*TACL 2015, Ischia*

*—autonomous categories and tensor products*

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An history of Linear Logic

Syntax

$\lambda$-calculus

Semantic

Normal functors
An history of Linear Logic

Syntax

\[ \lambda \text{-calculus} \]

Linearity

A vectorial setting

\[ f : A \rightarrow B \text{ linear} \]

\[ g : !A \rightarrow B \text{ non-linear} \]

Semantic

Normal functors

A smooth setting

S

Köthe spaces
An history of Linear Logic

Syntax

\(\lambda\)-calculus

Linear Logic
Girard 88

Linearity
A vectorial setting

Semantic

Normal functors

A *-autonomous category of weak spaces

Köthe spaces

Differential Linear Logic

Differential \(\lambda\)-calculus
Linear Logic, two implications

Grammar: \( A, B ::= 1 \mid \bot \mid T \mid 0 \mid A \otimes B \mid A \otimes B \mid A \oplus B \mid A \& B \mid !A \mid ?A \)

Cartesian closed category:
- Non-linear functions
  \( !A \twoheadrightarrow B, \&, \oplus \)

Monoidal closed category:
- Linear functions
  \( A \rightarrow B, \otimes, \& \)
Linear Logic, a linear negation

A model of Linear Logic must also be a *-autonomous category.

It is a monoidal closed category with a distinguished object \( \bot \), where the morphism

\[
d_A : A \to (A \to \bot) \to \bot
\]

is an isomorphism.

\( d_A \) is the transpose of

\[
eval_A : A \otimes (A \to \bot) \to \bot.
\]
An history of Linear Logic

Syntax
λ-calculus
Linear Logic

Linearity
A vectorial setting

Semantic
Normal functors

Relational model
Formulas as sets, Proofs as relations
Köthe spaces
Ehrhard 02
An history of Linear Logic

Syntax

- λ-calculus
- Linear Logic

Linearity

- A vectorial setting

Semantic

- Normal functors

Differentiation

- A smooth setting

Relational model

Köthe spaces
An history of Linear Logic

Syntax
- λ-calculus
- Linear Logic

Linearity
- A vectorial setting

Semantic
- Normal functors
  - Köthe spaces
  - Convenient spaces:
    - Blute, Ehrhard, Tasson 2010

Relational model
What do we want

I want to explain to my applied math colleague what is a model of LL.

Cartesian closed category:
Non-linear functions
!A → B, &, ⊕

Monoidal closed category:
Linear functions
A → B, ⊗, ⨿
What do we want

I want to explain to my applied math colleague what is a model of LL.

Cartesian closed category:
- Non-linear functions
  - \!(A \rightarrow B, \&, \oplus)

Monoidal closed category:
- Linear Functions
  - \!(A \rightarrow B, \otimes, \boxtimes)

The product
What do we want

I want to explain to my applied math colleague what is a model of LL.

Cartesian closed category:
Non-linear functions

\(!A \to B, \&, \oplus\)

Monoidal Closed Category:
Linear Functions

\(A \to B, \otimes, \bowtie\)

The product

The coproduct
What do we want

I want to explain to my applied math colleague what is a model of LL.

Cartesian closed category:
Non-linear functions
\(!A \rightarrow B, \&\)
The product
The coproduct

Monoidal closed category:
Linear functions
\(A \rightarrow \otimes \otimes\)
The tensor product
What do we want

I want to explain to my applied math colleague what is a *-autonomous category:

The following must be an isomorphism for every $A$:

$$d_A : A \to (A \multimap \bot) \multimap \bot$$

$d_A$ is the transpose of $\text{eval}_A$:

$$\text{eval}_A : A \otimes (A \multimap \bot) \to \bot$$

$$A \times \mathcal{L}(A, \mathbb{K}) \to \mathbb{K}$$

$x, f \mapsto f(x)$
What do we want

I want to explain to my applied math colleague what is a *-autonomous category:

The following must be an isomorphism for every $A$:

$$d_A : A \rightarrow (A \rightarrow \perp) \rightarrow \perp$$

$$A \rightarrow \mathcal{L}(\mathcal{L}(A, \mathbb{K}), \mathbb{K})$$

$$x \mapsto (\delta_x : f \mapsto f(x))$$

$d_A$ is the transpose of

$$eval_A : A \otimes (A \rightarrow \perp) \rightarrow \perp$$

$$A \times \mathcal{L}(A, \mathbb{K}) \rightarrow \mathbb{K}$$

$$x, f \mapsto f(x)$$
What do we want

I want to explain to my applied math colleague what is a *-autonomous category:

\[ d_A : x \mapsto (\delta_x : f \mapsto f(x)) \]

should be an isomorphism.

Exclamation
Well, this is a just a category of reflexive vector space.
What do we want

I want to explain to my applied math colleague what is a *-autonomous category:

\[ d_A : x \mapsto (\delta_x : f \mapsto f(x)) \]

should be an isomorphism.

**Exclamation**
Well, this is a just a category of reflexive vector space.

**Disapointment**
Well, the category of reflexive topological vector space is not closed.
Weak topologies

**Theorem**
The category of spaces endowed with their weak topology is a model of Linear Logic

*If the dual $E'$ of a topological vector space $E$ is endowed with its weak* topology, then $E''$ is isomorphic to $E$.*

*The reversible connectives are exactly those preserving the weak topology.*
A topology on the algebraic constructions

\begin{align*}
\text{Power Series} & \quad \! A \rightarrow B, \& , \oplus \\
\text{Linear Continuous Functions} & \quad A \rightarrow B, \otimes , \forall
\end{align*}
A topology on the algebraic constructions

Power Series

\( A \rightarrow B, \& , \oplus \)

Linear Continuous Functions

\( A \rightarrow B, \otimes, \bigcirc \)

The product topology

\( U \)
A topology on the algebraic constructions

\[ \text{Power Series} \quad \forall A \rightarrow B, \& , \oplus \]

\[ \text{Linear Continuous Functions} \quad A \rightarrow B, \otimes, \wedge \]

The product topology

The coproduct topology
A topology on the algebraic constructions

- Power Series: $!A \to B, \&, \oplus$
- Linear Continuous Functions: $A \to B, \otimes, \Upsilon$
- The product topology
- The coproduct topology
A topology on the algebraic constructions

\[ U \]

Power Series
\[ !A \rightarrow B, \& , \oplus \]

Linear Continuous Functions
\[ A \rightarrow B, \otimes , \vee \]

Weak (The product topology)

Weak (The coproduct topology)

Weak (A tensor product topology)
A choice for the tensor product

There are three canonical topologies on the tensor product of two topological vector spaces $E$ and $F$.

$$E \otimes_i F, E \otimes_\pi F, E \otimes_\epsilon F$$
A choice for the tensor product

There are three canonical topologies on the tensor product of two topological vector spaces $E$ and $F$.

\[ E \otimes_i F, \ E \otimes_\pi F, \ E \otimes_\epsilon F \]

- Identifying $\otimes_\pi$ and $\otimes_\epsilon$ defines Nuclear spaces.
A choice for the tensor product

There are three canonical topologies on the tensor product of two topological vector spaces $E$ and $F$.

$E \otimes_i F, E \otimes_\pi F, E \otimes_\epsilon F$

- Identifying $\otimes_\pi$ and $\otimes_\epsilon$ defines Nuclear spaces.
- Fréchet spaces are the complete metrizable spaces. In such a space, $\otimes_\pi$ and $\otimes_i$ correspond.
Nuclear Fréchet spaces are Reflexive spaces

Theorem
A Nuclear space which is also Fréchet or (DF) is reflexive.
Nuclear Fréchet spaces are Reflexive spaces

Theorem
A Nuclear space which is also Fréchet or (DF) is reflexive.

The category of Nuclear Fréchet or (DF) is monoidal closed.

Nuclear Fréchet or (DF) spaces preserve the cartesian product and coproduct.
Nuclear Fréchet spaces are Reflexive spaces

**Theorem**

A Nuclear space which is also Fréchet or (DF) is reflexive.

*The category of Nuclear Fréchet or (DF) is monoidal closed.*

*Nuclear Fréchet or (DF) spaces preserve the cartesian product and coproduct.*

**Theorem**

Nuclear Fréchet (or (DF)) spaces form a model of Polarized Multiplicative Additive Linear Logic.
A smooth Exponential?

Examples of Nuclear Fréchet or (DF) space:

\[ C_c^\infty(U), \mathcal{D}'(U), C^\infty(V), \mathcal{H}(V). \]

where \( U \) is an open subset of \( \mathbb{R}^n \) and \( V \) is a smooth or analytical manifold.

They verify:

\[ \mathcal{F}'(V) \hat{\otimes} \mathcal{F}'(U) = \mathcal{F}'(U \times V) \]
Thank you.