The computational meaning of differentiation

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The Differential λ-calculus

Terms of the λ-calculus can be linearized [1]: one has an operator D, such that $D(\lambda x.t)$ is the linearization of the program $\lambda x.t$.

$$\mathrm{D}(\lambda x.s) \cdot t \to_{\beta_D} \lambda x. \frac{\partial s}{\partial x} \cdot t.$$

in which $\frac{\partial s}{\partial x} \cdot t$ is the linear substitution of x by t in s. For example :

$$\frac{\partial y}{\partial x} \cdot T = \begin{cases} T \text{ if } x = y \\ 0 \text{ otherwise} \end{cases}$$

$$\frac{\partial}{\partial x}(su) \cdot v = (\frac{\partial s}{\partial x} \cdot v)u + (\mathrm{D}s \cdot (\frac{\partial u}{\partial x} \cdot v)u)$$

$\frac{\partial s}{\partial x} \cdot t$ is the sum of all possibilities for substituting linearly x by t in s.

Deterministic differential calculi

Computation	$\rightsquigarrow$	Object
Type A	$\rightsquigarrow$	Space A
λ-term $A \to B$	$\rightsquigarrow$	Function f from A to B
$\lambda x.xt$	$\rightsquigarrow$	Linear function
$D(\lambda x.t)$	$\rightsquigarrow$	Linear local approximation of f

In $\frac{\partial s}{\partial x} \cdot t$ we choose to subsitute by t the first occurrence of x encountered in the reduction process.

It leads to different determinisitic differential λ-calculi, depending on the chosen reduction strategy.For example, in a call-by-name reduction :

$$\left(\frac{\partial(\lambda z.xz)x}{\partial x}u\right)_n \to \left(\frac{\partial(x)x}{\partial x}u\right)_n \to ((u)x)_n$$

and in a call-by-value reduction :

$$\left(\frac{\partial(\lambda z.xz)x}{\partial x}u\right)_v \to \left((\lambda z.\frac{\partial xz}{\partial z}u)x\right)_v.$$

$$\frac{\partial s}{\partial x} \cdot t = \sum_\sigma \left(\frac{\partial s}{\partial x} \cdot t\right)_\sigma$$

where σ ranges over the reduction paths distinguishing over x.

These deterministic differential calculi generalize to a differential version of the enriched effect calculus, in which they embed. The enriched effects calculus distinguishes computations from values, with a typing system admitting a linearity condition. We get a general notion of what is the differentiation of a computation.

Linearity and Differentiation

A program is linear in a data x if it uses x only once during a computation.

Differentiating a program corresponds to making it linear in its input.

Towards differential equations

We want to compute the solutions of a differential equation of λ-terms. This is done by

- Understanding what the differentiation of a program really is, and in particular give meaning to the sum of terms in a linear substitution.
- Modeling Differential Linear Logic with objects from rich theoretical fields, such as differential geometry.

References

[1] Ehrhard, Regnier, The differential λ-calculus, 2004
[2] Blute, Ehrhard, Tasson, A convenient differential category, 2012
[3] K., Tasson, Mackey-complete spaces and power series - A topological model of Differential Linear Logic, preprint, 2015
[4] K., Weak topologies for Linear Logic 2015
[5] Egger, Møgelberg, Simpson, The Enriched Effect Calculus: Syntax and Semantics, 2012

Models of differential linear ogic

Differential Linear Logic is a Logic where proofs can be linear. It can be used to type the differentiation of a term.

In historical models of Differential Linear Logic, proofs are represented by power series $P = \sum_k P_k$ where P_k is k-linear. The differential typing rule is then represented by :

$$D_0 : P = \sum P_k \mapsto P_1$$

One would want smooth models of this logic, where the differential of proofs is the usual one from differential geometry

Smooth models of differential linear logic

Linear programs	$\rightsquigarrow$	Vector spaces
Differentiation of programs	$\rightsquigarrow$	Topologies
Models of simply-typed λ-calculus	$\rightsquigarrow$	Cartesian closed category
Involutive linear negation	$\rightsquigarrow$	reflexive spaces

Modelling Differential Linear Logic with smooth models has two main difficulties :

- It requires to have a good differentiation theory for infinite dimensional spaces : we want to have a cartesian closed category of spaces of functions.
- It asks for a closed category of reflexive vector spaces : modelling the involutive linear negation necessitates spaces which are isomorphic to their double dual.

Good old Banach spaces don't work : spaces of functions are not easily Banach spaces.

Solutions for the first point are for example Mackey-complete topological vector spaces [2] [3]. A trick for having a closed category of reflexive spaces is for example the use of weak topologies [4].

Nuclear spaces

A special category of topological vector spaces are the Nuclear Fréchet spaces. They have almost all the ingredients for being a good model of Differential Linear Logic, and are good candidates for spaces in which differential equations can be used.