TACL 2015, Ischia

# $\star$-autonomous categories and tensor products

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# An history of Linear Logic

**S**yntax

$\lambda$-calculus

**S**emantic

Normal functors

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**S**yntax

$\lambda$-calculus

**S**emantic

Normal functors

Linearity

A vectorial setting

$f : A \multimap B$ linear

$g : !A \multimap B$ non-linear

# An history of Linear Logic

**S**yntax

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Linear Logic
Girard 88

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# Linear Logic, two implications

Grammar : $A, B ::= 1 | \bot | \top | 0 | A \parr B | A \otimes B | A \oplus B | A \,\&\, B | !A | ?A$

U

Cartesian closed category : Non-linear functions $!A \multimap B$, &, $\oplus$

Monoidal Closed Category : Linear Functions $A \multimap B$, $\otimes$, $\parr$

!

# Linear Logic, a linear negation

A model of Linear Logic must also be a *-autonomous category.

It is a monoidal closed category with a distinguished object $\perp$, where the morphism

$$d_A : A \rightarrow (A \multimap \perp) \multimap \perp$$

is an isomorphism.

$d_A$ is the transpose of

$$eval_A : A \otimes (A \multimap \perp) \rightarrow \perp.$$

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Normal functors

Relational model

Formulas as sets, Proofs as relations

Köthe spaces

Ehrhard 02

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A vectorial setting

Normal functors

Linear Logic

Relational model

Differentiation

A smooth setting

Differential Linear Logic

Ehrhard Regnier 03

Köthe spaces

Differential $\lambda$-calculus

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Convenient spaces : Blute, Ehrhard, Tasson 2010

# What do we want

I want to explain to my applied math colleague
what is a model of LL.

U

**Cartesian closed category :**
Non-linear functions
$!A \multimap B$, $\&$, $\oplus$

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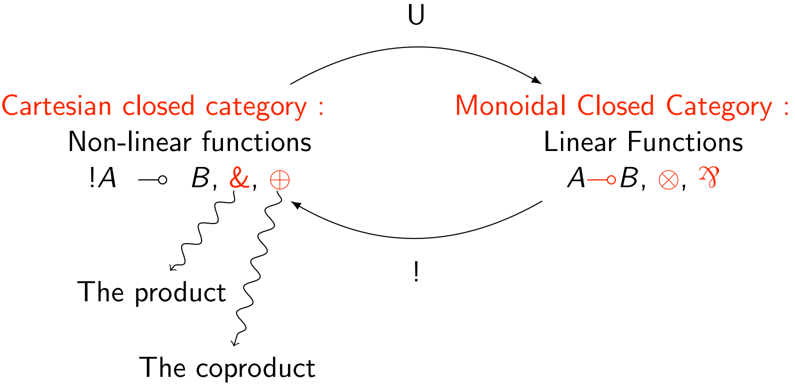

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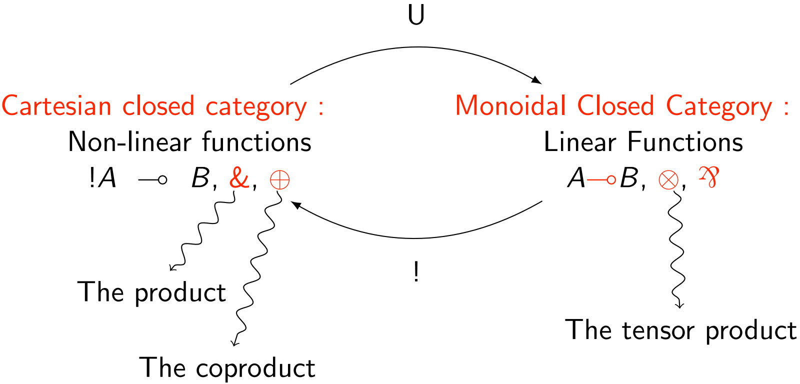

# What do we want

I want to explain to my applied math colleague what is a *-autonomous category:

The following must be an isomorphism for every $A$ :

$$d_A : A \to (A \multimap \bot) \multimap \bot$$

$d_A$ is the transpose of

$$\begin{aligned} eval_A : & A \otimes (A \multimap \bot) \to \bot \\ & A \times \mathcal{L}(A, \mathbb{K}) \to \mathbb{K} \\ & x, f \mapsto f(x) \end{aligned}$$

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## Exclamation

Well, this is a just a category of reflexive vector space.

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## Exclamation

Well, this is a just a category of reflexive vector space.

## Disapointment

Well, the category of reflexive topological vector space is not closed.

# Weak topologies

## Theorem

The category of spaces endowed with their weak topology is a model of Linear Logic

*If the dual $E'$ of a topological vector space $E$ is endowed with its weak* topology, then $E''$ is isomorphic to $E$.*

*The reversible connectives are exactly those preserving the weak topology .*

# A topology on the algebraic constructions

U

Power Series

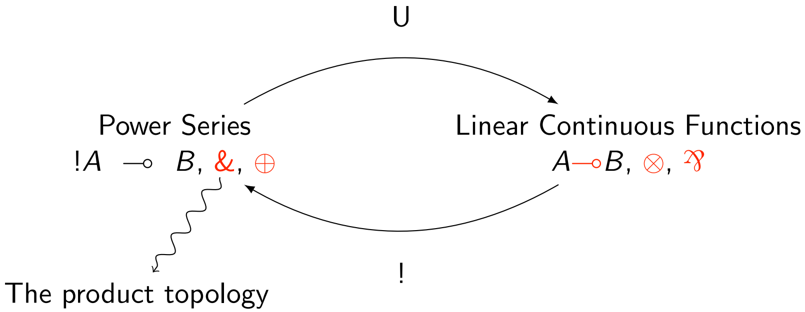
$!A \multimap B$, $\&$, $\oplus$

Linear Continuous Functions

$A \multimap B$, $\otimes$, $⅋$

!

# A topology on the algebraic constructions



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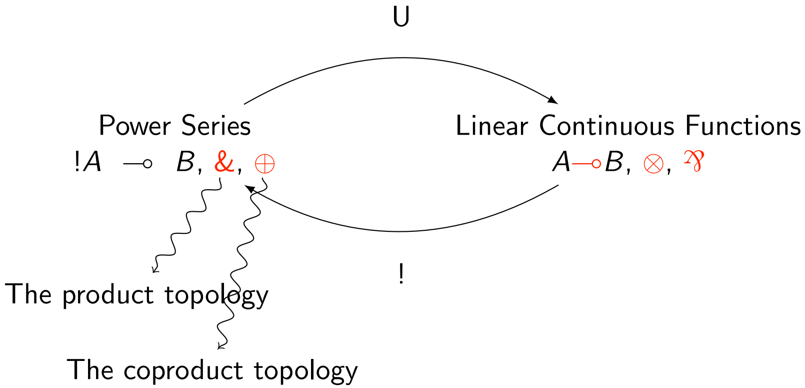

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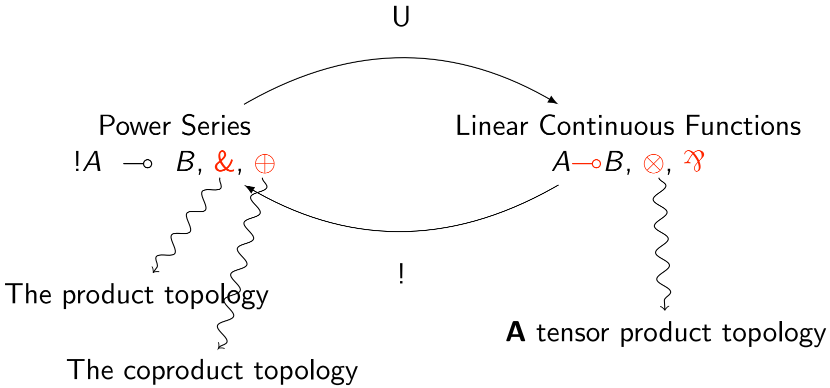

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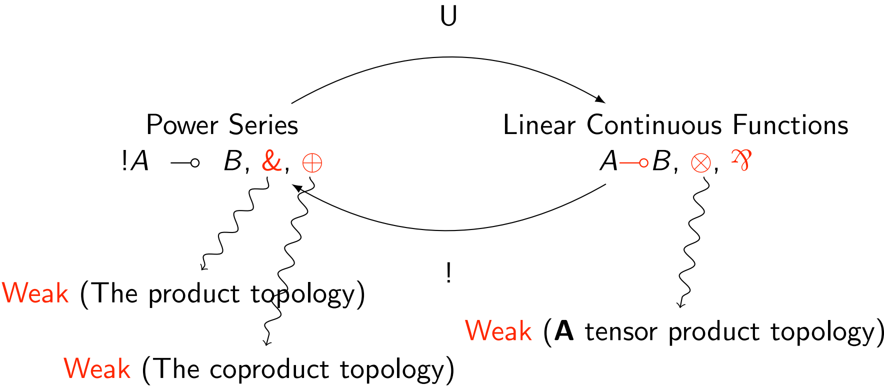

# A choice for the tensor product

There are three canonical topologies on the tensor product of two topological vector spaces $E$ and $F$.

$$E \otimes_i F, E \otimes_\pi F, E \otimes_\epsilon F$$

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- Identifying $\otimes_\pi$ and $\otimes_\epsilon$ defines Nuclear spaces.

# A choice for the tensor product

There are three canonical topologies on the tensor product of two topological vector spaces $E$ and $F$.

$$E \otimes_i F, E \otimes_\pi F, E \otimes_\epsilon F$$

- Identifying $\otimes_\pi$ and $\otimes_\epsilon$ defines Nuclear spaces.
- Fréchet spaces are the complete metrizable spaces. In such a space, $\otimes_\pi$ and $\otimes_i$ correspond.

# Nuclear Fréchet spaces are Reflexive spaces

**Theorem**

A Nuclear space which is also Fréchet or (DF) is reflexive.

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A Nuclear space which is also Fréchet or (DF) is reflexive.

*The category of Nuclear Fréchet or (DF) is monoidal closed.*

*Nuclear Fréchet or (DF) spaces preserve the cartesian product and coproduct.*

# Nuclear Fréchet spaces are Reflexive spaces

## Theorem

A Nuclear space which is also Fréchet or (DF) is reflexive.

> *The category of Nuclear Fréchet or (DF) is monoidal closed.*
>
> *Nuclear Fréchet or (DF) spaces preserve the cartesian product and coproduct.*

## Theorem

Nuclear Fréchet (or (DF)) spaces form a model of Polarized Multiplicative Additive Linear Logic.

# A smooth Exponential ?

Examples of Nuclear Fréchet or (DF) space :

$$\mathcal{C}_c^\infty(U),\ \mathcal{D}'(U),\ \mathcal{C}^\infty(V),\ \mathcal{H}(V).$$

where $U$ is an open subset of $\mathbb{R}^n$ and $V$ is a smooth or analytical manifold.

They verify :

$$\mathcal{F}'(V) \hat{\otimes} \mathcal{F}'(U) = \mathcal{F}'(U \times V)$$

Thank you.