Trends in Linear Logic and Applications & Linearity
July 2018

Smooth denotational models of Linear Logic based on Schwart'z ε product

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Proofs and smooth objects

$\parr = \varepsilon$

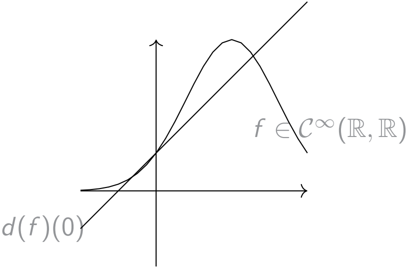
Duality and completion

Smooth functions and new topologies

Smoothness

Differentiation

Differentiating a function $f : \mathbb{R}^n \to \mathbb{R}$ at x is finding a linear approximation $d(f)(x) : v \mapsto d(f)(x)(v)$ of f near x.



A co-inductive definition

Smooth functions are functions which can be differentiated everywhere in their domain and whose differentials are smooth.

Differentiating proofs

- Differentiation was in the air since the study of Analytic functors by Girard :

$$\bar{d}(x) : \sum f_n \mapsto f_1(x)$$

- DiLL was developed after a study of vectorial models of LL inspired by coherent spaces : Finiteness spaces (Ehrhard 2005), Köthe spaces (Ehrhard 2002).

Normal functors, power series and λ-calculus. Girard, APAL(1988)

Differential interaction nets, Ehrhard and Regnier, TCS (2006)

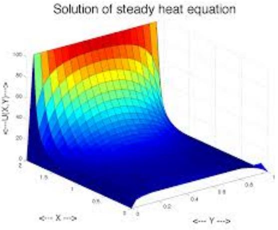
The computational content of differentiation

Historically, resource sensitive syntax and semantics:

- Quantitative semantics : $f = \sum_n f_n$
- Resource λ-calculus, Taylor formulas, probabilities and algebraic syntax (Ehrhard, Pagani, Tasson, Vaux ...) : $M = \sum_n M_n$

Differentiation in Physics and Mathematics takes part in the study of continuous systems :

- Differential Geometry and functional analysis
- Ordinary and Partial Differential Equations



Differential Linear Logic

The rules of DiLL are those of MALL + promotion + :

Syntax

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash}{\vdash !A}\, \bar{w} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\, \bar{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

What's not working

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

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The tentative to have a normed space of analytic functions fails (Girard's Coherent Banach spaces).

- We want to use power series.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can't be bounded on an unbounded space (Liouville's Theorem).
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Topological vector spaces

We work with Hausdorff topological vector spaces : real or complex vector spaces endowed with a Hausdorff topology making addition and scalar multiplication continuous.

Two layers: algebraic and topological constructions

- The topology on E determines the dual E' as a vector space.
- The topology on E' determines whether $E \simeq E''$.
- Many topologies on the tensor $E \otimes F$ which may or may not lead to a monoidal closed category, depending of the spaces (Grothendieck "problèmes des topologies").

We work within the category $\textsc{TopVect}$ of topological vector spaces and continuous linear functions between them.

Challenges

We encounter several difficulties in the context of topological vector spaces :

- ✓ Finding a category of tvs and smooth functions which is Cartesian closed. Requires some completeness.
- ✓ Interpreting the involutive linear negation $(E^{\perp})^{\perp} \simeq E$.

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Convenient differential category Blute, Ehrhard Tasson Cah. Geom. Diff. (2010) New: reflexive with the Mackey dual

Mackey-complete spaces and Power series, K. and Tasson, MSCS 2016.

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Weak topologies for Linear Logic, K. LMCS 2015.
Involves a topology which is an internal Chu construction.

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- *A model of LL with Schwartz' epsilon product*, Dabrowski and K., *Preprint.*
- *A logical account for PDEs*, K., *LICS18*

$$\mathscr{Y} = \varepsilon$$

A good $\parr$ is a glueing $\parr$

$$A^{\perp} \parr B \equiv A \multimap B$$

$$(!A)^{\perp} \parr B \equiv A \Rightarrow B$$

When proofs are interpreted by Smooth functions :

$$\mathcal{C}^{\infty}(E, \mathbb{R})\varepsilon F \simeq \mathcal{C}^{\infty}(E, F)$$

The ε product

Only one good $\parr$

$$E \varepsilon F := \mathcal{L}_{\varepsilon}(E'_c, F)$$

No surprises algebraically speaking, but the choice of topologies is important.

$$\mathcal{C}^\infty(E,F) \simeq \mathcal{C}^\infty(E,\mathbb{R})\varepsilon F \text{ when } E \text{ and } F \text{ are complete.}$$

A monoidal category by Schwartz

The ε is associative and commutative on quasi-complete spaces.

Théorie des Distributions à valeurs vectorielles, I Schwartz, (1957)

Negatives are interpreted by (quasi, k-, Mackey) complete spaces.
And $\uparrow$ is the completion.

A minimal condition for associativity

Reading back Schwartz's proof : to prove associativity, Schwartz only needs the fact that the absolutely convex closure of a compact is still compact.

Definition

We call **k-quasi-complete** the topological vector spaces verifying this property : $(\text{K-}\textsc{Compl}, \varepsilon, \mathbb{K})$ is a symmetric monoidal category.

What should we care about that ? Because this *weaker* completeness condition makes it possible for duality to preserve completeness.

Duality and completions

Duality in topological vector spaces

A subcategory of $\textsc{TopVect}$ is $\star$-autonomous *iff* its objects are reflexive $E \simeq E''$.

It's a mess.

- It depends of the topology E'_β, E'_c, E'_w, E'_μ on the dual.
- It is typically not preserved by $\otimes$.
- For the strong (and most used) topology on the dual, E'_β is not reflexive.

A good topology on the dual

When duality is an orthogonality, we have a closure operation making spaces reflexive :

$$E \hookrightarrow E^{\perp\perp} \simeq E^{\perp\perp\perp\perp}$$

When choosing on E' the topology compact open, one always has :

$$E'_c \simeq ((E'_c)'_c)'_c$$

This allows for the construction of a $\star$-autonomous category of spaces such that $E' \simeq (E'_c)'_c$.

A $*$-autonomous category of complete spaces

This allows for the construction of a $\star$-autonomous category of spaces such that $E' \simeq (E'_c)'_c$.

We have a completion procedure $\hat{\_}$ making spaces complete:

$$E \xrightarrow{(\_)'_c} E'_c \xrightarrow{\hat{\_}} \hat{E}'_c \xrightarrow{(\_)'_c} (\hat{E}'_c)'_c \xrightarrow{\hat{\_}} \cdots$$

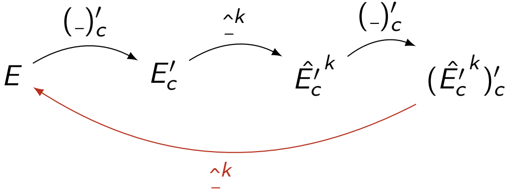
Completion makes the topology on $(\hat{E}'_c)'_c$ too fine to have $(\hat{E}'_c)'_c \simeq E$.

A $*$-autonomous category of complete spaces

For the k-quasi-completion $\hat{\cdot}^k$ we have :

Lemma

When $E \in \mathrm{K}\textsc{c}$, $(\hat{E'_c}^k)'_c$ is k-quasi-complete.



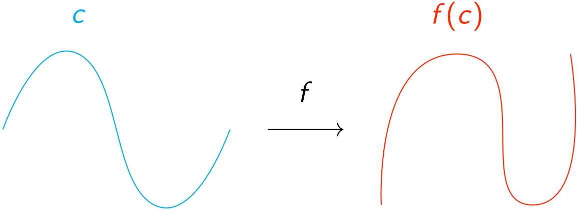
Theorem

K-REFL is a model of *MALL* with complete topological vector spaces and $\parr = \varepsilon$.

Smooth Functions
and topologies inherited from them

Smooth maps à la Frölicher,Kriegl and Michor

A smooth curve $c : \mathbb{R} \to E$ is a curve infinitely many times differentiable.



A smooth function $f : E \to F$ is a function sending a smooth curve on a smooth curve.

In Banach spaces, the definition coincides with the usual one (all iterated derivatives exists and are continuous).

A. Frölicher and A. Kriegl, Linear Spaces and differentiation Theory . 1988

A model with higher order smooth functions

A smooth curve $c : \mathbb{R} \to E$ is a curve infinitely many times differentiable.

A smooth function $f : E \to F$ is a function sending a smooth curve on a smooth curve.

A model of IDiLL

This definition leads to a cartesian closed category of Mackey-complete bornological spaces and smooth functions, and to a first smooth model of Intuitionist DiLL.

Convenient differential category Blute, Ehrhard Tasson Cah. Geom. Diff. (2010)

Functions smooth on compact sets

A smooth model of LL with ϵ

We adapt the notion of smooth function to $\mathcal{C}^{\infty}_{co}$ in order to have an exponential and a cartesian closed category.

- $C^{\infty}_{co}(X, F)$ is the space of infinitely many times Gâteaux-differentiable functions ...
- with derivative continuous on compacts with value in the space $L^{n+1}_{co}(E, F) = L_{co}(L^{n}_{co}(E, F))$ ..
- with at each stage the topology of uniform convergence on compact sets.

A cartesian closed category in K-REFL

If E and F are k-reflexive and G is k-quasi-complete, then

$$\mathcal{C}^{\infty}_{co}(E \times F, G) \simeq \mathcal{C}^{\infty}_{co}(E, \mathcal{C}^{\infty}_{co}(F, G)).$$

Towards a general construction for smooth models of LL

Consider $\mathcal{C}$ a small cartesian category contained in k-ref.

Smooth functions with parameters in $\mathcal{C}$

$\mathcal{C}^\infty_{\mathcal{C}}(E, F) :=$
$\{f : E \to F, \forall X \in \mathcal{C}, \forall c \in \mathcal{C}^\infty_{co}(X, E) \Rightarrow f \circ c \in \mathcal{C}^\infty_{co}(X, F)\}$

A new induced topology

For any tvs E, the dereliction forces an injection $E \hookrightarrow \mathcal{C}^\infty_{\mathcal{C}}(E'_\mu, \mathbb{R})$ which induces a new topology $\mathscr{S}_{\mathcal{C}}(E)$ on E.

Then when E is Mackey-complete :

$\mathcal{C}$	$\mathscr{S}_{\mathcal{C}}(E)$
Fin	The Schwartzification of E
Ban	The Nuclearification of E
$\{0\}$	The weak topology on E

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The topology $\mathscr{S}_C$ ensures that E is Mackey and thus reflexive.

Smooth and classical models of LL

This constructs two other models of DiLL : The Nuclear Mackey-complete spaces and the Schwartz Mackey-complete spaces.

They are also models of DiLL, but that's less pretty.

Conclusion

This work:

- Argues for a theory of functional analysis with reflexive spaces as a starting point.
- Presents several smooth models of Classical Linear Logic: LL really deals with analysis.

Further work on polarized approaches:

- Between convenient spaces and this work: a classical smooth model with good differentiation.
- Partial Differential Equations: LICS on Tuesday.

Thank you .