 0-neighbourhood

Keep calm and polarize everything

Convenient differential category Blute, Ehrhard Tasson Cah. Geom. Diff. (2010)

$\rightsquigarrow$ Mackey-complete and bornological spaces : a smooth model of IDiLL.

Mackey-complete spaces and Power series, K. and Tasson, MSCS 2016.

$\rightsquigarrow$ Mackey-complete spaces are enough for IDiLL.

Models based on ε, Dabrowski and K., 2017

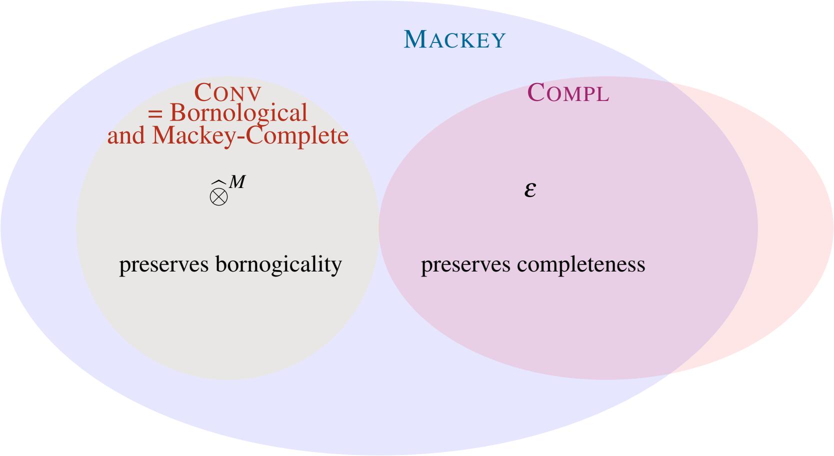
$\rightsquigarrow$ Mackey-complete and Schwartz or Nuclear spaces have duals with their Mackey topology.

Stability properties

Mackey topologies = positives Completeness = negatives.

MACKEY SPACES

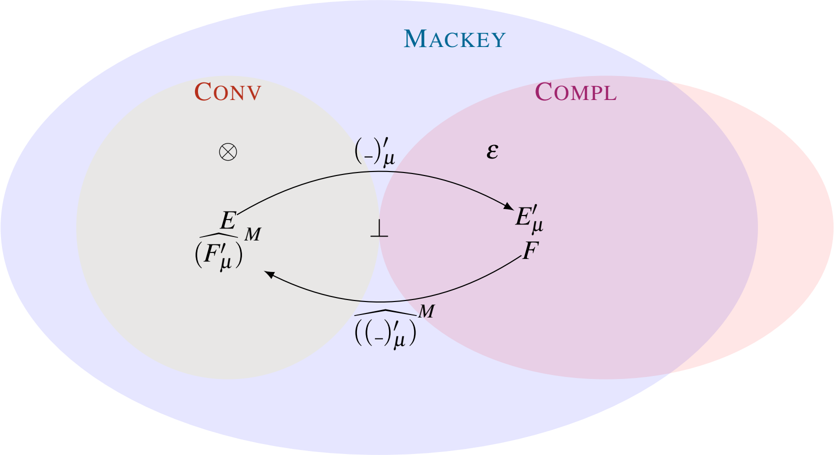
A model of MALL



Chapter 6.

Mackey spaces

A polarized model of MALL, with two negations

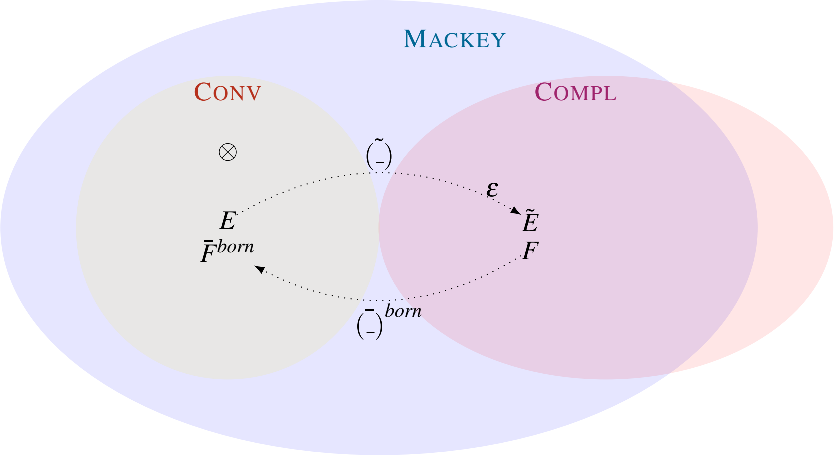


Chapter 6.

Mackey spaces

A polarized model of MALL, with shifts

although we do not have an adjunction.



Conclusion

Polarization

From mathematics to proof-theory and back.

Differential Equations

Research directions from now

- Extending the positive interpretation of Mackey spaces to distributions.
- A better higher-order for distributions.
- A categorical setting for polarized models of DiLL.
- An indexed proof sequent for differential operators.
- A deterministic differential calculus.

Conclusion

Polarization

From mathematics to proof-theory and back.

Differential Equations

Research directions from now

Let's go towards **non-linearity** and finer interpretations of LPDEs.

Thank you.