SD2014, Vienna

Weak topologies, duality and polarities

Marie Kerjean

PPS Laboratory
Paris 7 University

July 12, 2014

Introduction

Motivation : A model of *LL* whose objects are intuitive (general vector spaces) but were not constructed specifically for Linear Logic.

- A strong link between Linear Logic and Functional Analysis.
- A mathematical interpretation of connectives according to their polarities.

Spoilers

We have a model of propositional Linear Logic:

- The **formulas** are interpreted by the separated and locally convex topological vector spaces, endowed with their weak topology.
- **Linear proofs** are interpreted by the continuous linear maps.
- **Non-linear proofs** are interpreted by sequences of monomials.

Plan

- **Duality** in LL : How to interpret the involutive linear negation ? Orthogonalities and weak topologies.
- **Polarities** : the enforcement of the weak topology as a shift from positive connectives to negative connectives.

How can duality be interpreted?

Let us write $[A]$ for the semantic interpretation of a formula A of Linear Logic.

You want to have **reflexive** objects: $[\neg\neg A] = [A]$.

- In **Rel** : $[\neg A] = [A]$.
- In **Coherent spaces**, Finiteness spaces, Köthe spaces ... : $[\neg A] = [A]^\perp$

where $[A]^\perp$ is the orthogonal of the coherent space $[A]$.

Orthogonality relations

Definition

$\perp \subset \Omega_1 \times \Omega_2$ is a symmetric relation. If $X \subset \Omega_1$, then $X^\perp = \{y \in \Omega_2 \mid \forall x \in X, (x, y) \in \perp\}$.

Example

If A is a coherent space, if $a, b \subset A$, then $a \perp b$ iff $|a \cap b| \leq 1$.

A set is **bi-orthogonally closed** if $(X^\perp)^\perp = X$. If A is a coherent space, and $\mathcal{C}(A)$ the set of its cliques, then $\mathcal{C}(A)^{\perp\perp} = \mathcal{C}(A)$.

Duality and orthogonality

Double orthogonality completion

$X^{\perp}$ is always reflexive: $X^{\perp\perp\perp} = X^{\perp}$.

When an object is not reflexive, we can make it reflexive !

Example

If A and B are two coherent spaces

$$\mathcal{C}(A \otimes B) = \{a \otimes b \mid a \in \mathcal{C}(A), b \in \mathcal{C}(B)\}^{\perp\perp}$$

where $a \otimes b = \{(x, y) \mid x \in a, y \in b\}$

How can duality be interpreted?

You want to have **reflexive** objects : $[\neg\neg A] = [A]$.

Let us write $[A]$ from the semantical interpretation of a formula $[A]$

- In **Rel** : $[\neg A] = [A]$.
- In **Coherent spaces**, Finiteness spaces, Köthe spaces ...: $[\neg A] = [A]^{\perp}$. You restrict to spaces where a definition by orthogonality is possible.
- In $\mathbb{K}$-**vector spaces**, $[\neg A] = [A]^* = L([A], \mathbb{K})$ the algebraic dual of E.

The last point is intuitive: $A^{\perp} = A^{\perp} \parr \perp = A \multimap \perp$.

Duality in vector spaces

If $[A]$ is a vector space, $[A]^* = L([A], \mathbb{K})$ is its dual.

No reflexivity completion

If E is a vector space, E^* is not reflexive in general.

Definition

A topological vector space E is a vector space endowed with a topology making the addition and multiplication by a scalar continuous. E' is the **topological dual** of E.

Duality in topological vector spaces

Definition

A topological vector space E is a vector space endowed with a topology making the addition and multiplication by a scalar continuous. E' is the **topological dual** of E.

No reflexivity completion

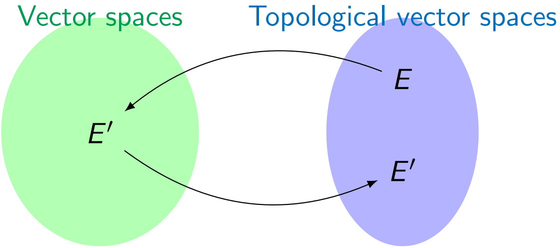
If E is a **topological vector space**, E' is not reflexive in general.

We are going to work with locally convex and separated topological vector spaces : E, F.

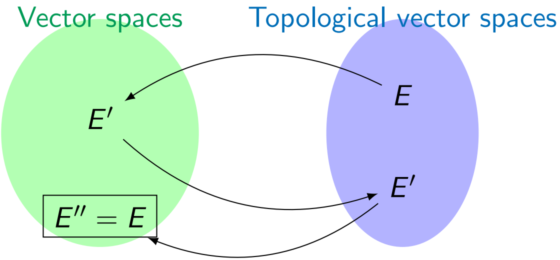
The weak topology on E'

A weak topology induced by E

We endow E' with the weak topology induced by E, that is the coarsest topology making all $ev_x : E' \to \mathbb{K}$ continuous.



The weak topology on E'



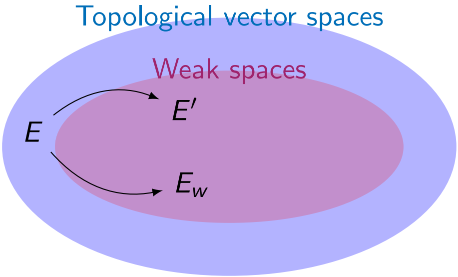
Fundamental property

When E' is endowed with the weak topology induced by E, then E'' and E are the same vector spaces.

The weak topology on E

A weak topology induced by E

We endow E with the weak topology induced by E', that is the coarsest topology making all $l \in E$ continuous. E_w is the vector space E endowed with its weak topology.

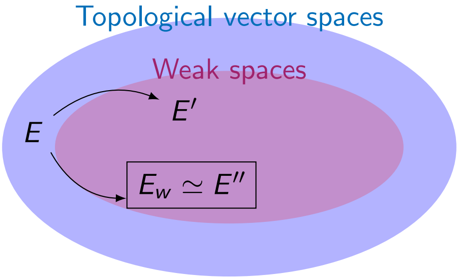


E' is already a weak space: the weak topologies induced by E or E'' corresponds.

The weak topology on E

A weak topology induced by E

We endow E with the weak topology induced by E', that is the coarsest topology making all $l \in E$ continuous. E_w is the vector space E endowed with its weak topology.



E'' and E_w are the same **topological** vector spaces.

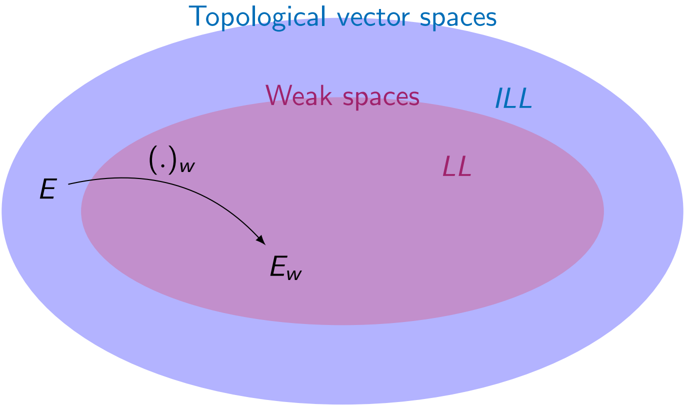
A model of LL

- $\otimes$ is interpreted by the inductive **tensor product**.
  - We have a monoidal closed category, thanks to the chosen topology and the fact that $\mathcal{L}_s(E, F_w)' = E \otimes F'$.
- $\parr$ is its dual. $E \parr F$ is the **space of separately continuous bilinear forms** on $E \times F$.
- $\oplus$ is the topological **co-product**, $\times$ is the topological **product**.

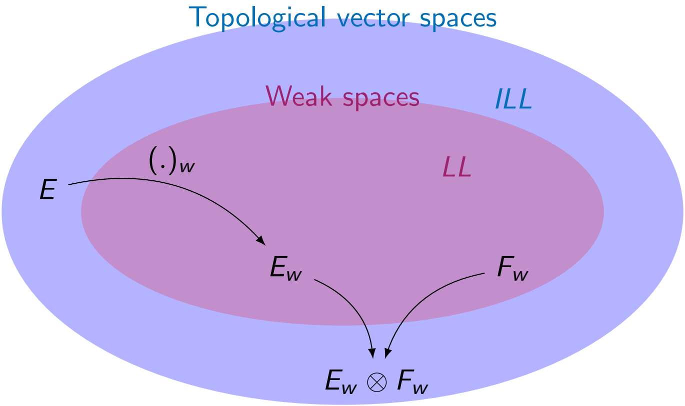
Quantitative semantics helps us finding a good exponential.

... and then we consider these spaces endowed with their weak topology.

Polarities

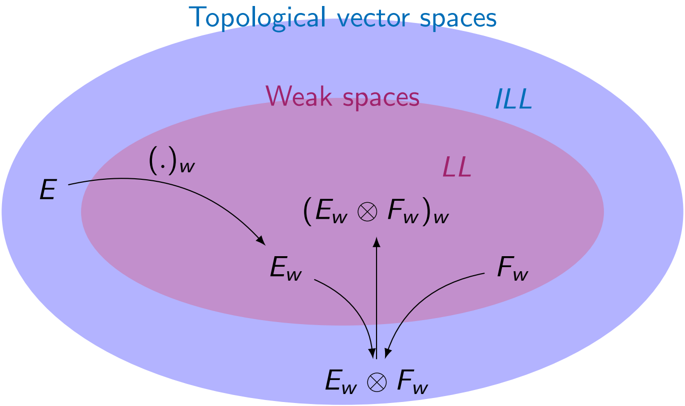


A positive connective



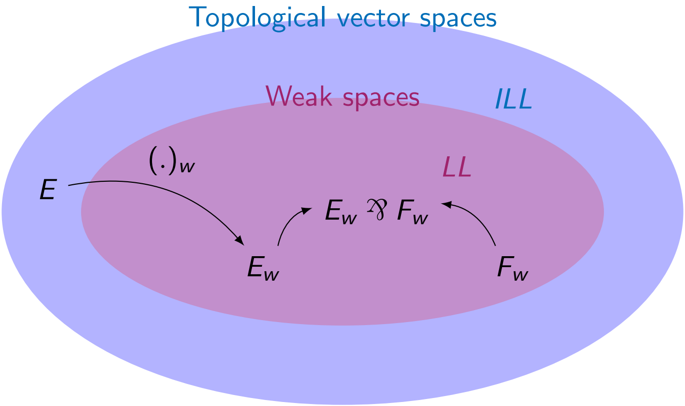
Positive connectives don't preserve the weak topology.

A positive connective



Positive connectives don't preserve the weak topology.

A negative connective



Negative connectives preserve the weak topology.

Polarities and weak topologies

If we write $\uparrow E$ for E_w, when E is a locally convex and separated topological vector space :

- $\uparrow(E \otimes F) \neq \uparrow E \otimes \uparrow F$ and $\uparrow(E \otimes F) = \uparrow(\uparrow E \otimes \uparrow F)$.
- $\uparrow(E \parr F) = \uparrow E \parr \uparrow F = E \parr F$.
- $\uparrow \oplus_{i \in \mathbb{N}} E_i \neq \oplus_{i \in \mathbb{N}} \uparrow E_i$ but $\uparrow \oplus_{i \in \mathbb{N}} E_i \neq \uparrow \oplus_{i \in \mathbb{N}} E_i$.
- $\uparrow \&_{i \in \mathbb{N}} E_i = \&_i \uparrow E_{i \in \mathbb{N}}$ but $\&_i \uparrow E_{i \in \mathbb{N}} \neq \&_{i \in \mathbb{N}} E_i$.
- $\uparrow ! E \neq ! \uparrow E$.
- $\uparrow ? E = ? \uparrow E$.

Shift and weak topologies

$(.) = \uparrow$

Negatives connectives are exactly those which preserve the weak topology.

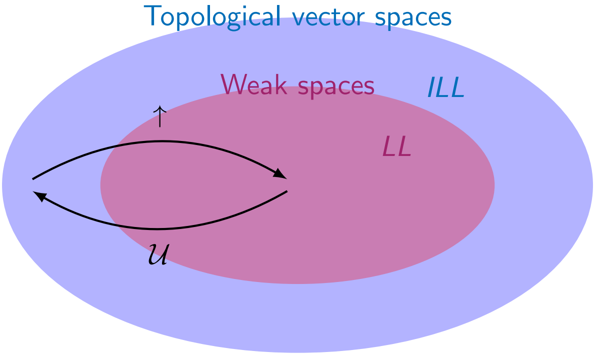
A loss of information

- $E \to E_w$ is always continuous but $E_w \to E$ is not. E_w has less open sets than E.
- The construction of the interpretation of a positive connective is a non-reversible operation.

An adjunction

Proposition

If E and F are tvs, $\mathcal{L}(E, F_w) \simeq \mathcal{L}(E_w, F_w)$. $\uparrow$ is left adjoint to $\mathcal{U}$.



(Discussion with T. Ehrhard).

Polarities and Orthogonalities

When using orthogonalities to interpret the involutive linear negation of LL, there is also a distinctive use of polarities.

Negative connectives in Coherent spaces

- If we write $\mathcal{C}(X) \otimes \mathcal{C}(Y) = \{x \otimes y | x \in \mathcal{C}(X), y \in \mathcal{C}(Y)\}$ with $x \otimes y = \{(a,b) \mid a \in x, b \in y\}$, then

$$\mathcal{C}(X \otimes Y) = (\mathcal{C}(X) \otimes \mathcal{C}(Y))^{\perp\perp}.$$

- If we write $!\mathcal{C}(X) = \{u \subset \mathcal{C}_{fin}(x) \mid \bigcup u \in \mathcal{C}_{fin}(X)\}$ then

$$\mathcal{C}(!X) = (!\mathcal{C}(X))^{\perp\perp}.$$

Positive connectives

If we write $\mathcal{C}(X) \parr \mathcal{C}(Y) = (\mathcal{C}(X) \otimes \mathcal{C}(Y))^{\perp}$, then $\mathcal{C}(X \parr Y) = \mathcal{C}(X) \parr \mathcal{C}(Y)$. Idem for $\oplus$ and ?.

For which orthogonality could we have:

$$(.)_w = (.)^{\perp\perp} \text{ ?}$$

Perspectives

- Barr's work: a similar model with the Mackey topology ?
- An interpretation of focused proof ? The downward shift could be interpreted by the enforcement of the weak* topology.
- Models with richer topological vector spaces ?

Thank you.

The exponential

Definition

$!E \simeq \bigoplus_{n\in\mathbb{N}} \mathcal{H}^n(E,\mathbb{K})'$ and if $f \in \mathcal{L}(E_w, F_w)$ we define

$$!f : \begin{cases} !E_w \to !F_w \\ \phi \mapsto ((g_n) \in \prod_n \mathcal{H}^n(F,\mathbb{K}) \mapsto \phi((g_n \circ f)_n) \end{cases}$$

The exponential

$$\epsilon_E \begin{cases} !E_w \to E_w \\ \quad \phi \mapsto \phi_1 \in E'' \simeq E \end{cases}$$

$$\delta_E \begin{cases} !E_w \to !!E_w \simeq \left(\prod_n \mathcal{H}^n([\prod_m \mathcal{H}^m(E, \mathbb{K})]', \mathbb{K}) \right)' \\ \quad \phi \mapsto \left[(g_n)_n \mapsto \phi(\left(x \in E \mapsto \sum_{k|p} g_k[(f_m)_m \mapsto f_{p|k}(x)] \right)_p) \right] \end{cases}$$