LICS 2018, Oxford

A Logical Account for Linear Partial Differential Equations

Marie Kerjean

IRIF, Université Paris Diderot
kerjean@irif.fr

Linear Logic is about joining Logic and Algebra.

Differential Linear Logic is about joining Logic and Differentiation.

In this talk, we join Logic and Mathematical Physics, Via Linear Partial Differential Equations and a generalization of Differential Linear Logic.

Linear Logic is about joining Logic and Algebra.

Differential Linear Logic is about joining Logic and Differentiation.

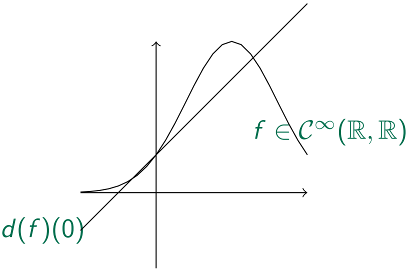
In this talk, we join Logic and Mathematical Physics, Via Linear Partial Differential Equations and a generalization of Differential Linear Logic.

This takes place in a more general setting: Computer Science is drifting from Discrete Mathematics to Analysis.

Smoothness

Differentiation

Differentiating a function $f : \mathbb{R}^n \to \mathbb{R}$ at x is finding a linear approximation $D(f)(x) : v \mapsto D_x(f)(v)$ of f near x.



A coinductive definition

Smooth functions are functions which can be differentiated everywhere in their domain and whose differentials are smooth.

Linear Logic

A decomposition of the implication

$$A \Rightarrow B \simeq\ !A \multimap B$$

A linear proof is in particular non-linear.

$A \vdash B$ is linear. $!A \vdash B$ is non-linear.

$$\frac{A \vdash \Gamma}{!A \vdash \Gamma} \text{ dereliction}$$

Linear Logic

A decomposition of the implication

$$A \Rightarrow B \simeq\ !A \multimap B$$

Usual non-linear implication

A linear proof is in particular non-linear.

$A \vdash B$ is linear. $!A \vdash B$ is non-linear.

$$\frac{A \vdash \Gamma}{!A \vdash \Gamma} \text{ dereliction}$$

Linear Logic

A decomposition of the implication

$$A \Rightarrow B \simeq\ !A \multimap B$$

Linear implication

A linear proof is in particular non-linear.

$A \vdash B$ is linear. $!A \vdash B$ is non-linear.

$$\frac{A \vdash \Gamma}{!A \vdash \Gamma}\ \text{dereliction}$$

Linear Logic

A decomposition of the implication

$$A \Rightarrow B \simeq !A \multimap B$$

Exponential: Usually, the duplicable copies of A.
Here the exponential is a space of Solution to a Differential Equation.

A linear proof is in particular non-linear.

$A \vdash B$ is linear. $!A \vdash B$ is non-linear.

$$\frac{A \vdash \Gamma}{!A \vdash \Gamma} \text{ dereliction}$$

Differential Linear Logic

$$\frac{\vdash \Gamma, A^\perp}{\vdash \Gamma, ?A^\perp}\, d$$

A linear proof is in particular non-linear.

$$\frac{\vdash \Delta, A}{\vdash \Delta, !A}\, \bar{d}$$

From a non-linear proof we can extract a linear proof

Differential Linear Logic

$$\frac{\vdash \Gamma, \ell : A^\perp}{\vdash \Gamma, \ell : ?A^\perp}\, d$$

A linear proof is in particular non-linear.

$$\frac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A}\, \bar{d}$$

From a non-linear proof we can extract a linear proof

Differential Linear Logic

$$\frac{\vdash \Gamma, \ell : A^\perp}{\vdash \Gamma, \ell : ?A^\perp}\, d$$

A linear proof is in particular non-linear.

$$\frac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A}\, \bar{d}$$

From a non-linear proof we can extract a linear proof

Cut-elimination:

$$\frac{\dfrac{\vdash \Gamma, v : !A}{\vdash \Gamma, !A}\, \bar{d} \qquad \dfrac{\vdash \Delta, A^\perp}{\vdash \Delta, ?A^\perp}\, d}{\vdash \Gamma, \Delta}\, \text{cut}$$

$$\rightsquigarrow$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, A^\perp}{\Gamma, \Delta}\, \text{cut}$$

Differential Linear Logic

$$\frac{\vdash \Gamma, \ell : A^{\perp}}{\vdash \Gamma, \ell : ?A^{\perp}}\ d$$

A linear proof is in particular non-linear.

$$\frac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A}\ \bar{d}$$

From a non-linear proof we can extract a linear proof

Cut-elimination:

$$\frac{\dfrac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(\_)(v) : !A}\ \bar{d} \qquad \dfrac{\vdash \Delta, \ell : A^{\perp}}{\vdash \Delta, \ell : ?A^{\perp}}\ d}{\Gamma, \Delta}\ \text{cut}$$

$$\rightsquigarrow \qquad \frac{\vdash \Gamma, x : A \qquad \vdash \Delta, \ell : A^{\perp}}{\Gamma, \Delta, D_0(\ell)(x) = \ell(x) : \mathbb{R} = \perp}\ \text{cut}$$

Just a glimpse at Differential Linear Logic

$$A, B := A \otimes B | 1 | A \parr B | \bot | A \oplus B | 0 | A \times B | \top | !A | !A$$

Exponential rules of DiLL_0

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash \Gamma, !A, \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\, \bar{c} \qquad \frac{\vdash}{\vdash !A}\, \bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

- *Normal functors, power series and λ-calculus.* Girard, APAL(1988)
- *Differential interaction nets*, Ehrhard and Regnier, TCS (2006)

Linear Partial Differential Equations with constant coefficient

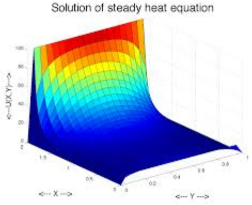
Consider D a LPDO with constant coefficients:

$$D = \sum_{\alpha, |\alpha| \leq n} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}.$$

The heat equation in $\mathbb{R}^2$

$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$

$u(x, y, 0) = f(x, y)$



Theorem (Malgrange 1956)

For any D LPDOcc, there is $E_D \in \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R})'$ such that $DE_D = \delta_0$, and thus for any $\phi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$:

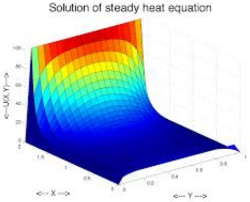
$$D(E_D * \phi) = \phi$$

Linear Partial Differential Equations with constant coefficient

Consider D a LPDO with constant coefficients:

$$D = \sum_{\alpha, |\alpha| \leq n} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}.$$

The heat equation in $\mathbb{R}^2$
$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$
$u(x, y, 0) = f(x, y)$



Theorem (Malgrange 1956)

For any D LPDOcc, there is $E_D \in \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R})'$ such that $DE_D = \delta_0$, and thus for any $\phi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$:

$$\text{output } D(E_D * \phi) = \phi \text{ input}$$

What this work is about: A new exponential $!_D$.

D is a Linear partial Differential Operator with constant coefficients:

DiLL

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

D – DiLL

$$\frac{\vdash \Gamma, ?_D A}{\vdash \Gamma, ?A}\, d_D \qquad \frac{\vdash \Gamma, !_D A}{\vdash \Gamma, !A}\, \bar{d}_D$$

What this work is about: A new exponential $!_D$.

D is a Linear partial Differential Operator with constant coefficients:

$\mathrm{DiLL} = D_0 - \mathrm{DiLL}$ Because of $A \equiv A^{\perp\perp}$

$$\frac{\vdash \Gamma, ?_{D_0} A}{\vdash \Gamma, ?A}\ d \qquad \frac{\vdash \Gamma, !_{D_0} A}{\vdash \Gamma, !A}\ \bar{d}$$

$D - \mathrm{DiLL}$

$$\frac{\vdash \Gamma, ?_{D} A}{\vdash \Gamma, ?A}\ d_D \qquad \frac{\vdash \Gamma, !_{D} A}{\vdash \Gamma, !A}\ \bar{d}_D$$

What this work is about: the same cut-elimination

Cut-elimination models resolution of the Linear Partial Differential Equations on Distributions $D\psi = \phi$.

$$\dfrac{\dfrac{\vdash \Gamma, !_D A}{\vdash \Gamma, !A}\ \bar{d}_D \qquad \dfrac{\vdash \Delta, ?_D A^\perp}{\vdash \Delta, ?A^\perp}\ d_D}{\vdash \Gamma, \Delta}\ \text{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\vdash \Gamma, !_D A \qquad \vdash \Delta, ?_D A}{\vdash \Gamma, \Delta}\ \text{cut}$$

It's all about semantics

And getting a smooth model of Differential Linear Logic with involutive linear negation.

It's all about semantics

And getting a smooth model of Differential Linear Logic with involutive linear negation.

$[\![A]\!] = [\![A]\!]''$, spaces are reflexive

$$[\![A \Rightarrow B]\!] = \mathcal{C}^\infty([\![A]\!], [\![B]\!])$$

Challenges

We encounter several difficulties in the context of topological vector spaces:

- ✓ Finding a category of tvs and smooth functions which is Cartesian closed. Requires some completeness, and a dual topology fine enough.
- ✓ Interpreting the involutive linear negation $(E^\perp)^\perp \simeq E$: Reflexive spaces, and a dual topology coarse enough.

Challenges

We encounter several difficulties in the context of topological vector spaces:

- ✓ Finding a category of tvs and smooth functions which is Cartesian closed. Requires some completeness, and a dual topology fine enough.
- × Interpreting the involutive linear negation $(E^{\perp})^{\perp} \simeq E$.

Convenient differential category Blute, Ehrhard Tasson Cah. Geom. Diff. (2010)

Mackey-complete spaces and Power series, K. and Tasson, MSCS 2016.

Challenges

We encounter several difficulties in the context of topological vector spaces:

- × Finding a category of tvs and smooth functions which is Cartesian closed. Requires some completeness, and a dual topology fine enough.
- ✓ Interpreting the involutive linear negation $(E^{\perp})^{\perp} \simeq E$: Reflexive spaces, and a dual topology coarse enough.

Weak topologies for Linear Logic, K. LMCS 2015.

Challenges

We encounter several difficulties in the context of topological vector spaces:

- ✓ Finding a category of tvs and smooth functions which is Cartesian closed. Requires some completeness, and a dual topology fine enough.
- ✓ Interpreting the involutive linear negation $(E^{\perp})^{\perp} \simeq E$: Reflexive spaces, and a dual topology coarse enough.

We construct in this paper a polarized solution to this problem.

Distributions are everywhere

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\phi_f : g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int fg.$$

- In a classical and Smooth model of Differential Linear Logic, the exponential is a space of Distributions.

$$!A \multimap \bot = A \Rightarrow \bot$$

Distributions are everywhere

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\phi_f : g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int fg.$$

- In a classical and Smooth model of Differential Linear Logic, the exponential is a space of Distributions.

$$!A \multimap \bot = A \Rightarrow \bot$$
$$\mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$

Distributions are everywhere

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\phi_f : g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int fg.$$

- In a classical and Smooth model of Differential Linear Logic, the exponential is a space of Distributions.

$$!A \multimap \bot = A \Rightarrow \bot$$
$$\mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$
$$(!E)'' \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$

Distributions are everywhere

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

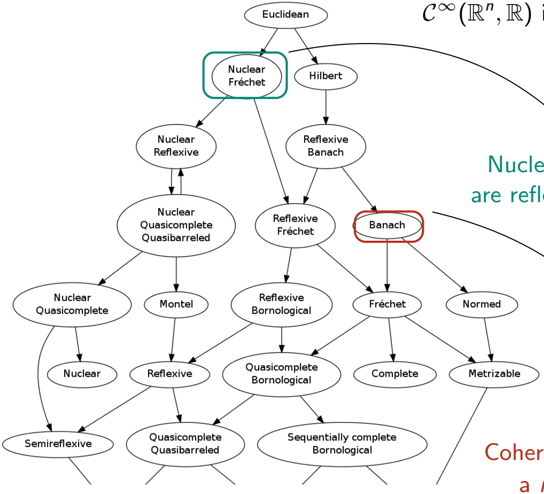
$$\phi_f : g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int fg.$$

- In a classical and Smooth model of Differential Linear Logic, the exponential is a space of Distributions.

$$!A \multimap \bot = A \Rightarrow \bot$$
$$\mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$
$$(!E)'' \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$
$$!E \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$

Topological models of DiLL

$\mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$ is not finite dimensional

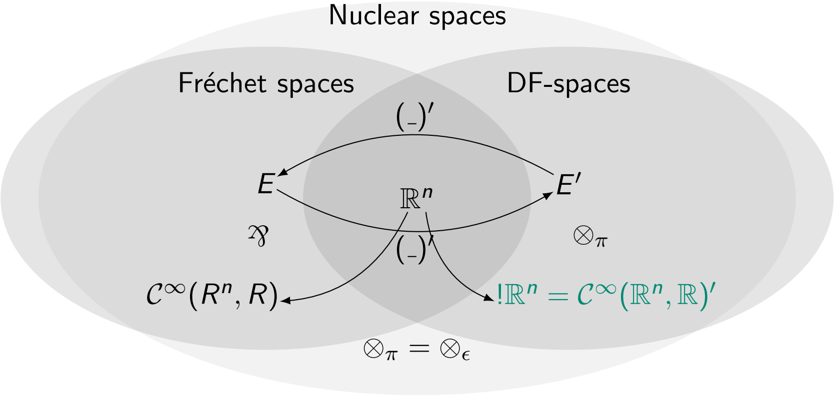


Nuclear Fréchet spaces
are reflexive and complete

Coherent Banach spaces, Girard 2004,
a *norm* is too restrictive

Let us take the other way around, through Nuclear Fréchet spaces.

A Smooth classical Differential Linear Logic with Distributions



Seely's isomorphism corresponds to Schwartz Kernel Theorem.

Getting a model with Higher-order was done in a recent collaboration with JS Lemay.

Another exponential is possible

D a Linear Partial Differential operator with constant coefficients:

$$!_D E = (D(\mathcal{C}^\infty(E, \mathbb{R}))'$$

that is $!_D\mathbb{R}^n = \{\phi \in (\mathcal{C}_c^\infty(\mathbb{R}^n))', D\phi \in !\mathbb{R}^n\}$.

$$\bar{d}_D : \begin{cases} !_D E \to D\phi \\ \quad \phi \mapsto (f \mapsto \phi(D(f))) \end{cases} \qquad d_D : \begin{cases} !E \to !_D E \\ \psi \mapsto \psi * E_D \end{cases}$$

E_D is the fundamental solution of D.

Getting back to LL when $D = D_0$

$!_{D_0} A \simeq \mathcal{L}(A, \mathbb{R})' \simeq A'' \simeq A$ by reflexivity.

When $D = Id$, $!_D A = !A$.

What this work is about: the same cut-elimination

Cut-elimination models resolution of the Differential Equations on Distributions $D\psi = \phi$.

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !_D A}{\vdash \Gamma, D\phi : !A}\ \bar{d}_D \qquad \dfrac{\vdash \Delta, g : ?_D A^{\perp}}{\vdash \Delta, E_D * g : ?A^{\perp}}\ d_D}{\vdash \Gamma, \Delta, D(\phi)(E_D * g) : \mathbb{R} = \perp}\ \text{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\vdash \Gamma, \phi : !_D A \qquad \vdash \Delta, g : ?_D(A)}{\vdash \Gamma, \Delta, \phi(g) : \mathbb{R} = \perp}\ \text{cut}$$

$$D(E_D * \phi)(g) = D(\phi)(E_D * g) = \phi(g)$$

$$\psi = E_D * \phi$$

What this work is about: the same cut-elimination

Cut-elimination models resolution of the Differential Equations on Distributions $D\psi = \phi$.

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !_D A}{\vdash \Gamma, D\phi : !A}\ \bar{d}_D \qquad \dfrac{\vdash \Delta, g : ?_D A^\perp}{\vdash \Delta, E_D * g : ?A^\perp}\ d_D}{\vdash \Gamma, \Delta, D(\phi)(E_D * g) : \mathbb{R} = \perp}\ \text{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\vdash \Gamma, \phi : !_D A \qquad \vdash \Delta, g : ?_D(A)}{\vdash \Gamma, \Delta, \phi(g) : \mathbb{R} = \perp}\ \text{cut}$$

$$D(E_D * \phi)(g) = D(\phi)(E_D * g) = \phi(g)$$

$$\psi = E_D * \phi$$

What this work is about: the same cut-elimination

Cut-elimination models resolution of the Differential Equations on Distributions $D\psi = \phi$.

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !_D A}{\vdash \Gamma, D\phi : !A}\ \bar{d}_D \qquad \dfrac{\vdash \Delta, g : ?_D A^\perp}{\vdash \Delta, E_D * g : ?A^\perp}\ d_D}{\vdash \Gamma, \Delta, D(\phi)(E_D * g) = \phi(g) : \mathbb{R} = \perp}\ \text{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\vdash \Gamma, \phi : !_D A \qquad \vdash \Delta, g : ?_D(A)}{\vdash \Gamma, \Delta, \phi(g) : \mathbb{R} = \perp}\ \text{cut}$$

$$D(E_D * \phi)(g) = D(\phi)(E_D * g) = \phi(g)$$

$$\psi = E_D * \phi$$

Intermediates rules for D

DiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, !A}\, \bar{w} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\, \bar{c} \qquad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x)!A}\, \bar{d}$$

$D - DiLL$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\, w \qquad \frac{\vdash \Gamma, ?A, ?_D A}{\vdash \Gamma, ?_D A}\, c \qquad \frac{\vdash \Gamma, ?_D A}{\vdash \Gamma, ?A}\, d_D$$

$$\frac{\vdash}{\vdash E_D : !_D A}\, \bar{w}_D \qquad \frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !_D A}{\vdash \Gamma, \Delta, \phi * \psi : !_D A}\, \bar{c}_D \qquad \frac{\vdash \Gamma, \psi : !_D A}{\vdash \Gamma, D\psi : !A}\, \bar{d}$$

A deterministic cut-elimination.

Logic in Computer Science: Curry-Howard-Lambek

This Talk: Linear Partial Differential Equations are the Semantics for $\mathrm{D-DiLL}$

Theorical computer science

Mathematical Physics

Conclusion

Take away

Linear Logic and DiLL extends to Linear Partial Differential Operators, in which $!A$ is interpreted by a space of distributions, and a space of solutions to a Differential Equation, and cut-elimination computes the solution.

Now that we've build a bridge with functional analysis, there's A LOT of exciting possibilities.

Two priorities

- Curry-Howard: a deterministic LPDE calculus.
- Most importantly: towards non-linear PDEs.