GT Geocal-LAC, November 2017

Partial Differential Equations in Differential Linear Logic

Marie Kerjean

IRIF, Université Paris Diderot
kerjean@irif.fr

Proofs and smooth objects

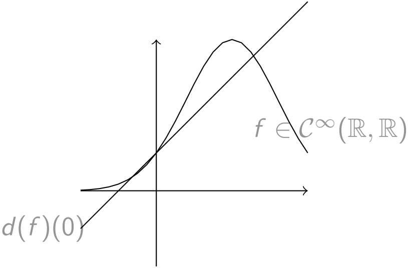
Models with topological vector spaces

Solutions to a LPDE in the syntax

Smoothness

Differentiation

Differentiating a function $f : \mathbb{R}^n \to \mathbb{R}$ at x is finding a linear approximation $d(f)(x) : v \mapsto d(f)(x)(v)$ of f near x.



Smooth functions are functions which can be differentiated everywhere in their domain and whose differentials are smooth.

Differentiating proofs

- Differentiation was in the air since the study of Analytic functors by Girard :

$$\bar{d}(x) : \sum f_n \mapsto f_1(x)$$

- DiLL was developed after a study of vectorial models of LL inspired by coherent spaces : Finiteness spaces (Ehrhard 2005), Köthe spaces (Ehrhard 2002).

Normal functors, power series and λ-calculus. Girard, APAL(1988)

Differential interaction nets, Ehrhard and Regnier, TCS (2006)

Differential Linear Logic

The rules of DiLL are those of MALL and :

co-dereliction

$$\bar{d} : x \mapsto f \mapsto df(0)(x)$$

Syntax

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash}{\vdash !A}\, \bar{w} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\, \bar{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

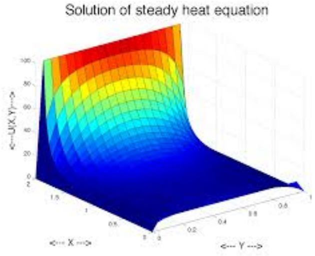
The computational content of differentiation

Historically, resource sensitive syntax and semantics

- Quantitative semantics : $f = \sum_n f_n$
- Resource λ-calculus and Taylor formulas : $M = \sum_n M_n$

Differentiation is inspired by the study of continuous systems :

- Differential Geometry and functional analysis
- Ordinary and Partial Differential Equations



The computational content of differentiation

Historically, resource sensitive syntax and semantics

- Quantitative semantics : $f = \sum_n f_n$
- Resource λ-calculus and Taylor formulas : $M = \sum_n M_n$

Differentiation is inspired by the study of continuous systems :

- Differential Geometry and functional analysis
- Ordinary and Partial Differential Equations

Can we match the requirement of models of LL with the intuitions of physics ?
(YES, we can.)

A smooth exponential

We want to interpret objects of Linear Logic as smooth objects :

- Formulas as $\mathbb{K}$-vector spaces with a caracterisation for limits (norm, metric, or topology)
- Proofs are some kind of smooth functions :

$$Hom(E, F) = \mathcal{C}^\infty(E, F)$$

- We want a $*$-autonomous category of linear maps well-behaving with a cartesian closed category of smooth maps.

$$!E \simeq \mathcal{C}^\infty(E, \mathbb{K})'$$

A typical inhabitant of $!E$ is $ev_x : f \in \mathcal{C}^\infty(E, \mathbb{K}) \mapsto f(x)$.

Topological vector spaces

We work with Hausdorff topological vector spaces : real or complex vector spaces endowed with a Hausdorff topology making addition and scalar multiplication continuous.

- The topology on E determines E'.
- The topology on E' determines whether $E \simeq E''$.

Many topological tensor product

$\otimes_\pi$, $\otimes_i$, $\otimes_\varepsilon$, $\otimes_\gamma$... *Some of which may form a monoidal closed category on some specific spaces.*

We work within the category TopVect of topological vector spaces and continuous linear functions between them.

Challenges

We encounter several difficulties in the context of topological vector spaces :

- Finding a category of general tvs and smooth functions which is Cartesian closed.
- Interpreting the involutive linear negation $(E^{\perp})^{\perp} \simeq E$

Challenges

We encounter several difficulties in the context of topological vector spaces :

- Finding a category of general tvs and smooth functions which is Cartesian closed.
- Interpreting the involutive linear negation $(E^{\perp})^{\perp} \simeq E$

Convenient differential category Blute, Ehrhard Tasson Cah. Geom. Diff. (2010)

Mackey-complete spaces and Power series, K. and Tasson, MSCS 2016.

Challenges

We encounter several difficulties in the context of topological vector spaces :

- Finding a category of smooth functions which is Cartesian closed.
- Interpreting the involutive linear negation $(E^{\perp})^{\perp} \simeq E$

Weak topologies for Linear Logic, K. LMCS 2015.
Involves a topology which is an internal Chu construction.

Challenges

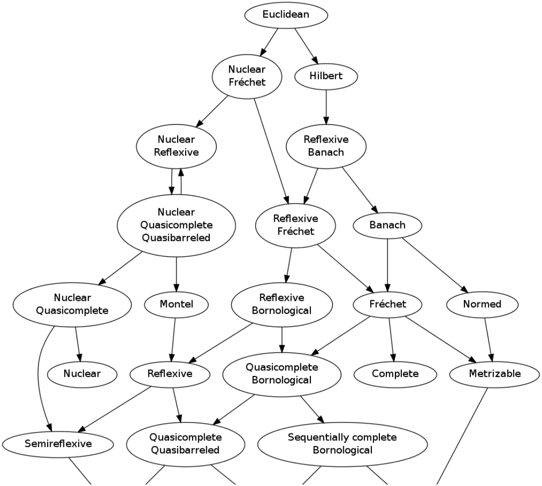
We encounter several difficulties in the context of topological vector spaces :

- Finding a category of general tvs and smooth functions which is Cartesian closed.
- Interpreting the involutive linear negation $(E^{\perp})^{\perp} \simeq E$

- *A model of LL with Schwartz' epsilon product*, K. and Dabrowski.
- *Distributions and Smooth Differential Linear Logic*, K.

Solving Linear Partial Differential Equations in Differential Linear Logic

Looking for a $*$-autonomous category of topological vector spaces :



A model arguing for a graded semantics

Nuclear spaces are either :

- Finite dimensional : $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^m$
- or non-normable, typically spaces of smooth functions and their duals :
  $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, $\mathcal{E}'(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, $\mathcal{D}(\mathbb{R}^n) = \mathcal{C}^\infty_{co}(\mathbb{R}^n, \mathbb{R})$,
  $\mathrm{D}'(\mathbb{R}^n) = \mathcal{C}^\infty_{co}(\mathbb{R}^n, \mathbb{R})'$

Theorem : a model of Smooth DiLL

- Nuclear Fréchet or Nuclear DF spaces are reflexive : we have a **polarized $\star$-autonomous category**.
- Spaces of distributions verify the Kernel Theorem :
  $!\mathbb{R}^n \otimes !\mathbb{R}^m \simeq !(\mathbb{R}^{n+m})$: we have **a monoidal functor for** $!$.

Without higher-order

We don't have an obvious way to construct $!E = \mathcal{C}^\infty(E, R)'$ when E is any Nuclear DF space.

Smooth DiLL, a failed exponential

A new graded syntax

Finitary formulas : $A, B := X | A \otimes B | A \parr B | A \oplus B | A \times B.$
Smooth formulas : $U, V := A | !A | ?A | U \otimes V | U \parr V | U \oplus V | U \times V$

For the old rules

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash}{\vdash !A}\, \bar{w} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\, \bar{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

The categorical semantic of smooth DiLL is the one of LL, but where ! is a monoidal functor and d and $\bar{d}$ are to be defined independently.

Linear Partial Differential Equations as Exponentials

Linear functions as solutions to a LPDE

$$
\begin{aligned}
f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \text{ is linear} \quad & \text{iff } \forall x, f(x) = D(f)(0)(x) \\
& \text{iff } f = \bar{d}(f) \\
& \text{iff } \exists g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), f = \bar{d} g
\end{aligned}
$$

Linear functions as solutions to a LPDE

$$
\begin{aligned}
f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \text{ is linear} \quad & \text{iff } \forall x, f(x) = D(f)(0)(x) \\
& \text{iff } f = \bar{d}(f) \\
& \text{iff } \exists g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), f = \bar{d} g
\end{aligned}
$$

- Finitary formulas are the type of solutions f to the PDE : $\exists g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})', f = \bar{d}(g)$
- Smooth formulas are the type of parameters $g \in \mathcal{C}^\infty(R^n, \mathbb{R})$, $\mathcal{C}^\infty(\mathbb{R}^n, R)'$ to the PDE : $f = \bar{d}(g)$.

A graded syntax

A new graded syntax

Finitary formulas parameters f to $f = \bar{d}(g)$
$A, B := X | A \otimes B | A \parr B | A \oplus B | A \times B$.
Smooth formulas the space in which we find g to differentiate:
$U, V := A | !A | ?A | U \otimes V | U \parr V | U \oplus V | U \times V$

Another definition for $\bar{d}$

A linear partial differential operator D acts on $\mathcal{C}^\infty(\mathbb{R}^n, R)$, and is extended on $\mathcal{C}^\infty(\mathbb{R}^n, R)'$:

$$D(g)(x) = \sum_{|\alpha| \leq n} a_\alpha(x) \frac{\partial^\alpha g}{\partial x^\alpha}.$$

A graded syntax

A new graded syntax

Finitary formulas parameters to $f = \bar{d}(g)$
$A, B := X | A \otimes B | A \parr B | A \oplus B | A \times B.$
Formulas parameter to $f = D(g)$
$A, B := X | A \otimes B | A \parr B | A \oplus B | A \times B.$
Smooth formulas the space in which we find g to differentiate:
$U, V := A | !A | ?A | U \otimes V | U \parr V | U \oplus V | U \times V$

Another definition for $\bar{d}$

A linear partial differential operator D acts on $\mathcal{C}^\infty(\mathbb{R}^n, R)$, and is extended on $\mathcal{C}^\infty(\mathbb{R}^n, R)'$:

$$D(g)(x) = \sum_{|\alpha| \leq n} a_\alpha(x) \frac{\partial^\alpha g}{\partial x^\alpha}.$$

Another exponential is possible

$$!_D A = (D(\mathcal{C}^\infty(A, \mathbb{R})))'$$

that is the space of linear functions acting on functions $f = Dg$, for $g \in \mathcal{C}^\infty(A, \mathbb{R})$, when $A \subset \mathbb{R}^n$ for some n.

$$\bar{d}_D : !_D A \to !A; \phi \mapsto (f \mapsto \phi(D(f)))$$

$$d_D : !A \to !_D A; \phi \mapsto \phi_{|D(\mathcal{C}^\infty(A)}$$

Functions	E'	$D(\mathcal{C}^\infty(A))$	$\mathcal{C}^\infty(A)$
!	$E'' \simeq E$	$!_D A = D(\mathcal{C}^\infty(A))'$	$!A = \mathcal{C}^\infty(A)'$
d	$\phi \mapsto \phi_{\|(A)'}$	$\phi \mapsto \phi_{\|D(\mathcal{C}^\infty(A))}$	
$\bar{d}$	$x \mapsto (f \mapsto d(f)(0)(x))$	$\phi \mapsto (f \mapsto \phi(D(f)))$	

Recall : The structural morphisms on $!E$

$!E = \mathcal{C}^\infty(E,\mathbb{R})'$ whose typical inhabitant : $ev_x \in !E : f \mapsto f(x)$

- The codereliction $\bar{d}_E : E \to !E = \mathcal{C}^\infty(E,\mathbb{R})'$ encodes the differential operator.
- In a $\star$-autonomous category $d_E : E \to ?E$ encode the fact that linear functions are smooth : $d_{E^\perp}^\perp : \phi \in \mathcal{C}^\infty(E,\mathbb{R})' \mapsto \phi_{E'} \in E'' \simeq E$.
- $c : !E \to !E \otimes !E \to$ is deduced from the Seely isomorphism and maps $ev_x \otimes ev_x$ to ev_x.
- $\bar{c} !E \otimes !E \to !E$ is the convolution $\star$ between two distributions
- $w : !E \to \mathbb{R}$ maps ev_x to 1.
- $\bar{w} : \mathbb{R} \to !E$ maps 1 to $ev_0 : f \mapsto f(0)$, the neutral for $\star$.

LPDE with constant coefficient

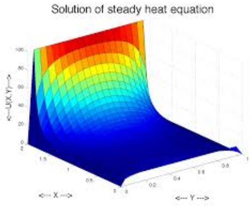
Consider D a LPDO with constant coefficients :

$$D = \sum_{\alpha, |\alpha| \leq n} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}.$$

The heat equation in $\mathbb{R}^2$

$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$

$u(x, y, 0) = f(x, y)$



Then we know how to solve : $\phi = D\psi, \psi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$ and this is done through an algebraic structure on $!_D\mathbb{R}^n$.

An algebraic structure on $!_D$

Co-weakening : Existence of a fundamental solution

For such D there is $E_0 \in \mathcal{C}^\infty(A)'$ such that $DE_0 = ev_0$.

$$\bar{w}_D : \mathbb{R} \to !E, 1 \mapsto E_0$$

Co-contraction : D commutes with convolution

If $f \in D(\mathcal{C}^\infty(A))$ and $g \in \mathcal{C}^\infty(A)$, then $f * g \in D(\mathcal{C}^\infty(A))$.

$$\bar{c}_D : !E \otimes !_D E \to !_D E, (\phi, \psi) \mapsto D(\phi) \otimes \psi$$

An algebraic structure

$D(E_0) * f = f$

$$\bar{c}_D(\phi \otimes E_0) = \phi$$

A coalgebraic structure on D

Weakening

$w :!_D E \to \mathbb{R}$ comes from $t : E \to \{0\}$.

If $E = \mathbb{R}^n$, define $\mathbb{R}^{n'}$ another copy of E. Then

$$
\begin{aligned}
D(\mathcal{C}^\infty(E,\mathbb{R})) &\to D(\mathcal{C}^\infty(E \times E,\mathbb{R})) \\
&= D(\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^{n'},\mathbb{R})) \\
&= D(\mathcal{C}^\infty(E,\mathbb{R}) \parr \mathcal{C}^\infty(\mathbb{R}^{n'},\mathbb{R})) \\
&= D(\mathcal{C}^\infty(E,\mathbb{R})) \parr \mathcal{C}^\infty(\mathbb{R}^{n'},\mathbb{R})
\end{aligned}
$$

Contraction

We thus have $c :!_D E \to !E \otimes !_D E$.

Intermediates rules for D

work in progress

Syntax

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash \Gamma, !A, !A}{\vdash \Gamma, !A}\, \bar{c} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, !A}\, \bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

Syntax for $!_D$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\, w \qquad \frac{\vdash \Gamma, ?A, ?_D A}{\vdash \Gamma, ?_D A}\, c \qquad \frac{\vdash \Gamma, ?_D A}{\vdash \Gamma, ?A}\, d_D$$

$$\frac{\vdash}{\vdash !_D A}\, \bar{w}_D \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !_D A}{\vdash \Gamma, \Delta, !_D A}\, \bar{c}_D \qquad \frac{\vdash \Gamma, !_D A}{\vdash \Gamma, !A}\, \bar{d}$$

The rules of Differential Linear Logic encode the resolution of a Linear Partial Differential Equation with constant coefficients.

What about the converse ? Can we characterize more generally the operators $D : \mathcal{C}^\infty \to \mathcal{C}^\infty$ for which we know how to solve $f = Dg$?

Conclusion

What we have :

- Several smooth models of Classical Linear Logic
- An interpretation of the exponential in terms of distributions.
- The encoding of the resolution of linear *PDE*'s with constant coefficient into the rules of DiLL.
- The first steps for a general understanding of smooth models of linear logic.

What we could get :

- A constructive Type Theory for differential equations.
- Logical interpretations of fundamental solutions, specific spaces of distributions, Fourier transformations or operation on distributions.
- A categorical framework for understanding smooth models of linear logic.