Smooth denotational models of Linear Logic based on Schwart’z $\varepsilon$ product

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Proofs and smooth objects

\[ \mathcal{R} = \varepsilon \]

Duality and completion

Smooth functions and new topologies
Smoothness

Differentiation

Differentiating a function $f : \mathbb{R}^n \to \mathbb{R}$ at $x$ is finding a linear approximation $d(f)(x) : v \mapsto d(f)(x)(v)$ of $f$ near $x$.

A co-inductive definition

Smooth functions are functions which can be differentiated everywhere in their domain and whose differentials are smooth.
Differentiating proofs

- Differentiation was in the air since the study of Analytic functors by Girard:

  \[ \tilde{d}(x) : \sum f_n \mapsto f_1(x) \]

- DiLL was developed after a study of vectorial models of LL inspired by coherent spaces: Finiteness spaces (Ehrhard 2005), Köthe spaces (Ehrhard 2002).

- Normal functors, power series and \( \lambda \)-calculus. Girard, APAL(1988)

- Differential interaction nets, Ehrhard and Regnier, TCS (2006)
The computational content of differentiation

Historically, resource sensitive syntax and semantics:

- Quantitative semantics: \( f = \sum_n f_n \)
- Resource \( \lambda \)-calculus, Taylor formulas, probabilities and algebraic syntax (Ehrhard, Pagani, Tasson, Vaux ...): \( M = \sum_n M_n \)

Differentiation in Physics and Mathematics takes part in the study of continuous systems:

- Differential Geometry and functional analysis
- Ordinary and Partial Differential Equations
Differential Linear Logic

The rules of DiLL are those of MALL + promotion + :

\[
\begin{align*}
\Gamma & \vdash w & \Gamma, ?A & \vdash \Gamma, ?A & \Gamma, A & \vdash \Gamma, ?A \\
\Gamma & \vdash \bar{w} & \Gamma, A & \vdash \Gamma, ?A & \Delta, !A & \vdash \Gamma, \Delta, !A & \Gamma, A & \vdash \Gamma, !A \\
\end{align*}
\]
What’s not working

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

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We can’t restrict ourselves to finite dimensional spaces.

The tentative to have a normed space of analytic functions fails (Girard’s Coherent Banach spaces).

- We want to use power series.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can’t be bounded on an unbounded space (Liouville’s Theorem).
- Thus functions must depart from an open ball, but arrive in a closed ball. Thus they do not compose.
- This is why Coherent Banach spaces don’t work.
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We can’t restrict ourselves to normed spaces.
Topological vector spaces

We work with Hausdorff topological vector spaces: real or complex vector spaces endowed with a Hausdorff topology making addition and scalar multiplication continuous.

Two layers: algebraic and topological constructions

- The topology on $E$ determines the dual $E'$ as a vector space.
- The topology on $E'$ determines whether $E \cong E''$.
- Many topologies on the tensor $E \otimes F$ which may or may not lead to a monoidal closed category, depending on the spaces (Grothendieck "problèmes des topologies").

We work within the category $\text{TopVect}$ of topological vector spaces and continuous linear functions between them.
Challenges

We encounter several difficulties in the context of topological vector spaces:

✓ Finding a category of tvs and smooth functions which is Cartesian closed. Requires some completeness.
✓ Interpreting the involutive linear negation $(E^\perp)^\perp \simeq E$. 
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*Weak topologies for Linear Logic, K. LMCS 2015.*
Involves a topology which is an internal Chu construction.
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▶ *A logical account for PDEs*, K., *LICS18*
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$\mathcal{F} = \varepsilon$
A good $\mathcal{R}$ is a glueing $\mathcal{R}$

\[ A^\perp \mathcal{R} B \equiv A \to B \]

\[ (!A)^\perp \mathcal{R} B \equiv A \Rightarrow B \]

When proofs are interpreted by Smooth functions:

\[ C^\infty(E, \mathbb{R}) \mathcal{R} F \simeq C^\infty(E, F) \]
The $\varepsilon$ product

Only one good $\varepsilon$

\[ E\varepsilon F := \mathcal{L}_\varepsilon(E'_c, F) \]

No surprises algebraically speaking, but the choice of topologies is important.

\[ C^\infty(E, F) \simeq C^\infty(E, \mathbb{R})\varepsilon F \text{ when } E \text{ and } F \text{ are complete.} \]

A monoidal category by Schwartz

The $\varepsilon$ is associative and commutative on quasi-complete spaces.

Théorie des Distributions à valeurs vectorielles, I Schwartz, (1957)

Negatives are interpreted by (quasi, k-, Mackey) complete spaces. And $\uparrow$ is the completion.
A minimal condition for associativity

Reading back Schwartz’s proof: to prove associativity, Schwartz only needs the fact that the absolutely convex closure of a compact is still compact.

Definition

We call **k-quasi-complete** the topological vector spaces verifying this property: \((K\text{-Compl}, \varepsilon, K)\) is a symmetric monoidal category.

What should we care about that? Because this *weaker* completeness condition makes it possible for duality to preserve completeness.
Duality and completions
Duality in topological vector spaces

A subcategory of \( \mathbf{TopVect} \) is \( \star \)-autonomous iff its objects are reflexive \( E \cong E'' \).

- It depends on the topology \( E'_\beta, E'_c, E'_w, E'_\mu \) on the dual.
- It is typically not preserved by \( \otimes \).
- For the strong (and most used) topology on the dual, \( E'_\beta \) is not reflexive.
A good topology on the dual

When duality is an orthogonality, we have a closure operation making spaces reflexive:

$$E \hookrightarrow E^\perp \simeq E^{\perp \perp \perp \perp}$$

When choosing on $E'$ the topology compact open, one always has:

$$E'_c \simeq ((E'_c)'_c)'_c$$

This allows for the construction of a $\star$-autonomous category of spaces such that $E' \simeq (E'_c)'_c$. 
A $\star$-autonomous category of complete spaces

This allows for the construction of a $\star$-autonomous category of spaces such that $E' \simeq (E'_c)'_c$.

We have a completion procedure $\hat{\cdot}$ making spaces complete:

\[
\begin{array}{ccccccc}
E & \xrightarrow{(\cdot)'_c} & E'_c & \xrightarrow{\hat{\cdot}} & \hat{E}'_c & \xrightarrow{(\cdot)'_c} & (\hat{E}'_c)'_c & \xrightarrow{\hat{\cdot}} & \ldots
\end{array}
\]

Completion makes the topology on $(\hat{E}'_c)'_c$ too fine to have $(\hat{E}'_c)'_c \simeq E$. 

A $\ast$-autonomous category of complete spaces

For the $k$-quasi-completion $\hat{k}$ we have:

Lemma

When $E \in K_c$, $(\hat{E}_c^k)'_c$ is $k$-quasi-complete.

Theorem

$K_{Refl}$ is a model of $MALL$ with complete topological vector spaces and $\mathcal{B} = \varepsilon$. 
Smooth Functions
and topologies inherited from them
Smooth maps à la Frölicher, Kriegl and Michor

A smooth curve $c : \mathbb{R} \to E$ is a curve infinitely many times differentiable.

A smooth function $f : E \to F$ is a function sending a smooth curve on a smooth curve.

In Banach spaces, the definition coincides with the usual one (all iterated derivatives exist and are continuous).

A. Frölicher and A. Kriegl, Linear Spaces and differentiation Theory. 1988
A smooth curve \( c : \mathbb{R} \to E \) is a curve infinitely many times differentiable.

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A model of IDiLL

This definition leads to a cartesian closed category of Mackey-complete bornological spaces and smooth functions, and to a first smooth model of Intuitionist DiLL.

Functions smooth on compact sets

A smooth model of LL with $\varepsilon$

We adapt the notion of smooth function to $C^\infty_{co}$ in order to have an exponential and a cartesian closed category.

- $C^\infty_{co}(X, F)$ is the space of infinitely many times Gâteaux-differentiable functions ...
- with derivative continuous on compacts with value in the space $L^{n+1}_{co}(E, F) = L_{co}(L^n_{co}(E, F))$ ..
- with at each stage the topology of uniform convergence on compact sets.

A cartesian closed category in $K$-$\text{Refl}$

If $E$ and $F$ are $k$-reflexive and $G$ is $k$-quasi-complete, then

$$C^\infty_{co}(E \times F, G) \simeq C^\infty_{co}(E, C^\infty_{co}(F, G)).$$
Towards a general construction for smooth models of LL

Consider $\mathcal{C}$ a small cartesian category contained in $k$-ref.

Smooth functions with parameters in $\mathcal{C}$

$$\mathcal{C}^\infty_{\mathcal{C}}(E, F) := \{ f : E \to F, \forall X \in \mathcal{C}, \forall c \in C^\infty_{co}(X, E) \Rightarrow f \circ c \in C^\infty_{co}(X, F) \}$$

A new induced topology

For any tvs $E$, the dereliction forces an injection $E \hookrightarrow C^\infty_{\mathcal{C}}(E_{\mu}', \mathbb{R})$ which induces a new topology $\mathcal{L}_\mathcal{C}(E)$ on $E$.

Then when $E$ is Mackey-complete:

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Towards a general construction for smooth models of LL

Then when $E$ is Mackey-complete:

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The topology $\mathcal{S}_\mathcal{C}$ ensures that $E$ is Mackey and thus reflexive.

Smooth and classical models of LL

This constructs two other models of DiLL: The Nuclear Mackey-complete spaces and the Schwartz Mackey-complete spaces.

They are also models of DiLL, but that’s less pretty.
Conclusion

This work:

▶ Argues for a theory of functional analysis with reflexive spaces as a starting point.
▶ Presents several smooth models of Classical Linear Logic: LL really deals with analysis.

Further work on polarized approaches:

▶ Between convenient spaces and this work: a classical smooth model with good differentiation.
▶ Partial Differential Equations: LICS on Tuesday.
Thank you.