MFPS 2018, Halifax

A logical account for Linear Partial Differential Equations

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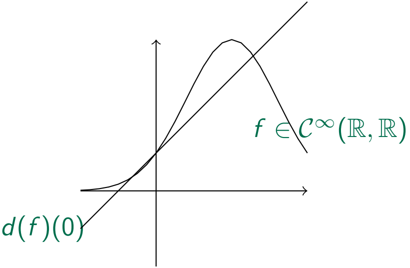
Differential Linear Logic

Smoothness

Differentiation

Differentiating a function $f : \mathbb{R}^n \to \mathbb{R}$ at x is finding a linear approximation $d(f)(x) : v \mapsto d(f)(x)(v)$ of f near x.



A coinductive definition

Smooth functions are functions which can be differentiated everywhere in their domain and whose differentials are smooth.

Linear Logic

A decomposition of the implication

$$A \Rightarrow B \simeq !A \multimap B$$

Denotational semantic

We interpret formulas as sets and proofs as functions between these sets.

Denotational semantic of LL

We have a cohabitation between linear functions and non-linear functions.

Differentiating proofs

- Differentiation was in the air since the study of Analytic functors by Girard :

$$\bar{d}(x) : \sum f_n \mapsto f_1(x)$$

- DiLL was developed after a study of vectorial models of LL inspired by coherent spaces : Finiteness spaces (Ehrhard 2005), Köthe spaces (Ehrhard 2002).

Normal functors, power series and λ-calculus. Girard, APAL(1988)

Differential interaction nets, Ehrhard and Regnier, TCS (2006)

Differential Linear Logic: Semantics

DiLL is a modification of the exponential rules of Linear Logic in order to allow differentiation.

Differentiation

For each $f :!A \multimap B \simeq \mathcal{C}^\infty(A, B)$, we have an interpretation for its differential at 0:

$$D_0 f : A \multimap B$$

Exponential connectives

$$?E \simeq \mathcal{C}^\infty(E', \mathbb{R})$$

$$!E \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$

A typical inhabitant of $!E$ is $ev_x : f \in \mathcal{C}^\infty(E, \mathbb{K}) \mapsto f(x)$.

(Differential) Linear Logic is classical

In Linear Logic, negation is linear :

$$A^{\perp} := A \multimap \perp.$$

Linear Logic and Differential Linear Logic are classical :

$$A^{\perp\perp} \simeq A$$

This classicality *must* translates into semantics. When formulas are interpreted by vector spaces it implies :

$$[\![A^{\perp}]\!] := \mathcal{L}([\![A]\!], \mathbb{R}) = [\![A]\!]'$$

$$[\![A]\!]'' \simeq [\![A]\!]$$

$$ev_x \mapsto x$$

We want a model of *reflexive* vector spaces.

Differential Linear Logic : Syntax

$$A, B := A \otimes B | 1 | A ⅋ B | \bot | A \oplus B | 0 | A \times B | \top | !A | ?A$$

Proofs

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash \Gamma, !A, \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\, \bar{c} \qquad \frac{\vdash}{\vdash !A}\, \bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

Interactions between linearity and non linearity

$$\bar{d} : \begin{cases} E \to !E \\ x \mapsto (f \mapsto D_0(f)(x)) \end{cases} \qquad d : \begin{cases} !E \to E \\ \psi \mapsto \psi_{E'} \in E'' \simeq E \end{cases}$$

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Interactions between linearity and non linearity

$$\bar{d} : \begin{cases} E'' \to \mathcal{C}^\infty(E, \mathbb{R})' \\ ev_x \mapsto (f \mapsto ev_x(D_0(f)) \end{cases} \qquad d : \begin{cases} \mathcal{C}^\infty(E, \mathbb{R})' \to E \\ \psi \mapsto \psi_{E'} \in E'' \simeq E \end{cases}$$

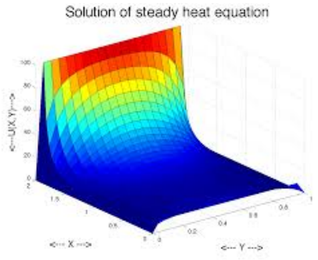
The computational content of differentiation

Historically, resource sensitive syntax and discrete semantics

- Quantitative semantics : $f = \sum_n f_n$
- Resource λ-calculus and Taylor formulas : $M = \sum_n M_n$

Nowadays, differentiation in computer science is motivated by the study of continuous data:

- Differential Geometry and functional analysis
- Ordinary and Partial Differential Equations



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Can we match the requirement of models of LL with the intuitions of physics ?

(YES, we can.)

Smooth and classical models of Differential Linear Logic

Topological vector spaces

We work with Hausdorff topological vector spaces : real or complex vector spaces endowed with a Hausdorff topology making addition and scalar multiplication continuous.

Two layers: algebraic and topological constructions

- The topology on E determines E' as a vector space.
- The topology on E' determines whether $E \simeq E''$.
- Many topologies on $E \otimes F$ which may or may not make it associative.

We work within the category $\textsc{TopVect}$ of topological vector spaces and continuous linear functions between them.

Challenges

We encounter several difficulties in the context of topological vector spaces :

- Finding a category of lcs and smooth functions which is Cartesian closed. Requires some completeness
- Interpreting the involutive linear negation $(E^{\perp})^{\perp} \simeq E$ The topology should not be too fine so as to not allow too many linear continuous scalar forms

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Convenient differential category Blute, Ehrhard Tasson Cah. Geom. Diff. (2010) New: reflexive with the Mackey dual

Mackey-complete spaces and Power series, K. and Tasson, MSCS 2016.

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Weak topologies for Linear Logic, K. LMCS 2015.
Involves a topology which is an internal Chu construction.

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- *A model of LL with Schwartz' epsilon product*, Dabrowski and K., *Preprint*.
- *A logical account for PDEs*, K., *LICS18*

What's not working

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

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The tentative to have a normed space of analytic functions fails (Girard's Coherent Banach spaces).

- We want to use power series.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can't be bounded on an unbounded space (Liouville's Theorem).
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Duality in topological vector spaces

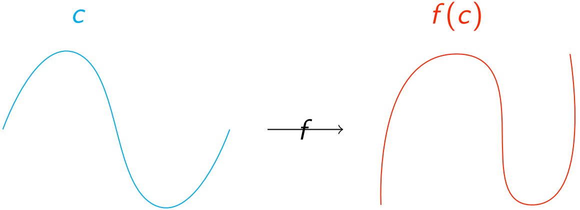
A subcategory of TopVect is $\star$-autonomous *iff* its objects are reflexive $E \simeq E''$.

It's a mess.

- It depends of the topology E'_β, E'_c, E'_w, E'_μ on the dual.
- It is typically not preserved by $\otimes$.
- It is in the canonical case not an orthogonality E'_β is not reflexive.

Smooth maps à la Frölicher,Kriegl and Michor

A smooth curve $c : \mathbb{R} \to E$ is a curve infinitely many times differentiable.



A smooth function $f : E \to F$ is a function sending a smooth curve on a smooth curve.

In Banach spaces, the definition coincides with the usual one (all iterated derivatives exists and are continuous).

A model with higher order smooth functions

A smooth curve $c : \mathbb{R} \to E$ is a curve infinitely many times differentiable.

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A model of IDiLL

This definition leads to a cartesian closed category of Mackey-complete bornological spaces and smooth functions, and to a first smooth model of Intuitionist DiLL [a].

[a] A Convenient differential category, Blute, Ehrhard Tasson Cah. Geom. Diff. (2010)

Nuclear spaces and distributions a smooth classical model

without higher order ... but it can be enhanced

Distributions are everywhere

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support :

$$\phi_f : g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int fg.$$

- In a classical model of Differential Linear Logic :

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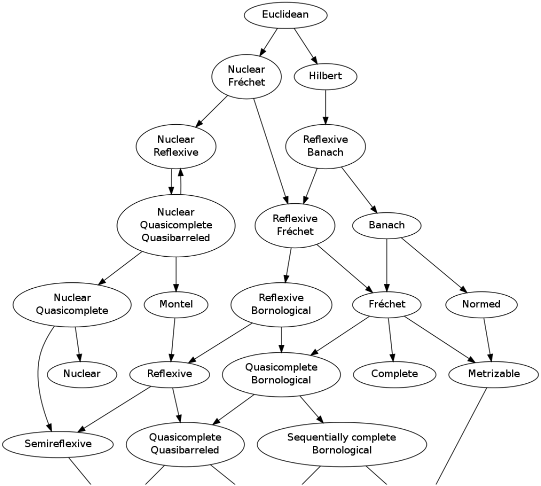
$$!E \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$

In KOTHE and CONV, distributions with compact support arise as a particular case.

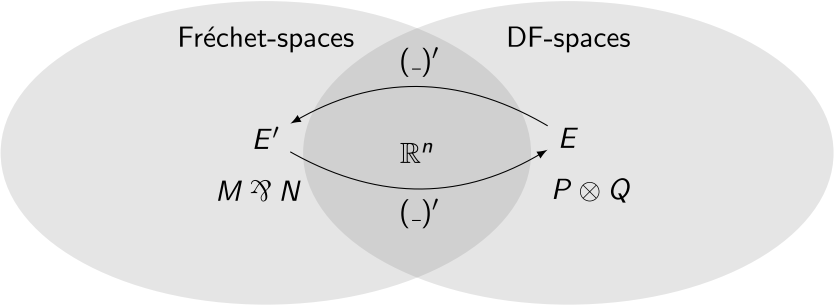
Topological models of DiLL



Let us take the other way around, through Nuclear Fréchet spaces.

Fréchet and DF spaces

- Fréchet : metrizable complete spaces.
- (DF)-spaces : such that the dual of a Fréchet is (DF) and the dual of a (DF) is Fréchet.

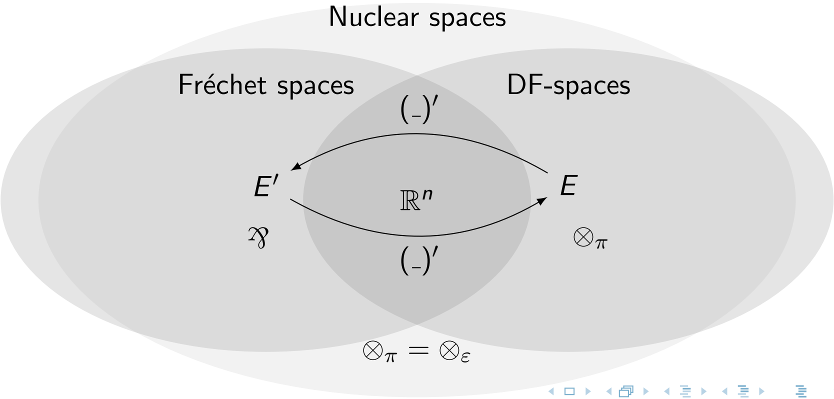


These spaces are in general not reflexive.

Nuclear spaces

Nuclear spaces are spaces in which one can identify the two canonical topologies on tensor products :

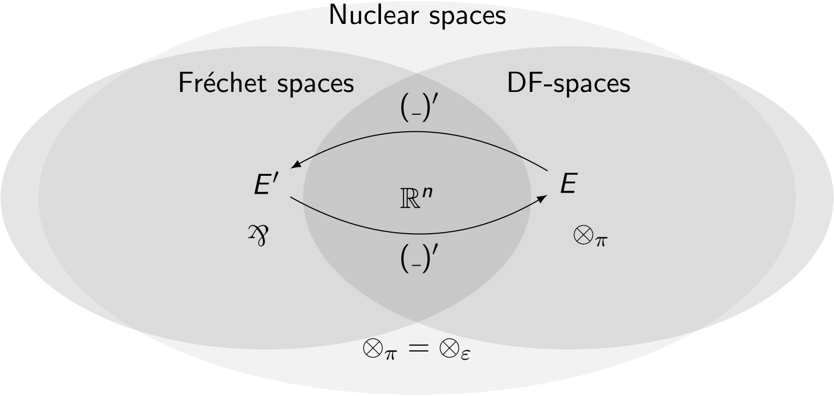
$$\forall F, E \otimes_\pi F = E \otimes_\varepsilon F$$



Nuclear spaces

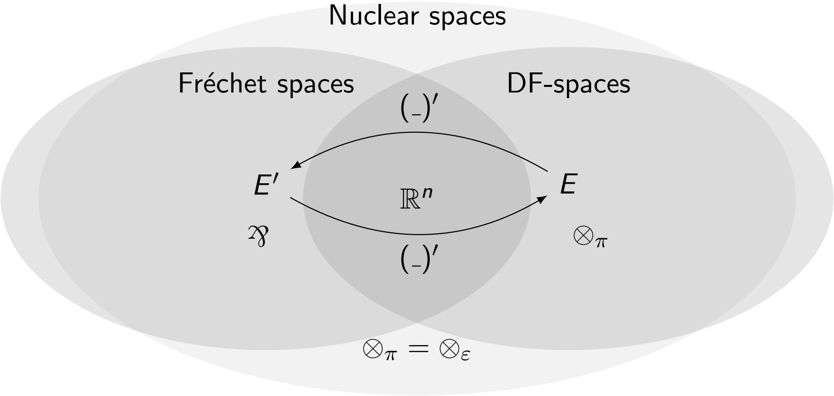
A polarized $\star$-autonomous category

A Nuclear space which is also Fréchet or dual of a Fréchet is reflexive.



Nuclear spaces

We get a polarized model of MALL : involutive negation $(\_)^{\perp}$, $\otimes$, $\parr$, $\oplus$, $\times$.



Distributions and the Kernel theorems

A typical Nuclear Fréchet space is the space of smooth functions on $\mathbb{R}^n$:

$$\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}).$$

A typical Nuclear DF spaces is Schwartz' space of distributions with compact support :

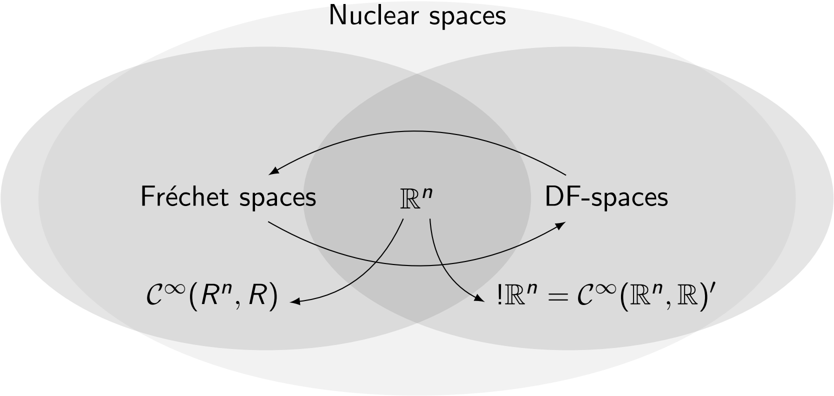
$$\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})' := \{\phi : f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \phi(f) \in \mathbb{R}\}.$$

The Kernel Theorems

$$\mathcal{C}^\infty(E, \mathbb{R})' \hat{\otimes} \mathcal{C}^\infty(F, \mathbb{R})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{R})'$$

$$!\mathbb{R}^n = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'.$$

A model of Smooth differential Linear Logic



A Smooth differential Linear Logic

Smooth DiLL

Finitary formulas Euclidean spaces:

$$A, B := X | A \otimes B | A \parr B | A \oplus B | A \times B.$$

Smooth formulas Nuclear F/DF spaces:

$$U, V := A | !A | ?A | U \otimes V | U \parr V | U \oplus V | U \times V.$$

A polarized model of Smooth DiLL

Functions are **smooth** and **exponential are distributions**.

No higher order : we don't have an obvious way to construct a Nuclear DF lcs $!E = \mathcal{C}^\infty(E, R)'$ when E is any Nuclear Fréchet lcs.

A toy semantics to understand the computational content of Partial Differential Equations.

A Type Theory
for Linear Partial Differential Equations

Linear functions as solutions to a Differential equation

$$
\begin{aligned}
f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \text{ is linear} \quad & \textit{iff } \forall x, f(x) = D(f)(0)(x) \\
& \textit{iff } f = \bar{d}(f) \\
& \textit{iff } \exists g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), f = \bar{d} g
\end{aligned}
$$

Linear functions as solutions to a Differential equation

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Another definition for $\bar{d}$

A linear partial differential operator D acts on $\mathcal{C}^\infty(\mathbb{R}^n, R)$, and is extended on $\mathcal{C}^\infty(\mathbb{R}^n, R)'$:

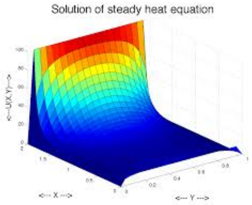
$$D(g)(x) = \sum_{|\alpha| \leq n} a_\alpha(x) \frac{\partial^\alpha g}{\partial x^\alpha}.$$

LPDE with constant coefficient

Consider D a LPDO with constant coefficients :

$$D = \sum_{\alpha, |\alpha| \leq n} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}.$$

The heat equation in $\mathbb{R}^2$
$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$
$u(x, y, 0) = f(x, y)$



Then we know how to solve : $\phi = D\psi, \psi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$ and this is done through an algebraic structure on a specific exponential $!_D$.

Another exponential is possible

$$!_D E = (D(\mathcal{C}_c^\infty(E,\mathbb{R})))'$$

that is the space of linear functions acting on functions $f = Dg$, for $g \in \mathcal{C}_c^\infty(E,\mathbb{R})$, when $E \subset \mathbb{R}^n$ for some n.

$$\bar{d}_D : \begin{cases} !_D E \to !E \\ \phi \mapsto (f \mapsto \phi(D(f))) \end{cases} \qquad d_D : \begin{cases} !E \to !_D E \\ \psi \mapsto \psi_{|D(\mathcal{C}^\infty(A))} \end{cases}$$

Getting back to LL when $D = D_0$

$!_{D_0} A \simeq \mathcal{L}(A,\mathbb{R})' \simeq A$ by reflexivity.

Another exponential is possible

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Getting back to LL when $D = D_0$

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An algebraic structure on $!_D A = (D(\mathcal{C}_c^\infty(A,\mathbb{R})))'$

Existence of a fundamental solution (Malgrange, Ehrhenpeis)

For such D there is $E_D \in \mathcal{C}_c^\infty(A)'$ such that $E_D \circ D = ev_0$.

$$\bar{w}_D : \mathbb{R} \to !_D E, 1 \mapsto E_D$$

D an LPDOcc commutes with convolution

If $f \in D(\mathcal{C}_c^\infty(A))$ and $g \in \mathcal{C}^\infty(A)$, then $f * g \in D(\mathcal{C}_c^\infty(A))$.

$$\bar{c}_D : !E \otimes !_D E \to !_D E, (\phi, \psi) \mapsto D(\phi) * \psi$$

Intermediates rules for D

DiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ d$$

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Syntax for $!_D$ in $D - DiLL$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\ w \qquad \frac{\vdash \Gamma, ?A, ?_D A}{\vdash \Gamma, ?_D A}\ c \qquad \frac{\vdash \Gamma, ?_D A}{\vdash \Gamma, ?A}\ d_D$$

$$\frac{\vdash}{\vdash !_D A}\ \bar{w}_D \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !_D A}{\vdash \Gamma, \Delta, !_D A}\ \bar{c}_D \qquad \frac{\vdash \Gamma, !_D A}{\vdash \Gamma, !A}\ \bar{d}$$

A deterministic cut-elimination.

Solving the LPDE

Consider $\psi \in \mathcal{C}^\infty(E,\mathbb{R})'$: the distribution $\phi \in !_D E$ such that

$$D\phi := \phi \circ D = \psi,$$

i.e. such that for any $f \in \mathcal{C}^\infty(E,\mathbb{R})$: $\phi(Df) = \psi(f)$, is

$$\phi = E_D * \psi.$$

$$\dfrac{\dfrac{\dfrac{\vdash \Gamma, \psi : !E \qquad \dfrac{}{\vdash E_D : !_D E}\ \bar{w}_D}{\vdash \Gamma, E_D * \psi : !_D E}\ \bar{c}_D}{\vdash \Gamma, (E_D * \psi) \circ D : !E}\ \bar{d}_D \qquad \vdash \Delta, f : ?E^\perp}{\vdash \Gamma, \Delta}\ \text{cut} \quad \rightsquigarrow$$

$$\dfrac{\vdash \Gamma, \psi : !E \qquad \vdash \Delta, f : ?E^\perp}{\vdash \Gamma, \Delta}\ \text{cut}$$

Conclusion

Take aways

- What is done in DiLL with differentiation can be done with *any* Linear Partial Differential Operator with constant coefficients.
- Differentiation in logic is linear classical and polarized.

Further work: Theorical computer science and Analysis

- Higher order with distributions : ongoing with JS Lemay. Also Dabrowski, K.
- Curry-Howard : a deterministic PDE calculus.
- Most importantly : towards non-linear PDEs.
- Fourier transformation, Sobolev spaces, Subtyping.

A coalgebraic structure on D

Weakening

$w :!_D E \to \mathbb{R}$ comes from $t : E \to \{0\}$.

If $E = \mathbb{R}^n$, define $\mathbb{R}^{n'}$ another copy of E. Then

$$\begin{aligned} D(\mathcal{C}^\infty(E,\mathbb{R})) \to D(\mathcal{C}^\infty(E \times E,\mathbb{R})) \\ = D(\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^{n'},\mathbb{R})) \\ = D(\mathcal{C}^\infty(E,\mathbb{R}) \parr \mathcal{C}^\infty(\mathbb{R}^{n'},\mathbb{R})) \\ = D(\mathcal{C}^\infty(E,\mathbb{R})) \parr \mathcal{C}^\infty(\mathbb{R}^{n'},\mathbb{R}) \end{aligned}$$

Contraction

We thus have $c :!_D E \to !E \otimes !_D E$.

What's typable with D-DiLL

Consider D a Smooth Linear Partial Differential Operator : D : $\mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$. D acts on $E \times E$:

$$\hat{D} = (D \otimes Id_F)\mathcal{C}^\infty(E \times E, \mathbb{R}) \to \mathcal{C}^\infty(E \times E, \mathbb{R})$$

Then Green's function is the operator $K_{x,y} :!E \; to!E$ such that :

$$K_{x,y} \circ (\hat{D})' = \delta_{x-y}$$

$$\dfrac{\dfrac{\vdash \Gamma, ?_D E^\perp, ?E^\perp}{\vdash ?_D E^\perp}\, c_D \qquad \dfrac{\vdash \Delta, ?_D E \qquad \dfrac{\vdash}{\vdash !_D E}\, \bar{w}_D}{\vdash ?_D \Delta, !_D E}\, c_D}{\vdash \Gamma, \Delta}\, cut$$

A closer look to Kernels

A answer to a well-known issue :

- Any $k \in (\mathrm{L}_p(\mu \otimes \eta))'$ gives rise to a compact operator $T_k : \mathrm{L}_p(\mu) \to \mathrm{L}_{p^*}(\eta) \simeq (\mathrm{L}_p(\eta))' : \; T_k(f)(g) = k(f.g)$.
- This is not a surjection : if $p = p^* = 2$, for $T_k = Id$ one should have $k = \delta_{x-y}$, which is *not a function*.
- The above morphism $k \mapsto T_k$ is an isomorphism on spaces of distributions spaces, generalizing L_p :

Kernel theorems

$$\mathcal{L}(\mathcal{C}^\infty(E,\mathbb{R})', \mathcal{C}^\infty(F,\mathbb{R})'') \simeq \mathcal{C}^\infty(E,\mathbb{R})' \hat{\otimes} \mathcal{C}^\infty(F,\mathbb{R})'$$
$$\simeq \mathcal{C}^\infty(E \times F, \mathbb{R})'$$
$$T_k \mapsto K_{x,y}$$

A closer look to Kernels

A answer to a well-known issue :

- Any $k \in (\mathrm{L}_p(\mu \otimes \eta))'$ gives rise to a compact operator $T_k : \mathrm{L}_p(\mu) \to \mathrm{L}_{p^*}(\eta) \simeq (\mathrm{L}_p(\eta))' : \; T_k(f)(g) = k(f.g)$.
- This is not a surjection : if $p = p^* = 2$, for $T_k = Id$ one should have $k = \delta_{x-y}$, which is *not a function*.
- The above morphism $k \mapsto T_k$ is an isomorphism on spaces of distributions spaces, generalizing L_p :

Kernel theorems

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Nuclearity

A closer look to Kernels

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Kernel theorems

$$\mathcal{C}^\infty(E,\mathbb{R})'\hat{\otimes}\mathcal{C}^\infty(F,\mathbb{R})' \simeq \mathcal{L}(\mathcal{C}^\infty(E,\mathbb{R})',\mathcal{C}^\infty(F,\mathbb{R})'')$$

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Density