Internship report

Convenient Spaces and Real Analytic Maps Between Them

Under the direction of R. Blute,
At Ottawa Logic and Foundations of Computing Group,
University of Ottawa, Canada.
Marie Kerjean - M1 mathematics at the ENS Lyon - marie.kerjean@ens-lyon.fr

Introduction

This internship was realized as a completion of my first year of Master at the ENS Lyon. It lasted six weeks, from the begining of June 2012 to mid July 2012. It was a very good experience. It was a really interesting opportunity for me to learn about subjects I've always been willing to aproach, and to try and find original solutions to a problem. Hence, these six weeks felt too short. I have learned a lot and also produced a litltle, and I'm eager to do more some day. Of course, I would like to thank R. Blute for everything he taught me : he is a remarkable supervisor.

The story of Linear Logic (LL) is paved with successful exchanges between syntax and semantic. LL itself comes from a semantic perspective : a study of coherent spaces by Girard in [Gir87]. Studying topological models of Linear Logic, Ehrard discovered in [Ehr02, Ehr05] that is was perfectly possible to supplement lambda-calculus and linear logic with linear applications and differential constructions. Differential Linear Logic (DiLL) was then formalized, and models of it were constructed. Recently, Richard Blute, Christine Tasson and Thomas Ehrhard found out in convenient spaces (see [BET10]) a particularly relevant differential category which is also a model of Intuitionistic Linear Logic (hence, very close to a model of DiLL). The important point is the amount of analysis you can do on convenient spaces (as the book by Kriegl and Michor [KM97] shows) : we are finally reaching a model of DiLL where the differentation on interpretations of proofs is almost the one we have always worked with in mathematics.

The aim of this internship was to understand and maybe work on [BET10]. Hence, we tried to extend the result of the article to real analytic maps, and to understand the possible links between this model and the syntax of ressource calculus and differential lambda-calculus. The first goal was achieved, but the second is still on hold. In this report, I will first state the rules of LL and $DiLL$, as well as the definition of a differential category. Then, I will explain what convenient spaces are and detail some of the constructions on them, referring to [FK88] a little, but mainly to [KM97]. In the third and most important part of the report I will build on convenient spaces the constructions necessary to a model of Intuitionistic Linear Logic (ILL) and to a differential category. Finally, I will discuss whether convenient spaces can form a model of $DiLL$.

The proofs and constructions presented in the third part are very close to the one dones in [BET10]. The only differences come from the variations between the definitions by Frölicher and Kriegl and those by Kriegl and Michor, some particular points on real analytic maps, and the discussion on models of $DiLL$.

Contents

1 Differential categories

In this first part, we will define differential categories. They were introduced by Blute, Cockey and Seely in [BCS06], after the introduction by Ehrhard and Regnier of differential λ-calculus ([ER03, ER06]). In this calculus, there is a linear application, working as a mathematical derivation : $D.\lambda x.s.t \to_{\beta_D} \lambda x.\frac{\partial s}{\partial x}.t$. As can be imagined, differential categories somehow capture the semantics of a linear application. That is why the notion of derivation in differential categories should meet with the traditional meaning of derivation : from a smooth function f, we would like to obtain at least its derivative at 0, a linear function. That is, in topological models of linar logic (finiteness spaces, Köthe spaces, convenient spaces), we would like to obtain $df(0) : A \to B$ from $f : !A \to B$.

1.1 Differential Linear Logic

In the first figure are the sequent calculus rules for Linear Logic (Figure 1). We do recall also the rules of Intuitionnistic Linear Logic (Figure 2), constructed with a one sided sequent version of the multiplicative group of Linear Logic. In the search of denotational models for DiLL, it will be useful to know that here the use of negation is avoided thanks to two-sided sequent, and $\parr$ disapear in order to give birth to $\multimap$ as a connector. In the Figure 3 we expose the presentation of Differential Linear Logic one can find in Tasson PhD thesis [Tas09] or in Ehrhard introduction to DiLL [Ehr11]. Note the existence of a vector group of rules, coming from the fact that in Differential lambda-calculus, the linearity of some applications is intuitively traduced by a non-deterministic reduction (see [ER03]).

Note also that : $\dfrac{\Gamma \vdash A}{\Gamma \vdash !A}$

allows us to obtain, through composition to the left, a linear application $A \to B$ from a common application $!A \to B$.

Then, in a categorical model of linear logic, we'll look for a way to go from $f : !A \to B$ to $Df : !A \to (A \Rightarrow B)$. If we want to generalize this condition to non closed categories : $Df : A \otimes !A \to B$.

1.2 Differential categories

In [BCS06], differential categories were built to modelize more than differential λ-calculus or differential linear logic, so they are neither closed nor $\star$-autonomous. Note for instance that the fundamental category of finite-dimensional real vector spaces and smooth maps between them is not closed at all, even though we would like them to be modelized by some differential category. Precisely, the purpose

- Identity group :

$$\frac{}{\vdash A, A^{\perp}}\ \text{axiom} \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ \text{cut}$$

- Multiplicative group :

$$\frac{}{\vdash 1}\ 1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp$$

$$\frac{\vdash \Gamma, A_1, A_2}{\vdash \Gamma, A_1 \parr A_2}\ \parr \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes$$

- Additive group :

$$\frac{}{\vdash \Gamma, \top}\ \top \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \&$$

$$\frac{\vdash \Gamma, A_1}{\vdash \Gamma, A_1 \oplus A_2}\ \oplus_L \qquad \frac{\vdash \Gamma, A_2}{\vdash \Gamma, A_1 \oplus A_2}\ \oplus_R$$

- Exponential group :

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \text{contraction} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \text{Weakening}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \text{Derilection} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ \text{Promotion}$$

Figure 1: Rules of LL

$$\frac{}{A \vdash A^{\perp}}\ (\text{axiom}) \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2, A \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash, \Delta}\ (\text{cut})$$

$$\frac{}{\vdash 1}\ (1_R) \qquad \frac{\Gamma \vdash A}{\Gamma, 1 \vdash A}\ (1_L)$$

$$\frac{\Gamma_1, \vdash A \qquad \Gamma_2, B \vdash \Delta}{\Gamma_1, \Gamma_2, A \multimap B \vdash \Delta}\ (\multimap_L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta}\ (\multimap_R)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ (\otimes_L) \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \otimes B, \Delta_1, \Delta_2}\ \otimes_R$$

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ \text{Promotion} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ \text{Dereliction}$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \text{Weakening} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \text{Contraction}$$

Figure 2: Rules of Intuitionistic Linear Logic

- Identity group :

$$\frac{}{\vdash A, A^{\perp}} \text{(axiom)} \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta} \text{(cut)}$$

- Multiplicative group :

$$\frac{}{\vdash 1} (1) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp} (\perp)$$

$$\frac{\vdash \Gamma, A_1, A_2}{\vdash \Gamma, A_1 \parr A_2} (\parr) \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} (\otimes)$$

- Exponential group :

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} \text{(contraction)} \qquad \frac{\vdash \Gamma, !A, !A}{\vdash \Gamma, !A} \text{(co-contraction)}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \text{(weakening)} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, !A} \text{(co-weakening)}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \text{(derilection)} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A} \text{(co-derilection)}$$

Figure 3: Rules of DiLL (see [Ehr11] for a complete presentation of *DiLL*)

of the study of convenient spaces by Frölicher and Kriegl in [FK88] was to find a cartesian closed category on which to do a substantial amount of analysis.

Definition 1. A differential category is a symmetric monoidal category, with an additive law on the Hom-sets, a coalgebra modality, and a differential combinator.

A coalgebra modality is a comonad $(!, \rho, \epsilon)$ such that each object $!X$ is equipped with a good co-algebra structure,of which we won't give the details. See [BCS06] for a precise explanation, and recall that we need two arrows for each $!X$: $\Delta : !X \to !X \otimes !X$ and $e : !X \to \mathcal{I}$, as well as an arrow : $\phi : !A \otimes !B \to !(A \otimes B)$, $\phi : 1 \to !1$.

A way to have an additive structure on the Hom-sets is to require a biproduct law on the category, and then arrows : $\triangledown : !X \otimes !X \to !X$ and $\nu : \mathcal{I} \to !A$. We'll use biproducts for the differential categoriy on convenient spaces (Con_ω).

Let's study the differential operator. It's supposed to be a combinator which naturaly transforms $f : !A \to B$ into $Df : A \otimes !A \to B$. In fact, it's enough for us to have, for each object A, a deriving transformation $d_A : A \otimes !A \to !A$ which would correspond to $D[1_{!A}]$. Indeed, suppose we have a differential operator which is natural in A and in B for each object A, B . Then if the first diagram commutes, the second will :

$$\begin{tikzcd}
!A \arrow[r, "f"] \arrow[d, "!u"'] & B \arrow[d, "v"] \\
!C \arrow[r, "g"'] & D
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes !A \arrow[r, "D[f["] \arrow[d, "u\otimes !u"'] & B \arrow[d, "v"] \\
C \otimes !C \arrow[r, "D[g["'] & D
\end{tikzcd}$$

Write d_A for $D[1_{!A}]$. Then one has two commuting diagrams :

$$\begin{tikzcd}
!A \arrow[r, "!1"] \arrow[d, "!1"'] & !A \arrow[d, "f"] \\
!A \arrow[r, "f"'] & B
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes !A \arrow[r, "d_A"] \arrow[d, "1\otimes !1"'] & !A \arrow[d, "f"] \\
A \otimes !A \arrow[r, "D[f["'] & B
\end{tikzcd}$$

and then can get d_A from D and D from d_A for all objects A. If biproducts exists, things are even simpler, as d_A can be written as $d_A = (coder_A \otimes 1); \triangledown$, where $coder_A : A \to !A$.

Of course, we'll want D to be additive, null on constant maps, to be the identity on linear maps and to verify the chain rule and Leibniz rule. According to Fiore (see [**?**]), it's summarized in the two following diagrams on $coder_A$:

- Strenght :

$$\begin{tikzcd}
A \otimes !B \arrow[rr, "coder_A \otimes 1"] \arrow[drr, "1 \otimes \epsilon_A"] & & !A \otimes !B \arrow[rr, "\phi"] & & !(A \otimes B) \\
& & A \otimes B \arrow[urr, "coder_{A \otimes B}"] & &
\end{tikzcd}$$

- Comonad :

$$\begin{tikzcd}
A \arrow[r, "coder_A"] \arrow[d, "1"'] & !A \arrow[dl, "\epsilon"] \\
A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "coder_A"] \arrow[d, "\simeq"] & !A \arrow[r, "\rho"] & !!A \arrow[d, "\triangledown"] \\
A \otimes 1 \arrow[r, "coder_A \otimes \nu"] & !A \otimes !A \arrow[r, "coder_A \otimes \rho"] & !!A \otimes !!A
\end{tikzcd}$$

Again, see [BCS06] for a complete presentation. Following this definition, Blute Cocket and Seely explain how differentials categories capture the smooth structure of Vec_K on Vec_K^{op}, beyond other simpler differentiation theory. After developing more on the adaptation of differentiation theory to category theory, the authors define the notion of categorical model of the differential calculus, which a differential category whose co-algebra modality is a storage modality, *i.e.* a comonad who modelize the exponential laws of Linear Logic.

2 Convenient vector spaces

The first goal of Frölicher and Kriegl in [FK88] was to find cartesian closed categories on which substantial analysis could be done. Even though the requirement of cartesian-closedness seems to be reasonable for a category, it is definitely uncommon. Neither the category of real vector spaces and linear morphisms, nor the one of smooth manifolds and smooth maps have all exponentials. Beyond being cartesian closed, the category of convenient spaces and bornological maps between them is also symmetric monoidal closed, and can be provided with a comonad whose coKleisli category would be the category of smooth maps. This will lead to the construction of a model of linear logic. Convenient spaces being very easy to work with, it will not be difficult to fit a differential structure on them, and build a model of $DiLL$.

2.1 Introduction to convenient vector spaces

In this section, I will present convenient vector spaces as they are explained in the two books [FK88] and [KM97]. [FK88] was written by Frölicher and Kriegl in 1988, whereas [KM97] was written in 1997 by Kriegl and Michor.

Bornologies

A locally convex vector space E is a topological vector space whose topology is locally convex. Here, every locally convex vector space comes with a *bornology*, *i.e.* a set of sets containing the finite ones which would be the bounded sets.

Definition 2. In a locally convex vector space, a set will be call *bounded* if it's absorbed by each 0-neighborhood.

Once you have a bornology on E, you can define an associated topology, whose 0-neighbourhood basis is the set of absolutely convex bornivorous disk of E. E will then be called bornological if this topology is the same as the initial one. Going from the least complete to the most detailled, see 52.1 in [KM97], section 1.2 in [FK88], or [HN77] for further explanations.

Here we have a hitch. Frölicher and Kriegl clearly ask to a convenient space to be a bornological vector space, whereas Kriegl and Michor don't. Moreover, in 2.14 of [KM97], it is stated that this book won't require the bornological topology, and said that the results of the two books are equivalent but non-isomorphic categories. Note that if the topology on E is bornological, then the notion of convergence on E is the same than the notion of Mackey-convergence, which is defined next. F We won't ask a convenient space to have a bornological topology. Indeed, everything seems to work without that hypothese, and moreover we will use the same topology on $\mathcal{C}^\infty(\mathbb{R}, E)$ than the one defined in [KM97], and I didn't prooved that if E is endowed with a bornological topology, then so is $\mathcal{C}^\infty(\mathbb{R}, E)$.

As we are using results from both books here, and without any demonstration of the equivalence, we will require to convenient spaces to be bornological.

Mackey-convergence

For various reason, some of them being found in section 2.2, we will want to work with bornologies rather than with topologies, or at least to consider a convergence criterium closer to the essential. Notice for example that in a locally convex vector space, the difference quotient $\frac{1}{t}(\frac{1}{t}((c(t) - c(0)) - c'(0))$ of a Lip^1 curve c takes bounded sets to bounded sets (see 1.7 of [KM97]).

Furthermore, locally convex vector spaces are a particularly relevant start point for a good category in analysis. We have a scalar testing for bounded sets (see 52.19 of [KM97]), and they fits with the testing on smooth curves. See 2.11 of [KM97] : A linear mapping $l : E \to F$ between locally convex vector spaces maps bounded sets to bounded ones iff it maps smooth curves in E to smooth curves in F. Such a linear mapping will be called a *bornological* one.

We will hence work with a particular notion of convergence and completeness, linked to bornologies. A sequence $(x_n)_{n\in\mathbb{N}}$ in E will be called *Mackey-Cauchy* if there is real numbers $(\lambda_n)_{n\in\mathbb{N}}$ decreasing to 0 and a bounded set B such that $x_n \in \lambda_n B$ for each n. Easily enough, a space is then *Mackey-complete* when every Mackey-Cauchy sequence converges. This definition leads to an important property, described just after the even more important definition :

Definition 3. A convenient space is a locally convex vector space which is Mackey-complete.

Proposition 1. Consider E a locally convex vector space. The two following assertions are equivalent :
(i) E is Mackey-complete.
(ii) Consider c a curve $c : \mathbb{R} \to E$. Then c is smooth iff for all linear smooth map $l : E \to \mathbb{R}$, $l \circ c : \mathbb{R} \to \mathbb{R}$ is smooth.

Boman's Theorem

Let us denote Con the category of convenient spaces and bornological maps between them.

The fundamental idea of Frölicher and Kriegl in [FK88] was to look at Boman theorem, a particular case of which being prooved in theorem 3.4 of [KM97]).

Theorem 1. Consider f a function from $\mathbb{R}^n$ to $\mathbb{R}$. Then f is smooth iff $f \circ c$ is smooth on $\mathbb{R}$ for every $c \in C^\infty(\mathbb{R}, \mathbb{R}^n)$.

Frölicher decided to take the smoothness on smooth curves as the definition of smoothness at every level of the construction of convenient spaces. Kriegl and Michor will do the same in [KM97] for holomorphic and real analytic, which are the point of this part.

Categorical work

The point of my internship was to extend what had been done on smooth maps in [BET10] to real analytic maps. I will summarize in this part the work of Blute and Tasson on the category of convenient spaces and smooth maps.

Given the category of conventient spaces and bounded linear maps between them, the authors of [BET10] built a comonad $!_\infty$ on it, so as to have as a co-Kleisli category the category of convenient spaces and smooth maps between them. We will denote $\mathcal{C}$ the category of convenient spaces and bornological linear maps between them, and by $\mathcal{C}^\infty$ coKleisli category of $!_\infty$.

The first objective is to built a model of intuitionnistic linear logic. Happily enough, a big part of the work has been done in [FK88]. They have to modify a little bit the usual tensor product on vector spaces so as to find in $\mathcal{C}$ a symmetric monoidal categories. The closedness isn't essential to a be a differential category, but fundamental in the definition of $!_\infty$, and to approach a model of intuitionnistic linear logic. Hence, $\mathcal{C}^\infty$ is cartesian closed, Seely Law is verified between $\mathcal{C}$ and $\mathcal{C}^\infty$ (Lemma 5.2.4 of [FK88]), and a biproduct is constructed (Lemma 2.5.6 of [FK88]). $\mathcal{C}$ equiped whith $!_\infty$ happen then to be a model of intuitionnistic linear logic
Concerning the differential part, the classical notion of derivation of a smooth appear to be a good derivation operator, and makes $\mathcal{C}$ a differential category. We found as a conclusion in $\mathcal{C}$ equipped with $!_\infty$ a model of DiLL.

2.2 Real analytic maps on convenient vector spaces

In all this section, E or F denotes a real convenient space. Moreover, E' refers to the set of bornological linear map $l : E \to \mathbb{R}$, and not to the set of continuous linear map from E to $\mathbb{R}$.

Smoothness

The study of holomorphic maps and real analytic maps between convenient vector spaces is done by Kriegl and Michor in the second chapter of [KM97]. A lot of results on real analytic maps come from those on holomorphic maps, by Lemma 9.5 in [KM97] : a curve is topologically real analytic in a convenient space iff in the complexification of his target space, it extends locally in an holomorphic maps.

We will recall here the major results of [KM97] which will lead to the construction of another differential category and model of the intuitionistic logic, with real analytic maps.

Definition 4. A curve $c : \mathbb{R} \to E$ is said to be smooth if it is smooth for the topology of E. The space of smooth curves is denoted $\mathcal{C}^\infty(\mathbb{R}, E)$, or sometimes $\mathcal{C}^\infty_E$.

A map $f : E \to F$ is said to be smooth if it maps smooth curves of E to smooth curves of F. The space of smooth maps between E and F is denoted $\mathcal{C}^\infty(E,F)$.

One of the evidences of the fact that this definition is particularly relevant is the following lemma, found in 2.11 of [KM97] : a linear map between convenient spaces is smooth if and only if it is bornological.

Definition 5. We denote by $\mathcal{C}^\infty(\mathbb{R},E)$ the space of smooth curves in E, with the topology of uniform convergence on compact sets of each derivative separately.
We provide then $\mathcal{C}^\infty(E,F)$ with initial topology with respect to all mappings $c^\star : \mathcal{C}^\infty(E,F) \to \mathcal{C}^\infty_F$ for $c \in \mathcal{C}^\infty_F$.

With our definition of the bornology on E, the characterization of bounded sets follows from the definition : a set $U \subseteq \mathcal{C}^\infty(E,F)$ is bounded if and only if $c^\star(U)$ is bounded in $\mathcal{C}^\infty_F$ for every $c \in \mathcal{C}^\infty_E$.

Observe that E isn't always metrizable (it is necessary and sufficient for a locally convex vector space to have a countable basis of neighborhoof of 0 to be metrizable), thus the the topology of uniform continuity on $\mathcal{C}^\infty(\mathbb{R},E)$ is not straightforward. As every topological group, E is a uniform space, whose entourages are the subset of $E \times E$ containing $\{(x,y), x-y \in \Omega\}$ where Ω is some neighborhood of 0 in E. Hence, the open sets for the topology of uniform convergence on compact sets of each derivative separately in $\mathcal{C}^\infty(\mathbb{R},E)$ are the :

$$O_{f,\Omega} = \{g \in \mathcal{C}^\infty(\mathbb{R},E), \forall k \in \mathbb{N}, \forall K \text{ a compact} \subset \mathbb{R}, \forall x \in K, f^{(k)}(x) - g^{(k)}(x) \in \Omega\}$$

where Ω is a neighborhood of 0 in E, and f is a smooth curve on E.

Real analytic curves and maps

The definition of real analytic maps has a similar construction, but some subtilities need to be solved. The first property we're looking for is the cartesian closedness, and especially for a real analytic function $f : \mathbb{R}^2 \to \mathbb{R}$ we would like to find a corresponding function $\breve{f} : \mathbb{R} \to \mathcal{C}^\omega(\mathbb{R},\mathbb{R})$. Kriegl and Michor show in example 9.1 of [KM97] that there is a function from $\mathbb{R}^2$ to $\mathbb{R}$ which is real analytic in the classical sense but would never allow such a $\breve{f}$ to come, as long as we ask to the topology on $\mathcal{C}^\omega(\mathbb{R},\mathbb{R})$ to make the evalutaion application $ev_t : \mathcal{C}^\omega(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ to be linear and bornological.
Hence, it's necessary to precise the definition of real analytic.

Definition 6. A curve $c : \mathbb{R} \to E$ is said to be real analytic if, for every $l \in E'$, $l \circ c : \mathbb{R} \to \mathbb{R}$ is real analytic (*i.e.* is locally given by its convergent Taylor series).
A map $f : E \to F$ is said to be real analytic if it's smooth (takes smooth maps to smooth maps) and if it takes real analytic maps to real analytic maps.

Of course, there is much more to say about this notion (see the second chapter of [KM97]). Let us just describe a link between the convergent Taylors series and our definition of real analytic maps.
A curve c into a convenient space E will be said to be topologically real analytic if and only if it is locally given by its convergent Taylor serie. Suppose now there is a Baire vector space topology on E' such taht the point evalutaion δ_x are continuous for every $x \in E$. Then any real analytic curve is locally given by its Mackey-convergent Taylor series. As we asked for a convenient space to have a bornological topology, this says that any real analytic curve on such a convenient space is topologically real analytic. See Theorem 9.6 of [KM97] for the demonstration.

Now a useful property of bornological linear maps :

Lemma 1. A linear map between convenient spaces is real analytic if and only if it is bornological

Proof. Consider E, F convenient spaces, and $l : E \to F$ linear. See Lemma 2.11 of [KM97] for the proof that l is bornological if and only if it is smooth. The proof uses a notion of fast convergence allowing a sequence to be parametrized by a smooth curve.
In fact, the proof of the fact that l is bornological if and only if it takes real analytic curves to real analytic curves has been done in Theorem 9.7 of [KM97]. It uses mainly the fact that for a function c from $\mathbb{R}$ to $\mathbb{R}$, c is real analytic if and only if for each compact $K \in \mathbb{R}$, there exists M, ρ such that, for every $k \in \mathbb{N}$ and for every $a \in K$, we have $|\frac{c^{(k)}(a)}{k!}| \leq M\rho^k$.

□

Theorem 2. Adaptation of Hartog's theorem. Consider $f : E \to F$. Then f is real analytic iff f is smooth and $l \circ f$ is real analytic along each affine in E, this being true for all $l \in F'$. See 10.4 in [KM97].

Topology on real analytic maps spaces

To understand the topology on real analytic spaces, it is useful to know that the part on real analytic maps in [KM97] comes right after a similar theory on holomorphic mapping (it's likely that one can find in this theory another model for DiLL, maybe bringing interesting properties to enrich DiLL). A lot of results on real analytic maps derive from those on holomorphic mappings. Precisely, the complexification of the real vector space $\mathcal{C}^\omega(\mathbb{R}, \mathbb{R})$ is the complex vector space of the holomorphic functions from $\mathbb{C}$ to $\mathbb{C}$. Hence, $\mathcal{C}^\omega(\mathbb{R}, \mathbb{R})$ is the real part of complex vector space of the holomorphic function from $\mathbb{C}$ to $\mathbb{C}$, which is a closed subspace of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ (See 7.21 and 11.2 of [KM97]). Indeed, the locally convex topology on $\mathcal{C}^\omega(\mathbb{R}, \mathbb{R})$ comes from these inclusions, and let's just notice :

Lemma 2. See 11.5. A subset $\mathcal{B} \in \mathcal{C}^\omega(\mathbb{R},\mathbb{R})$ is bounded if and only if there is an $r > 0$ such that $\{\frac{f^{(a)}}{a!}r^a : f \in \mathcal{B}, a \in \mathbb{N}\}$ is bounded in $\mathbb{R}$.

We are now able to define a topology on $\mathcal{C}^\omega(E,\mathbb{R})$ and $\mathcal{C}^\omega(E,F)$ when E and F are convenient spaces. Remember that we're using the definition of [FK88] which is stronger than the one of [KM97] as it require to the topology of E to be bornological. Itsi then essential to us that the following it prooved in [FK88].

Definition 7.

- The topology on $\mathcal{C}^\omega(E,\mathbb{R})$ is the one induced by the families of functions :

$$c^\star : \mathcal{C}^\omega(E,\mathbb{R}) \to \mathcal{C}^\omega(\mathbb{R},\mathbb{R}) \text{ for } c \in \mathcal{C}^\omega(\mathbb{R},E)$$

$$c^\star : \mathcal{C}^\infty(E,\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \text{ for } c \in \mathcal{C}^\infty(\mathbb{R},E).$$

- The topology on $\mathcal{C}^\omega(E,F)$ is the one induced by the family of functions :

$$l^\star : \mathcal{C}^\omega(E,F) \to \mathcal{C}^\omega(E,\mathbb{R}) \text{ for } l \in F')$$

The topology defined as above is a locally convex topology. To proove it makes those spaces convenient, the following lemma helps.You can find it in Theorem 2.15 of [KM97].

Lemma 3. If $(E_j)_{j\in J}$ is a family of convenient spaces, then $\prod_{j\in J} E_j$ is a convenient space.

Moreover, from the topology of $\mathcal{C}^\omega(U,\mathbb{R})$ it follows that :

Proposition 2. Let E be a convenient spaces. Then $\mathcal{C}^\omega(U,\mathbb{R})$ can be considered as a closed subspace of $\prod_{c\in\mathcal{C}^\omega(\mathbb{R},E)} \mathcal{C}^\omega(\mathbb{R},F\mathbb{R}) \times \prod_{c\in\mathcal{C}^\infty(\mathbb{R},E)} \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ consisting of the morphisms $(f_c)_{c\in\mathcal{C}^\omega(\mathbb{R},E)}$ such that $f_{g;c} = g; f_c$ for each $g \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, as well as of the morphisms $(f_c)_{c\in\mathcal{C}^\infty(\mathbb{R},E)}$ such that $f_{g;c} = g; f_c$ for each $g \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$.

Thanks to the previous lemma, it is now immediate that assuming that E and F are convenient spaces, so are $\mathcal{C}^\omega(E,\mathbb{R})$ and $\mathcal{C}^\omega(E,F)$. From the previous proposition, we can also deduce :

Corollary 1. Let f be a map from $\mathbb{R}$ to $\mathcal{C}^\omega(E,F)$. Then f is real analytic iff forall $c \in \mathcal{C}^\omega(\mathbb{R},E)$ (resp. $\mathcal{C}^\infty(\mathbb{R},E)$), we have $c^\star(f) : \mathbb{R} \to F$ is real analytique (resp. smooth).

3 $\mathcal{C}$ and $\mathcal{C}^\omega$, a model of DiLL ?

In this section, we will try and approach a model of $DiLL$ through categorical constructions on $\mathcal{C}$. First of all, we will construct a Seely category on $\mathcal{C}$ and then define $\otimes$ to have a symmetric monoidal category, $\times$ to have a cartesian co-Kleisli category, and finally a comonad $!_\omega$ so that this co-Kleisli category will be the one of real analytic maps. Then, the intuitive definition of derivation will be appropriate to make $\mathcal{C}$ a differential category. Eventually, we will try to proove that $\mathcal{C}$ is a model of $DiLL$, with the definition used by Ehrhard in [Ehr11], section 2.

3.1 Monoidal structure and Cartesian closedness

We will proove that $\mathcal{C}$ is a differential category when equiped with $!_\omega$. This proof is mainly an adaptation of [BET10]. Note that it will follow from [BCS09] that we have then a cartesian differential cartegory. For a complete exposure of what is a model of intuitionnistic linear logic, see [Mel08], chapter 7.

Let's just mention a few facts :

- On locally convex vector space you have a classical operation of Mackey-closure, Mackey-completing the vector space.
- If you want a bornological topology on any vector space, it's enough to define a bornology and to take as topology the radial bornivorous subset.
- A bornology on E defines E', but given a set L of linear forms on E you can also define a bornology. A set B will be bounded if and only if for every $l \in L$, $l(B)$ is bounded in $\mathbb{R}$. With the topology described in the point above, you get a convenient space.

We can now begin to build the structure of the model of DiLL.

Proposition 3. (See 3.1 of [BET10]) $\mathcal{C}$ equipped with the following tensor product is a symmetric monoidal closed category : $E \otimes F$ is the Mackey-closure of the algebraic tensor $E\hat{\otimes}F$, equipped with the bornology generated by $(E\hat{\otimes}F)' = \{h : E\hat{\otimes}F \to \mathbb{R} | \hat{h} : E \times F \to \mathbb{R}$ is bornological $\}$.

The second essential point is the cartesian closedness of $\mathcal{C}^\omega$. This one is prooved in [KM97], 11.18. Mainly, it consists of reducing to the case $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, through curves in $\mathcal{C}^\infty(\mathbb{R}, E_i)$ or $\mathcal{C}^\omega(\mathbb{R}, E_i)$, and bornological forms. Hence the truly technicals proofs are found in 11.7 for the real analytic part and in 3.2 for the smoothness.

Theorem 3. The category $\mathcal{C}^\omega$ of convenient spaces and real analytic maps between them is cartesian closed.

The cartesian closedness is not only one the main results of Frölicher, Kriegl and Michor, but is only key to many constructions in our model of Dill, and especially in the comonad.

3.2 The exponential modality

As explained in the appendix, one can build a comonad out of an adjunction. We will now explain how the dirac distribution give rise to a comonad which will modelize the exponential modality.

Definition 8. We note δ the dirac distribution : $\delta : E \to \mathcal{C}^\omega(E,\mathbb{R})'$, defined by $\delta_x(f) = f(x)$.

Proposition 4. δ is well defined and real analytic.

Proof. It is clear that for every $x \in E$, δ_x is linear. It remains to show that it is bornological. Consider U a bounded subset of $\mathcal{C}^\omega(E,\mathbb{R})$, that is $c^\star(U)$ is bounded in $\mathbb{R}$ for every $c \in \mathcal{C}^\omega(\mathbb{R},E)$ and $c \in \mathcal{C}^\infty(\mathbb{R},E)$. Let us denote $const_x$ for the constant curve $const_x(t) = x\ \forall t \in \mathbb{R}$. We have $const_x \in \mathcal{C}^\omega(E,\mathbb{R})$ and $const_x \in \mathcal{C}^\infty(E,\mathbb{R})$, then $const_x^\star(U) = \delta_x(U)$ is bounded in $\mathbb{R}$. Let us show now that δ is real analytic. We're going to use cartesian closedness : if $\hat{\delta} : E \times \mathcal{C}^\omega(E,\mathbb{R}) \to \mathbb{R}$ is real analytic, then so is $\check{\hat{\delta}} = \delta$.

Consider then $(c,f) \in \mathcal{C}^\omega(\mathbb{R}, E \times \mathcal{C}^\omega(E,\mathbb{R}))$. Then we have $c\ in\mathcal{C}^\omega(\mathbb{R},E)$ and $f \in \mathcal{C}^\omega(\mathbb{R},\mathcal{C}^\omega(E,\mathbb{R}))$. Indeed, consider $l \in E'$. Then $\bar{l} : \mathbb{R}, E\times\mathcal{C}^\omega(E,\mathbb{R}) \to \mathbb{R}; (x,g) \mapsto l(x)$ is also linear, and bornological by defintion of the product topology. Because (c,f) is real analytic, so is $\bar{l} \circ (c,f)$ which corresponds to $l \circ c$. This being true for every $l \in E'$ we are able to conclude that c is real analytic. A similar demonstration shows that f is real analytic too. Note that the scalar testing is possible here only because we are un a Mackey-complete space.

Keep in mind that we want to show that $\hat{\delta} \circ (c,f)$ is a real analytic. Now for every $t \in \mathbb{R}$, $\hat{\delta} \circ (c,f)(t) = f(t)(c(t))$. But f being real analytic curve in $\mathcal{C}^\omega(E,\mathbb{R})$, Corollary 1 tells us that for every γ real analytic curve in E $\gamma^\star(f)$ is real analytic too. As c is real analytic, so is $c^\star(f) = \hat{\delta} \circ (c,f)$. $\hat{\delta}$ takes real analytic curves to real analytic curve.

Is is easy to show that $\hat{\delta}$ also takes smooth curves to smooth curves. The map is then real analytic, and so is δ by cartesian closedness.

□

Definition 9. We will write $!E$ for the Mackey-closure of the linear span of $\delta(E)$ in $\mathcal{C}^\omega(E,\mathbb{R})'$ [1].

It's necessary to Mackey-close and linear expand $\delta(E)$ in order to find on $!E$ a convenient space. Now a useful tool to define functions over $!E$:

Lemma 4. See lemma 5.3 of [BET10]. Let $v_1, v_2, \dots, v_n$ be pairwise distinct vectors in E. Then the δ_{v_i} are linearly independant in $\mathcal{C}^\omega(E,\mathbb{R})'$.

Proposition 5. Endowed with the bornological linear maps $\phi_I : I \to !I$ defined with $\phi_I(1) = \delta_1$ and $\phi : !E \otimes !F \to !(E \otimes F)$ defined on basis elements by $\phi(\delta_x \otimes \delta_y) = delta_{x \otimes y}$, expanded linearly and completed to be defined then on $!E$, the functor ! is symmetric monoidal.

The adjonction from which the comonad on Con is constructed is the classical adjunction between a forgetful functor and a constructive one. Hence, as in [BET10], we will only proove the following bijection, leaving the natural transformations checkings to the reader (or see Theorem 5.1.1 in [FK88] for the smooth case).

Theorem 4. We have an adjunction between $! : \mathcal{C}^\omega \to Con$ and $\mathbf{U} : Con \to Com$. Indeed :

$$\mathcal{C}^\omega(E, \mathbf{U}F) \cong Con(!E, F)$$

Proof. We're prooving the bijection above. Let $l : !E \to F$ be a linear bornological map. Then when we define $f : E \to F$ by $f(e) = l(delta_e)$, we have a real analytic map. Indeed, l is real analytic according to Lemma 1, as δ. It's easy to show that the composition of two real analytic maps is still real analytic with our definition. Conversely, consider $f : E \to F$ a real analytic map. Define on $!E$ $l(\delta_e) = f(e)$, and then expand linearly (this is possible thanks to Lemma 4), and Mackey-complete to define l on E. l is clearly linear. Consider U a bounded subset of $< \delta(E) >$. Then $l(U) = U(\{l\})$ wich is bounded as the image by a bounded subset of function of a singleton. The completion will preserve the bornological criteria, and l is bornological on $!E$. □

The natural transformations associated to this adjunction are :

- The counit $\epsilon : !E \to E$ defined on the basis elements by the intuitive $\epsilon(\delta_x) = x$ and then extended linearly and completed.

[1] It is unecessary here but interesting to know the content of Proposition 5.1.5 and 5.1.8 of [FK88] : when E is finite dimensional, $!E$ corresponds to $\mathcal{C}^\omega(E,\mathbb{R})'$, which is in this case the space of distributions of compact support

- The unit : $\iota : E \to !E$, defined by $\iota(x) = \delta_x$.

For the comonad, ! comes then with ϵ and the comultiplication $\rho : !E \to !!E$ defined on basis elements by $\rho(\delta_x) = \delta_{\delta_x}$. This endows the category Con with a symmetric monoidal comonad.

From now on, we denote Con_ω the category of convenient spaces and bornological linear maps equiped with the monad defined above.

3.3 A model of Intuitionistic Linear Logic

We can construct on Con_ω a biproduct by defining that $E \oplus F$ has as underlying set the cartesian product $E \times F$. To modelize the multiplicative part of linear logic, it remains to show the following isomorphism :

Theorem 5. $!(E \oplus F) \cong !E \otimes !F$

Proof. We need to demonstrate that $!(E \times F)$ satistfies the universal property of the tensor product. Consider then $!E \times !F \to G$ a bilinear bornological map, G being a convenient space. To factor f as we want, we need first to show that f is real analytic. See [BET10], proposition 5.6, for the fact that $f \circ \iota \times \iota$ takes smooth curves to smooth curves. The proof uses the topology of [FK88]. Let us show now that it is real analytic. In fact, the second Theorem of this report simplifies everything.
Consider $\lambda \in G'$, $(a, b) \in E \times F$, and $(v, w) \in E \times F$. Let us denote c the affine line $c(t) = (a + tv, b + tw)$. Then for every $t \in \mathbb{R}$, $\lambda \circ f \circ c(t) = \lambda[f(a, b) + tf(a, w) + tf(v, b) + t^2 f(v, w)]$. This is clearly real analytic, then so is $f \circ \iota \times \iota$. Now, we have through the adjonction describe in Theorem 4 that $f \circ \iota \times \iota$ give rise to a linear map $\bar{f} : !(E \times F) \to G$. If we write $m_{E,F}$ for the bilinear map : $!E \times !F \to !(E \times F)$, we have that $f = \bar{f} \circ m$. $!(E \times F)$ is isomorphic to $!E \otimes !F$ and we have the Seely isomorphism wanted.

□

From all this conclusion, we conclude the following theorem. See Melliès's panorama, [Mel08], for a proper demonstration (proposition 24 especially).

Theorem 6. The category Con_ω is a model of intuitionnistic linear logic.

3.4 A differential category

We have still to work on the differential part. First of all let us note the bialgebra structure on $!E$ for every convenient space E :

- $\Delta : !E \to !E \otimes !E$ is defined by $\Delta(\delta_x) = \delta_x \otimes \delta_x$, then extented linearly and completed.
- $e : !X \to \mathcal{I}$ is $e(\delta_x) = 1$.
- $\triangledown : !E \otimes !E \to !E$ is $\triangledown(\delta_x \otimes \delta_y) = \delta_{x+y}$.
- $\nu : \mathcal{I} \to !A$ is $\nu(1) = \delta_0$.

Let us now define a codereliction map $coder : E \to !E$ by

$$coder(v) = \lim_{x \to 0} \tfrac{\delta_{tv} - \delta_0}{t}$$

Theorem 7. Codereliction is an arrow in $\mathcal{C}_\omega$. It verifies the strenght and comonad diagrams.

For the demonstration, we need a technical lemma :

Lemma 5. If $h : \mathbb{R}^2 \to !E$ is real analytic then $\partial_1 h|_{s=0} : \mathbb{R} \to !E$ is real analytic.

Proof. Consider $l \in (!E)'$. Then $l \circ \partial_1 h|_{s=0}(t) = \lim_{x \to 0} \frac{l \circ h(s,t) - l \circ h(0,t)}{s}$. $l \circ h$ being real analytic, for every real analytic curve $c : \mathbb{R} \to \mathbb{R}^2$ $l \circ h \circ c$ is real analytic too. Let $t \in \mathbb{R}$ be fixed, and defined $c_t : \mathbb{R} \to \mathbb{R}^2$ by $c_t(s) = (s,t)$. In a neighbourhood of 0 we have $l \circ h \circ c_t = \sum_{k \in \mathbb{N}} a_k(t) s^k$, hence $l \circ \partial_1 h|_{s=0}(t) = a_0(t)$. Let us proove now that $t \mapsto a_0(t)$ is real analytic. Define a real analytic curve $\gamma_0 : \mathbb{R} \to \mathbb{R}^2$ by $\gamma_0(t) = (0,T)$. Then in a neighbourhood of 0 we have

$$\begin{aligned} l \circ h \circ \gamma_0(t) &= l \circ h(0,0) + \sum_{k \in \mathbb{N}} b_k t^k \\ &= l \circ h \circ c_t(0) \\ & a_0(t) \end{aligned}$$

$l \circ h \circ \gamma_0(t)$ being real analytic, so is $a_0(t)$ and $l \circ \partial_1 h|_{s=0}(t)$. We conclude that $\partial_1 h|_{s=0}$ is real analytic. ☐

Proof. Proof of Theorem 7 First note that, δ being smooth and real analytic, *coder* is well defined. Let us show that *coder* is linear. It's clearly homogeneous. For the additive property, consider $v, w \in E$, $g : \mathbb{R} \times \mathbb{R} \to !E$ by $g(t,s) = \delta_{tv+sw}$. Then :

$$\begin{aligned} coder(v+w) &= (t \mapsto g(t,t))'(0) \\ &= \partial_1 g(0,0) + \partial_2 g(0,0) \\ &= coder(v) + coder(w) \end{aligned}$$

Now for the real analytic part. You will find in [BET10] the demonstration of the smoothness of *coder*, which is very similar to the one we are going to do. Consider $c \in \mathcal{C}^\omega(\mathbb{R}, E)$, and define $h : \mathbb{R} \times \mathbb{R} \to !E$ by $h(s,t) = \delta_{sc(t)}$. δ is real analytic, c is *realanalytic*, then so is h. Thanks to the previous lemma $c * (coder) : t \mapsto coder(c(t)) = \partial_1 h|_{s=0}(t)$ is real analytic. It remains to check that *coder* veryfies the differential equations presented in the first section . Consider A and B convenient spaces, as in the diagrams in section 1.2.

- The strenght equation is verified. Indeed, beginning with the upper leg of the diagram, we have :
  $\phi[coder_a \otimes 1(x \otimes \delta_y)]] = \phi[\lim\limits_{x\to 0} \frac{\delta_{tx}-\delta_0}{t} \otimes delta_y] = \lim\limits_{x\to 0} \frac{\delta_{tv\otimes y}-\delta_{0\otimes y}}{t} = \lim\limits_{x\to 0} \frac{\delta_{t(v\otimes y)}-\delta_0}{t} = coder_{A\otimes B[1\otimes\epsilon_A(x\otimes\delta_y)]}$.
- The comonad equation is verified : the first one follows from the continuity and the linearity of ϵ. Concerning the second diagram, let us calculate the way from A to $!!A$ through $coder_A$ and ρ . We obtain $\rho \circ coder_A(x) = \lim\limits_{x\to 0} \frac{\delta_{\delta_{tv}}-\delta_{\delta_0}}{t}$. Through the other way, we obtain $\lim\limits_{s,t\to 0} \frac{\delta_{[\frac{s}{t}(\delta_{tv}-\delta_0)+\delta_0]}-\delta_0}{s}$. This limit is unique, so we get the same result when taking $s = t \to 0$. Simplifying the above equation when $s = t$, we find the result we had through *coder* and ρ.

□

We have now everything we need to conclude :

Theorem 8. Con_ω is a differential category.

3.5 A model of DiLL ?

At the end of my internship, I wanted to show that besides being a differential category and a model of intuitionnistic linear logic, *Con* was also a denotational model of DiLL. The only definition of such a model was in [Ehr11], section 2, and asked to a model to ba $\star$-autonomous. In the case of $\mathcal{C}$, this raises an issue. Indeed, asking to $\eta_X : X \to ((X \multimap \perp) \multimap \perp)$ to be an isomorphism for each convenient space would amount to ask each convenient space to be reflexive.
Let us explain why. Suppose $\mathcal{C}$ is a $\star$-autonomous category, with a convenient space $\perp$ as dualizing object. Then, as said by Mellies in section 4.8 of [Mel08], $\perp \multimap \perp$ is isomorphic to the unit for $\otimes$, which $\mathbb{R}$ here. $\perp$ being a real vector space, $\mathcal{L}(\perp, \perp) \cong \mathbb{R}$ implies $\perp = \mathbb{R}$. Then $X \cong ((X \multimap \perp) \multimap \perp)$ if and only if X is reflexive (that is, $E \cong E''$ where X' is the space of bornological linear maps from X to $\mathbb{R}$. I didn't have time to find a non reflexive convenient space, althought it is stated in the begining of section 5.4 of [FK88] that it exists.

In case convenient spaces are indeed not always reflexive, we have to think of another denotational model for DiLL. We could maybe consider a slightly different version of DiLL, closer to the presentation of Intuitionnistic Linear Logic. Without a dualizing object, it is impossible to work with one sided sequent, and far more easy to modelize the linear implication $\multimap$ than $\parr$. Remember that in classical linear logic : $A \mapsto B = A^{\perp} \parr B$, so when the dualizing object exists, you can define $\parr$ thanks to $\multimap$. See the next figure for the first draft of this sequent calculus. It remains to show the coherence of this calculus, and it should be straightforward to see that Con is a denotational model of it.

As a last remark, one could note that if Con_{ω} is a denotational model of the intuitionistic version of DiLL, then Con_{∞} should be too.

- Identity group :

$$\frac{}{A \vdash A}\,(\text{axiom}) \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2, A \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash, \Delta}\,(\text{cut})$$

- Multiplicative group :

$$\frac{}{\vdash 1}\,(1_R) \qquad \frac{\Gamma \vdash A}{\Gamma, 1 \vdash A}\,(1_L)$$

$$\frac{\Gamma_1, \vdash A \qquad \Gamma_2, B \vdash \Delta}{\Gamma_1, \Gamma_2, A \multimap B \vdash \Delta}\,(\multimap_L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta}\,(\multimap_R)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\,(\otimes_L) \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \otimes B, \Delta_1, \Delta_2}\,(\otimes_R)$$

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\,(\text{Mix}) \qquad \frac{}{\vdash \bot}\,(Mix_0)$$

- Vector group :

$$\frac{}{\vdash \Gamma}\,(0) \qquad \frac{\vdash \Gamma \qquad \vdash \Gamma}{\vdash \Gamma}\,(+)$$

- Exponential group :

$$\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,(\text{contraction}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\,(\text{weakening}) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,(\text{derilection})$$

$$\frac{\Gamma_1 \vdash !A \qquad \Gamma_2 \vdash !A}{\Gamma_1, \Gamma_2 \vdash !A}\,(\text{co-contraction}) \qquad \frac{}{\vdash !A}\,(\text{co-weakening}) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash !A}\,(\text{co-derilection})$$

Figure 4: Intuitionistic DiLL

Conclusion

Finally, the category of convenient spaces and real analytic maps between them works as the one of smooth maps. Power series were already there in early studies of DiLL (see Köthe spaces or Finiteness spaces) : returning from smooth maps to real analytic ones is a progress toward the understanding of the syntaxic Taylor formula. A lot of questions are still open. It remains to know whether Con is a denotational model of DiLL (see section 3.5), and what are the links between real analytic functions between convenient spaces, classical real analytic functions, and the Taylor development of terms in ressource lambda-calculus[2]. Moreover, can the canonical isomorphism of 11.20 in [KM97] be interesting ? The work on holomorphy and integration done by Kriegl and Michor can surely result in a good category too. Above all, the most interesting would be to try and understand the possible links between this model of differential linear logic, ressource calculus, and the Taylor formula as described by Erhrard and Regnier.

[2]As for the first interaction, one can see Theorem 9.6 in [KM97] : under the condition that E' is endowed with a Bair topology making the δ_x continuous for every $x \in E$, every real analytic curve is locally given by its Mackey-convergent Taylor serie

A Appendice : Elements of category theory

I will detail here the definitions of category theory essential to this internship report. To learn more about category theory, see [Mac98] or [Awo06] for a simpler introduction.

Categories A category $\mathcal{C}$ consists of :

- A collection of objects : $A, B, \ldots$
- Arrows : $f, g, \ldots$

each arrows commig with two objects : $Dom(f)$ and $cod(f)$. We write : $f cod(f) \to dom(f)$ .Two laws must be verified :

- Each object A comes with an arrow $1_A : A \to A$ verifying unity equations to the right and to the left.
- For $f : A \to B$ and $g : B \to C$, there is an arrow in $\mathcal{C}$ called $f \circ g : A \to C$, with $\circ$ veryfying the associativity constraint.

The next fundamental definition is the one a functor . A functor F is a morphism of category, taking objects to objects and arrows to arrows. Indeed, take a functor $F : \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$. It must verify :

- For each $f : A \to B \in \mathcal{C}$, then we have $F(f)$ has for codomain $F(A)$ and for domain $F(B)$.
- For each $A \in \mathcal{C}$, $F(1_A) = 1_{F(A)}$.
- $F(f \circ g) = F(f) \circ F(g)$.

Without additional precision, the notation $\mathcal{C}$ will represent a random category. One last notation : we will write $\mathcal{C}(A, B)$ for the collection of arrows going from A to B.

Duality Usually, for each structure in category theory, you can get a costructure, with the same objects, but reversed arrows. Examples of costructures will abound during the rest of the appendice, so here we are just going to define the cocategory of a category. Let $\mathcal{C}$ be a category. Define $\mathcal{C}^{op}$ by :

- The objects of $\mathcal{C}^{op}$ are the objects of $\mathcal{C}$.
- $f : A \to B$ is an arrow in $\mathcal{C}^{op}$ if and only if there is an arrow $f^{op} : B \to A$ in $\mathcal{C}$.

Then, $(1_A)^{op} = 1_{A^{op}}$, $(f \circ g)^{op} = f^{op} \circ g^{op}$, and of course $\mathcal{C}^{op^{op}} = \mathcal{C}$.

Natural transformations Natural transformations are the next step to understand mutations and functions into categories. When a functor is a morphism of categories, a natural transformation is a morphism of functor. Consider $F, G : \mathcal{C} \to \mathcal{D}$ two functors between the same two categories. Formally, a natural transformation $\nu : F \to G$ is a family of arrows in $\mathcal{D}$ $(\nu_C : F(C) \to G(C))_{C \in \mathcal{C}}$, such that for every $C, C' \in \mathcal{C}$, and $f : C \to C'$ arrow in $\mathcal{C}$ the following diagram commutes :

$$\begin{tikzcd}
FC \arrow[r, "\nu_C"] \arrow[d, "F(f)"'] & GC \arrow[d, "G(f)"] \\
FC' \arrow[r, "\nu_{C'}"'] & GC'
\end{tikzcd}$$

When we talk about naturality, it refers to a natural transformation somewhere.

Products, initial and terminal objects Once you have this, you can start to look in categories for some classical structures. Let us first define an initial object as an object 0 in $\mathcal{C}$ such that for every $C \in \mathcal{C}$, there is a unique arrow $0 \to C$. Conversely, a terminal object 1 in $\mathcal{C}$ such that for every $C \in \mathcal{C}$, there is a unique arrow $C \to 1$.
Consider now two objects A and B. We say that the object P endowed with the arrows $p_1 : P \to A$ and $p_2 : P \to B$ is a product for A and B if it verifies the universal maping property of the product. That is to say, for every diagram :

$$\begin{tikzcd}
A & X \arrow[l, "x_1"'] \arrow[r, "x_2"] & B
\end{tikzcd}$$

there is a unique arrow $u : X \to P$ malking the following diagram commute:

$$\begin{tikzcd}
 & X \arrow[dl, "x_1"'] \arrow[d, "u"] \arrow[dr, "x_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

Thanks to the uniqueness of u, we have that the product is unique up to isomorphism (an isormorphism being an arrow $f : X \to Y$ such that there is $g : Y \to X$ verifying $f \circ g = 1_Y$ and $g \circ f = 1_X$). Usually, we write $A \times B$ for the product of A and B. Hence, when we are talhing about cartesian closedness, it refers to closedness with respect to the cartesian product defined above. Of course, we do only define cartesian closedness only for categories in which every product exists.

If $\mathcal{C}$ is a category with binary product (i.e. every product of two objects exists), we define a new category $\mathcal{C} \times \mathcal{C}$; which has for object the products of object of $\mathcal{C}$, and for arrows the "product" of arrows in $\mathcal{C}$. Let us detail this : If $f : A \to C$ and $g : B \to D$ are arrows, then we have the diagram :

$$C \xleftarrow{f \circ p_1} A \times B \xrightarrow{g \circ p_2} D$$

By definition of $C \times D$ there is an arrow $u : A \times B \to C \times D$ such that the next diagram. Of course, we will define u as the product of f and g. It is unique, so we will write it $f \times g$.

$$\begin{tikzcd}
 & A \times B \arrow[dl, "f \circ p_1"'] \arrow[d, "u"] \arrow[dr, "g \circ p_2"] & \\
C & C \times D \arrow[l, "q_1"'] \arrow[r, "q_2"] & D
\end{tikzcd}$$

Monoidal categories A monoidal category is a category endowed with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object I and natural isomorphisms $\alpha_{A,B,C}(A \otimes B) \otimes \simeq A \otimes (B \otimes C)$, $\rho_A : A \otimes I \simeq A$, and $\lambda_A : I \otimes A \simeq A$. Moreover, $\otimes$ is asked to make two diagrame commute , the pentagon diagram and the triangl diagram. I won't detail them, just keep in mind that there are here to ensure that I and associativity work well.

A symmetric monoidal categorie is a monoidal category where $A \otimes B \simeq B \otimes A$ for every $A, B \in \mathcal{C}$. Here to, a few more properties (the hexagon diagram) is required. Finally, a symmetric monoidal closed category is a symmetric monoidal category such that for every A and B, there is an object $A \Rightarrow B$ such that there is a natural bijection $CC(C \otimes A, B) \simeq \mathcal{C}(C, A \Rightarrow B)$ for every object C. Another way to say it is that $\Rightarrow$ is right adjoint to $\otimes$. What a perfect transition to the next paragraph !

Monads and adjunctions Keep reading, this is becoming interesting. A monad on a category $\mathcal{C}$ consists of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ and natural transformations $\nu : 1_{\mathcal{C}} \to T$, and $\mu : T^2 \to T$ satisfying :

- the associativity law : $\mu \circ \mu_T = \mu \circ T\mu$
- the unity law : $\mu \circ \nu_T = 1 = \mu \circ T\nu$.

The notation $1_{\mathcal{C}}$ refers to the unit arrow attached to $\mathcal{C}$ in the category of categories and functors. For a formal definition of the composition between natural transformation and functors, see [Mac98] or [Awo06]. I will just give the example : for an object $X \in \mathcal{C}$, $(\mu \circ \mu_T)_X = \mu_X \circ \mu_{T(X)}$ (a composition of arrows in $\mathcal{C}$) and

$(\mu \circ T\mu)_X = \mu_X \circ T(\mu_X)$. Moreover, this is only a glossary, you must go to the previous referenced books to understant this notions through the various examples presented there.

We will be specially interested in comonads, so let us just write what happen in a comonad. $\mathcal{C}$ is a category, $\delta : \mathcal{C} \to \mathcal{C}$ is an endofunctor , the counit ϵ is a natural transformation going from δ to $1_{\mathcal{C}}$ and the comultiplication ρ is a natural transformation going from δ to δ^2.

Then, an adjunction is a strong relation between two functors : $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Formally, F is left adjoint to G and G is right adjoint to F if and only if there is a natural transformation $\nu : 1_{\mathcal{C}} \to G \circ F$ which verify the UMP of the unit. That is :

For every $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $f : C \to U(D)$ there is a unique $g : F(C) \to D$ such that $f = G(g) \circ \nu_C$.

In fact, this implies also the existence of a counit $\epsilon : F \circ G \to 1_{\mathcal{D}}$, existence that we will use later.

An equivalent (and useful) definition is the following one.

For every $C \in \mathcal{C}$, $D \in \mathcal{D}$ there is an isomorphism : $\phi : \mathcal{D}(F(C), D) \simeq \mathcal{C}(C, G(D))$ which is natural in $\mathcal{C}$ and in $\mathcal{D}$.

Again, see [Mac98] for examples.

Kleisli categories and algebras There is an incredibly strong link between adjunctions and algebras. From every adjunction between the functors F and G, you get a monad on $\mathcal{C}$ by composing F and G : $G \circ F : \mathcal{C} \to \mathcal{C}$. With the above notations, the unit is ν, the same that in the adjunction, and the multiplication is $\rho = G\epsilon_F : (G \circ F)^2 \to (G \circ F)$.

Can we take the path in the other direction ? In fact, given a monad T, there is a several adjunctions that could fit. In the category of the adjunctions fitting with T, there is a terminal object and an initial object, the two giving rise to extremely intersting applications.

The initial object is the Kleisli category of T. We will denote it $\mathcal{C}_T$. Its objects are the same as the one of $\mathcal{C}$, but $f_T : A \to B$ is an arrow in $\mathcal{C}_T$ if and only if there is $C \in \mathcal{C}$ such that $B = T(C)$ and $f : A \to C$ is an arrow in $\mathcal{C}$. For $A_T \in \mathcal{C}_T$, its unit arrow is ν_A. For two arrows $f : A \to T(B)$ and $g : B \to T(C)$ in $\mathcal{C}$, the composition in this category of f and g is the composition $\mu_C \circ T(g) \circ f$ of arrows in $\mathcal{C}$.

The other category is the Eilenberg-Moore category of T, denoted $\mathcal{C}^T$, whose objects are T-algebras and whose arrows correspond exactly to arrows in $\mathcal{C}$, although they are not applied to the same objets. Briefly, a T-algebra is a pair (A, α), where A is an object in $\mathcal{C}$ and $\alpha : T(A) \to A$ an arrow in $\mathcal{C}$. α does verify $1_a = \alpha \circ \nu_A$ nad $\alpha \circ \nu_A = \alpha \circ T(\alpha)$. $\mathcal{C}^T$ is very common in usual mathematics, see [Mac98] for examples.

Seely categories A Seely category is a symmetric monoidal closed category $(\mathcal{L}, \otimes, 1)$ with binary product and a binary product denoted & and $\top$, endowed with :

- a comonad $(!, \delta, \epsilon)$.
- two natural isomorphism $m^2_{A,B} : !A \otimes !B \to !(A \,\&\, B)$ and $m_0 : 1 \to \top$ such that $(!, m) : (\mathcal{L}, \&, \top) \to (\mathcal{L}, \otimes, 1)$ is a symmetric monoidal functor.

Moreover, we ask the following diagram to commute for all $A, B \in \mathcal{L}$:

$$\begin{tikzcd}
!A \otimes !B \arrow[r, "m"] \arrow[dd, "\delta_A \otimes \delta_B"'] & !(A \,\&\, B) \arrow[d, "\delta_{!(A\&B)}"'] \\
 & !!(A \,\&\, B) \arrow[d, "!<!\pi_1, !\pi_2>"'] \\
!!A \otimes !!B \arrow[r, "m"] & !(!A \,\&\, !B)
\end{tikzcd}$$

As a result of this defintion, every Seely category is a model of intuitionnistic linear logic.

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