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Smooth models of Linear Logic : Towards a Type Theory for Linear Partial Differential Equations

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Wanted

A model of Classical Linear Logic where proofs are interpreted as smooth functions.

Obtained

A Smooth Differential Linear Logic where exponentials are spaces of solutions to a Linear Partial Differential Equation.

Plan

Proofs and smooth objects

An interpretation for ! and ¬

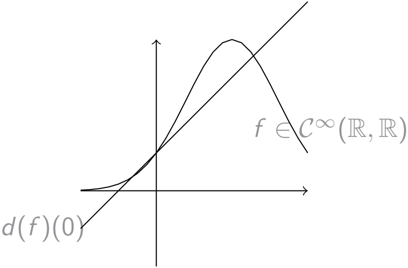
A model with Distributions

Linear PDE as exponentials

Smooth models of Linear Logic

Differentiation

Differentiating a function $f : \mathbb{R}^n \to \mathbb{R}$ at x is finding a linear approximation $d(f)(x) : v \mapsto D(f)(x)(v)$ of f near x.



Smooth functions are functions which can be differentiated everywhere in their domain and whose differentials are smooth.

Smooth models of Linear Logic

We work through denotational models of Linear Logic. Specifically:

Computation	Logic	Category
Term	Proof	Morphism
Type	Formula	Object
Evaluation	Normalization	Equality

Smooth models of Linear Logic

We work through denotational models of Linear Logic. Specifically:

Computation	Logic	Vector spaces
Term	Proof	Function
Type	Formula	Top. vector space
Evaluation	Normalization	Equality

Smooth models of Linear Logic

$$A, B := A \otimes B | 1 | A \parr B | \bot | A \oplus B | 0 | A \times B | \top | !A | ?A$$

A decomposition of the implication

$$A \Rightarrow B \simeq !A \multimap B$$

A decomposition of function spaces

$$\mathcal{C}^\infty(E, F) \simeq \mathcal{L}(!E, F)$$

The dual of the exponential : smooth scalar functions

$$\mathcal{C}^\infty(E, \mathbb{R}) \simeq \mathcal{L}(!E, \mathbb{R}) \simeq !E'$$

Smooth models of Classical Linear Logic

A Classical logic

$\neg A = A \Rightarrow \perp$ and $\neg\neg A \simeq A$.

Linear Logic features an involutive linear negation :

$$A^{\perp} \simeq A \multimap 1$$

$$A^{\perp\perp} \simeq A$$

$$E'' \simeq E$$

The exponential is the dual of the space of smooth scalar functions

$$!E \simeq (!E)'' \simeq \mathcal{C}^{\infty}(E, \mathbb{R})'$$

Smooth models of Differential Linear Logic

Semantics

For each $f :!A \multimap B \simeq \mathcal{C}^\infty(A, B)$ we have $Df(0) : A \multimap B$

The rules of DiLL are those of MALL and :

co-dereliction

$$\bar{d} : x \mapsto f \mapsto Df(0)(x)$$

Syntax

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ d$$

$$\frac{\vdash \Gamma, !A, !A}{\vdash \Gamma, !A}\ \bar{c} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, !A}\ \bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \bar{d}$$

Why differential linear logic ?

- Differentiation was in the air since the study of Analytic functors by Girard :

$$\bar{d}(x) : \sum f_n \mapsto f_1(x)$$

- DiLL was developed after a study of vectorial models of LL inspired by coherent spaces : Finiteness spaces (Ehrard 2005), Köthe spaces (Ehrhard 2002).

Normal functors, power series and λ-calculus. Girard, APAL(1988)

Differential interaction nets, Ehrhard and Regnier, TCS (2006)

Smoothness of proofs

- Traditionally proofs are interpreted as graphs, relations between sets, power series on finite dimensional vector spaces, strategies between games: those are discrete objects.
- Differentiation appeals to differential geometry, manifolds, functional analysis : we want to find a denotational model of DiLL where proofs are general smooth functions.

Mathematical challenges : interpreting ! and $A^{\perp}$

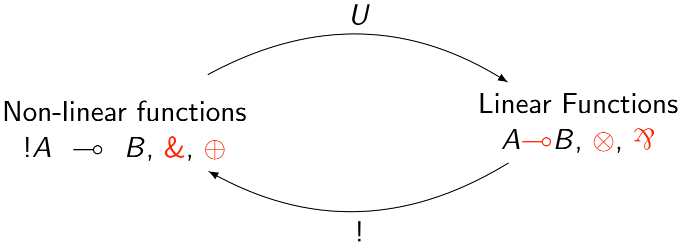
A categorical model

Every connective of Linear Logic is interpreted as a (bi)functor within the chosen category : transforming sets into sets, vector spaces into vector spaces, complete spaces into complete spaces.

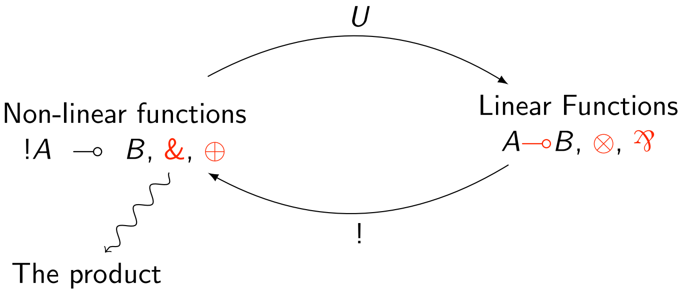
Linearity and Smoothness

We work with vector spaces with some notion of continuity on them : for example, normed spaces, or complete normed spaces (Banach spaces).

Interpreting LL in vector spaces



Interpreting LL in vector spaces



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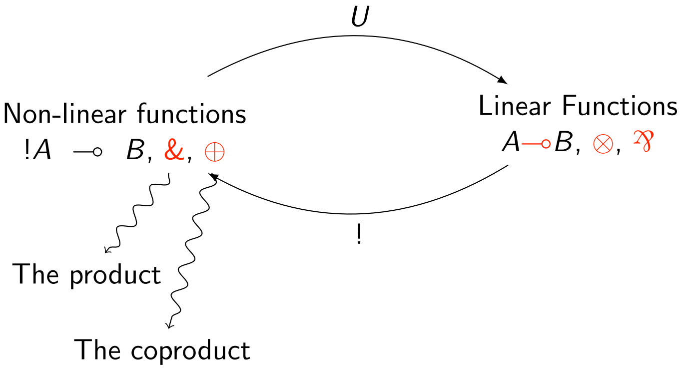

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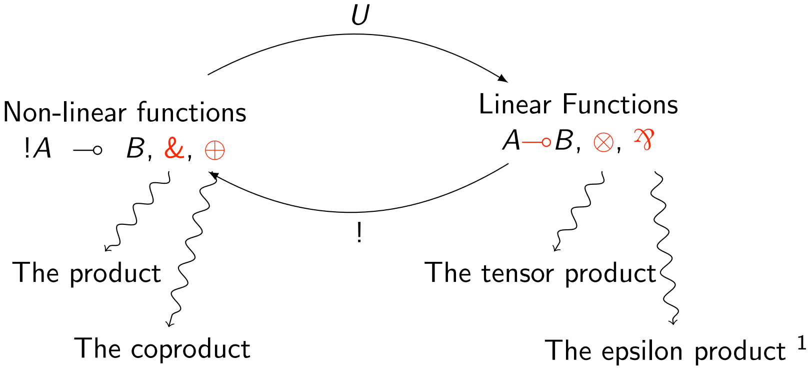

[1]Work with Y. Dabrowski

Challenges

We encounter several difficulties in the context of topological vector spaces :

- Finding a good topological tensor product.
- Finding a category of smooth functions which is Cartesian closed.
- Interpreting the involutive linear negation $(E^{\perp})^{\perp} \simeq E$

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Convenient differential category Blute, Ehrhard Tasson Cah. Geom. Diff. (2010)

Mackey-complete spaces and Power series, K. and Tasson, MSCS 2016.

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Weak topologies for Linear Logic, K. LMCS 2015.

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- *A model of LL with Schwartz' epsilon product*, K. and Dabrowski, *In preparation*.
- *Distributions and Smooth Differential Linear Logic*, K., *In preparation*

The categorical semantics of an involutive linear negation

Linear Logic features an involutive linear negation :

$$A^{\perp} \simeq A \multimap 1$$

$$A^{\perp\perp} \simeq A$$

*-autonomous categories are monoidal closed categories with a distinguished object 1 such that $E \simeq (E \multimap \perp) \multimap \perp$ through d_A.

$$d_A : \begin{cases} E \rightarrow (E \multimap \perp) \multimap \perp \\ x \mapsto ev_x : f \mapsto f(x) \end{cases}$$

$*$-autonomous categories of vector spaces

I want to explain to my math colleague what is a *-autonomous category: $\bot$ neutral for $\parr$, thus $\bot \simeq \mathbb{R}$, $A \multimap \bot$ is $A' = \mathcal{L}(A, \mathbb{R})$.

$$d_A : \begin{cases} E \to E'' \\ x \mapsto ev_x : f \mapsto f(x) \end{cases}$$

should be an isomorphism.

Exclamation

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Disapointment

Well, the category of reflexive topological vector space is not closed (eg: Hilbert spaces).

Internal completeness

A way to resolve this is to work with pairs of vector spaces : the Chu Construction (used for coherent Banach spaces), or its internalization through topology (Weak or Mackey spaces).

The Chu construction

- Objects : (E_1, E_2).
- Morphisms : $(f_1 : E_1 \to F_1, f_2 : F_2 \to E_2) : (E_1, E_2) \to (F_1, F_2)$
- Duality : $(E_1, E_2)^\perp = (E_2, E_1)^\perp$.

These model are disapointing, even as $\star$-autonomous categories : any vector space can be turned into an object of this category.

We want reflexivity to be an internal property of our objects

An exponential for smooth functions

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

$$\dim \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m) = \infty.$$

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Girard's tentative to have a normed space of analytic functions fails.

- We want to use functions.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can't be bounded on an unbounded space (Liouville's Theorem).
- Thus functions must depart from an open ball, but arrive in a closed ball. Thus they do not compose.
- This is why Coherent Banach spaces don't work.

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We can't restrict ourselves to normed spaces.

Curryfication for linear and smooth functions

In a model of LL, you have

- Monoidal closed : $\mathcal{L}(E \otimes F, G) \simeq \mathcal{L}(, \mathcal{L}(F, G))$.
- Cartesian closed : $\mathcal{C}^\infty(E \times F, G) \simeq \mathcal{C}^\infty(E, \mathcal{C}^\infty(F, G))$.

Once you have monoidal closedeness, this sums up to a rule on exponentials :

Seely's formula

$!E \otimes !F \simeq !(E \times F)$

Thus, in a category of reflexive real vector spaces,

$$\mathcal{C}^\infty(E, \mathbb{R})' \otimes \mathcal{C}^\infty(F, \mathbb{R})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{R})'.$$

An exponential for differentiation

- The codereliction $\bar{d}_E : E \to !E = \mathcal{C}^\infty(E, \mathbb{R})'$ encodes the possibility to differentiate.
- In a $\star$-autonomous category $d_E : E \to ?E$ encode the fact that linear functions are smooth.

$$d_E : \begin{cases} !E = \mathcal{C}^\infty(E,R)' \to E'' \simeq E \\ \phi \in \mathcal{C}^\infty(E,R)' \mapsto \phi_{\mathcal{L}(E,\mathbb{R})} \end{cases}$$

Differentiation's slogan

"A linear function is its own differential"

$$d_E \circ \bar{d}_E = Id_E$$

A model with Distributions

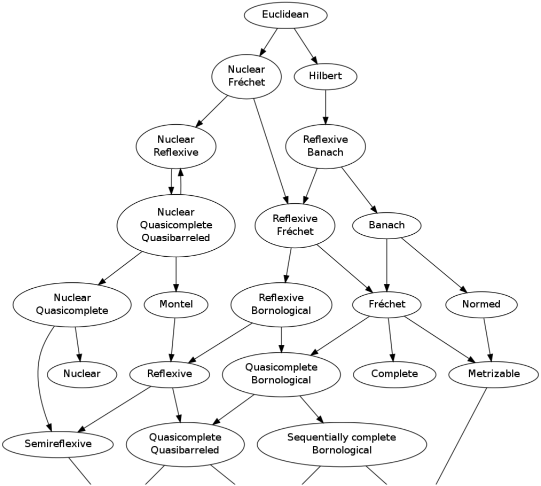
Topological vector spaces

We work with Hausdorff topological vector spaces : real or complex vector spaces endowed with a Hausdorff topology making addition and scalar multiplication continuous.

- The topology on E determines E'.
- The topology on E' determines whether $E \simeq E''$.

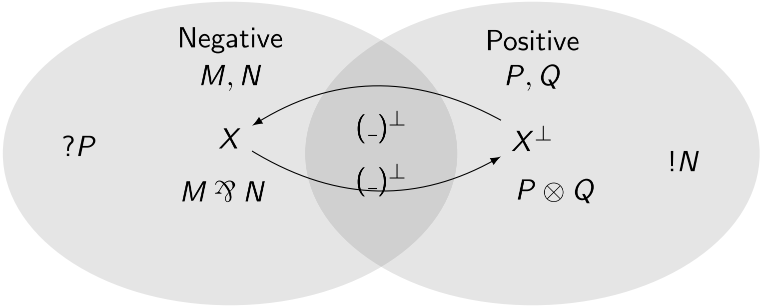
We work within the category $\textsc{TopVect}$ of topological vector spaces and continuous linear functions between them.

Topological models of DiLL



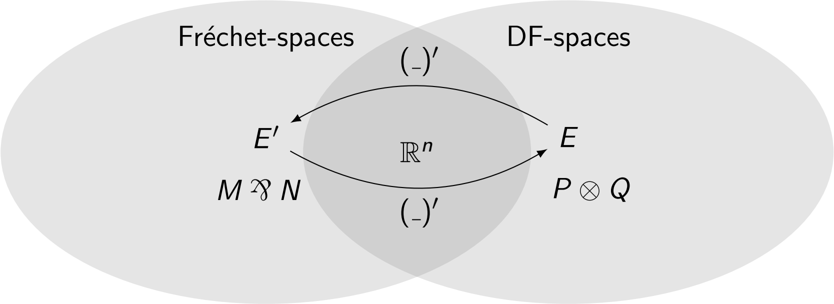
Let us take the other way around, through Nuclear Fréchet spaces.

Polarized models of LL

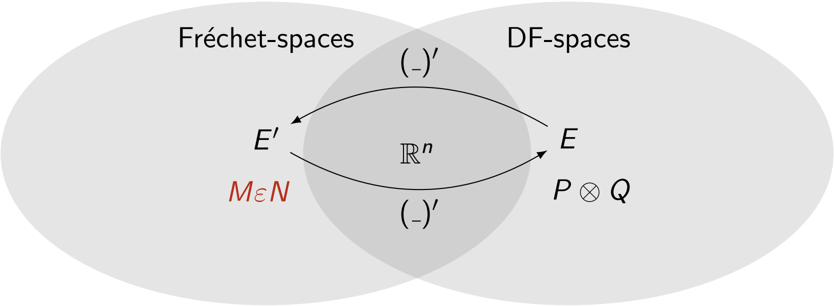


Fréchet and DF spaces

- Fréchet : metrizable complete spaces.
- (DF)-spaces : such that the dual of a Fréchet is (DF) and the dual of a (DF) is Fréchet.



Fréchet and DF spaces



These spaces are in general not reflexive.

The ε product

$E \varepsilon F = (E'_c \otimes_{\beta e} F'_c)'$ with the topology of uniform convergence on products of equicontinuous sets in E', F'.

The ε-product is designed to glue spaces of scalar continuous functions to a codomain :

$$\mathcal{C}(X, \mathbb{R})_c \varepsilon F \simeq \mathcal{C}(X, F)_c.$$

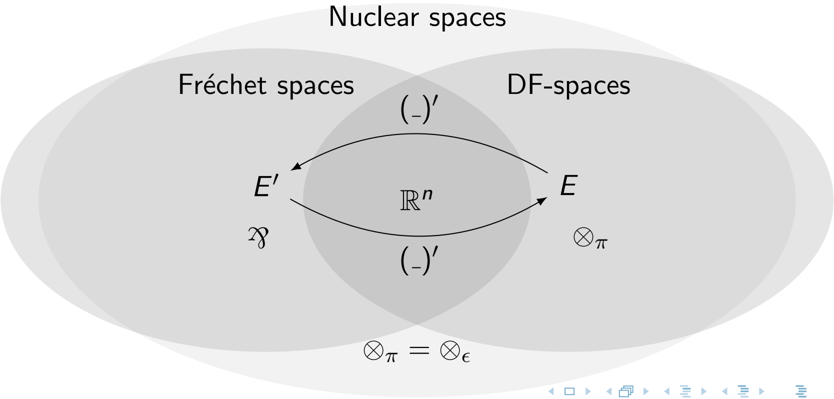
Theorem (Dabrowski)

The ε product is associative in the $\star$-autonomous category of Mackey Mackey-complete Schwartz tvs.

Nuclear spaces

Nuclear spaces are spaces in which one can identify the two canonical topologies on tensor products :

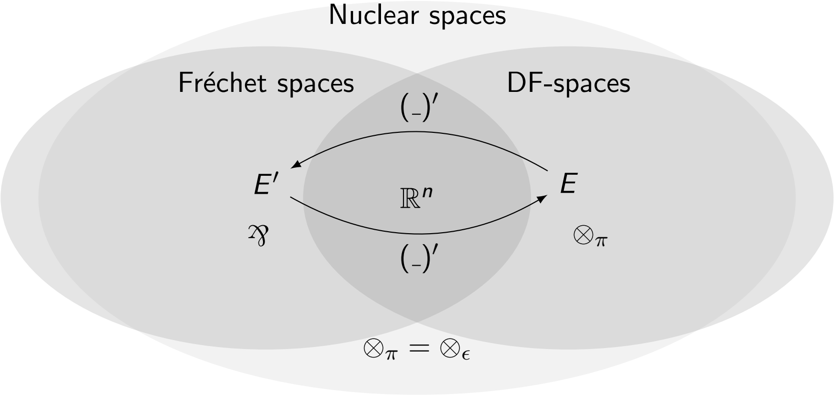
$$\forall F, E \otimes_\pi F = E \otimes_\epsilon F$$



Nuclear spaces

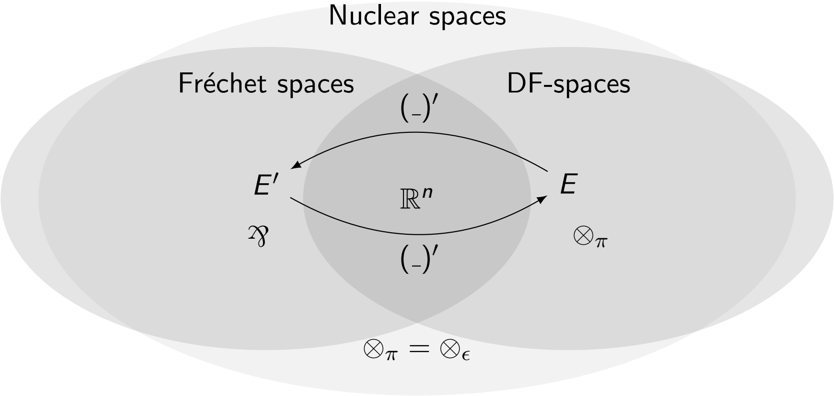
A polarized ⋆-autonomous category

A Nuclear space which is also Fréchet or dual of a Fréchet is reflexive.



Nuclear spaces

We get a polarized model of MALL : involutive negation $(\_)^\perp$, $\otimes$, $⅋$, $\oplus$, $\times$.



Distributions and the Kernel theorems

Examples of Nuclear Fréchet spaces includes :

$$\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}),\ \mathcal{C}_c^\infty(\mathbb{R}^n,\mathbb{R}),\ \mathcal{H}(\mathbb{C},\mathbb{C}),\ ..$$

Typical Nuclear DF spaces are distributions spaces Schwartz' generalized functions :

$$\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})',\ \mathcal{C}_c^\infty(\mathbb{R}^n,\mathbb{R})',\ \mathcal{H}'(\mathbb{C},\mathbb{C}),\ ..$$

The Kernel Theorems

$$\mathcal{C}_c^\infty(E,\mathbb{R})' \otimes \mathcal{C}_c^\infty(F,\mathbb{R})' \simeq \mathcal{C}_c^\infty(E\times F,\mathbb{R})'$$

Distributions and the Kernel theorems

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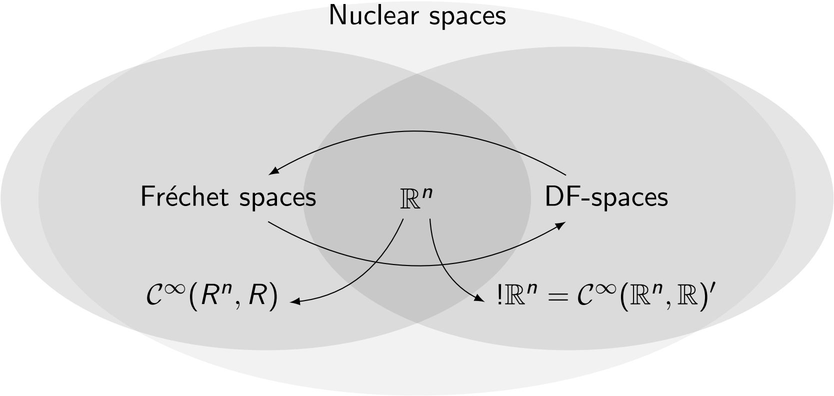
The Kernel Theorems

$$\mathcal{C}^\infty(E, \mathbb{R})' \otimes \mathcal{C}^\infty(F, \mathbb{R})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{R})'$$

$$!\mathbb{R}^n = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'.$$

A model of Smooth differential Linear Logic



A Smooth differential Linear Logic

A graded semantic

Finite dimensional vector spaces:

$$R^n, R^m := \mathbb{R} | R^n \otimes R^m | R^n \parr R^m | R^n \oplus R^m | R^n \times R^m.$$

Nuclear spaces :

$$U, V := R^n | !R^n | ?R^n | U \otimes V | U \parr V | U \oplus V | U \times V.$$

$$!\mathbb{R}^n = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})' \in \textsc{Nucl}$$

$$!\mathbb{R}^n \otimes !\mathbb{R}^m \simeq !(\mathbb{R}^{n+m})$$

We have obtained a smooth classical model of DiLL, to the price of Higher Order and Digging $!A \multimap !!A$.

Smooth DiLL

A new graded syntax

Finitary formulas : $A, B := X | A \otimes B | A \parr B | A \oplus B | A \times B.$
General formulas : $U, V := A | !A | ?A | U \otimes V | U \parr V | U \oplus V | U \times V$

For the old rules

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, !A}\, \bar{w} \qquad \frac{\vdash \Gamma, !A, !A}{\vdash \Gamma, !A}\, \bar{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

The categorical semantic of smooth DiLL is the one of LL, but where ! is a monoidal functor and d and $\bar{d}$ are to be defined independently.

Linear Partial Differential Equations as Exponentials

Work in progress

Intermediate ranks in the syntax

Finitary formulas : $A, B := X | A \otimes B ...$ and linear maps.

...

General formulas : $U, V := A | !A | U \otimes V | ...$ and smooth maps.

Intermediate ranks in the syntax

Finitary formulas : $A, B := X|A \otimes B ...$ and linear maps.

...

$U, V := A|S(A)'|U \otimes V|...$ and solutions to a differential equation.

...

General formulas : $U, V := A|!A|U \otimes V|...$ and smooth maps.

Differential Linear Logic

Syntax

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash \Gamma, !A, !A}{\vdash \Gamma, !A}\, \bar{c} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, !A}\, \bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

Semantic of the co-dereliction

$$\bar{d} : x \mapsto f \mapsto Df(0)(x)$$

Semantic of the dereliction

$$d : E \to ?E = (!E')'$$

$$E \multimap 1 \subset !E \multimap 1$$

$$\mathcal{L}(E, \mathbb{R}) \subset \mathcal{C}^\infty(E, \mathbb{R})$$

.

Spaces of solutions to a differential equations

A linear partial differential operator on $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ with constant coefficient

$$\mathrm{D} = \sum_{|\alpha| \leq n} a_\alpha \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdot \partial^{\alpha_n} x_n}$$

For example : $\mathrm{D}(f) = \frac{\partial^n f}{\partial x_1 \ldots \partial x_n}$.

Theorem(Schwartz)

Under some considerations on D, the space $S_{\mathrm{D}}(E, \mathbb{R})'$ of functions solution to $\mathrm{D}(f) = f$ is a Nuclear Fréchet space of functions.

Thus $S_{\mathrm{D}}(E, \mathbb{R})'$ is an exponential.

A new exponential

Spaces of Smooth functions	Exponentials
$\mathcal{C}^\infty(E,\mathbb{R})$	${\mathcal{C}^\infty}'(E,\mathbb{R})$
$S_{\mathrm{D}}(E,\mathbb{R})$	$S'_{\mathrm{D}}(E,\mathbb{R})$
$E' \simeq \mathcal{L}(E,\mathbb{R})$	$E'' \simeq E$

Linear functions are exactly those in $\mathcal{C}^\infty(E,\mathbb{R})$ such that for all x : $f(x) = D(f)(0)(x)$.

$$\forall x, ev_x(f) = ev_x(\bar{d})(f).$$

Dereliction and co-dereliction adapt to LPDE

For linear functions

$$\bar{d} : E \xrightarrow{\text{linear}} \mathcal{C}^\infty(E, \mathbb{R})', x \mapsto (f \mapsto D(f)(x)).$$

$$d : \mathcal{C}^\infty(E, \mathbb{R})' \to S'(E, \mathbb{R}), \phi \mapsto \phi_{|\mathcal{L}(E,\mathbb{R})}$$

For solutions of $\mathrm{D}f = f$

$$\bar{d}_{\mathrm{D}} : E \xrightarrow{\text{smooth}} \mathcal{C}^\infty(E, \mathbb{R})', x \mapsto (f \mapsto \mathrm{D}(f)(x)).$$

$$d_{\mathrm{D}} : \mathcal{C}^\infty(E, \mathbb{R})' \to S'(E, \mathbb{R}), \phi \mapsto \phi_{|S_{\mathrm{D}}(E,\mathbb{R})}$$

The map $\bar{d}_{\mathrm{D}}$ represents the equation to solve, while d_{D} represents the fact that we are for looking solutions in $\mathcal{C}^\infty(E, \mathbb{R})$.

Exponentials and invariants

Spaces of Smooth functions	Exponentials	Equations
$\mathcal{C}^\infty(E, \mathbb{R})$	$\mathcal{C}^\infty(E, \mathbb{R})$	
$S_{\mathrm{D}}(E, \mathbb{R})$	$S'_{\mathrm{D}}(E, \mathbb{R})$	
$E' \simeq \mathcal{L}(E, \mathbb{R})$	$E'' \simeq E$	$d \circ \bar{d} = Id$

Exponentials and invariants

Spaces of Smooth functions	Exponentials	PDE
$\mathcal{C}^\infty(E,\mathbb{R})$	$\mathcal{C}^\infty(E,\mathbb{R})'$	
$S_{\mathrm{D}}(E,\mathbb{R})$	$S'_{\mathrm{D}}(E,\mathbb{R})$	$S'(E,\mathbb{R}) \xrightarrow{\bar{d}_{\mathrm{D}}} !E \xrightarrow{d_{\mathrm{D}}} S'(E,\mathbb{R})$
$E' \simeq \mathcal{L}(E,\mathbb{R})$	$E'' \simeq E$	$E \xrightarrow{\bar{d}} !E \xrightarrow{d} E''$, $\quad ev_E : E \to E''$

The logic of linears PDE's

Rules

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d \qquad \frac{\vdash \Gamma, ?_{\mathrm{D}} A}{\vdash \Gamma, ?A}\, d_{\mathrm{D}}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d} \qquad \frac{\vdash \Gamma, !_{\mathrm{D}} A}{\vdash \Gamma, !A}\, \bar{d}_{\mathrm{D}}$$

The logic of linears PDE's

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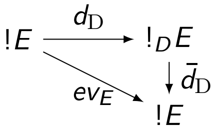
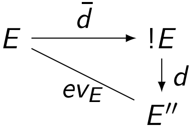
$$?_{\mathrm{D}} E = S_{\mathrm{D}}(E', \mathbb{R}) \text{ and } \bar{d}_{\mathrm{D}} : f \mapsto x \mapsto \mathrm{D}(x)(f)$$

Cut elimination (work in progress)





The analogy is not perfect

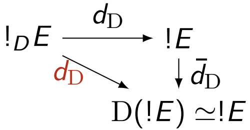
Rules

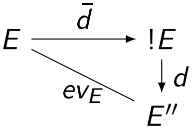
$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d \qquad \frac{\vdash \Gamma, ?_{\mathrm{D}}A}{\vdash \Gamma, ?A}\, d_{\mathrm{D}}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d} \qquad \frac{\vdash \Gamma, {\color{red}!A}}{\vdash \Gamma, !A}\, \bar{d}_{\mathrm{D}}$$

$$\bar{d}_{\mathrm{D}} : \phi \in !E \mapsto (\mathrm{D}\phi : f \in \mathcal{C}^\infty(E, \mathbb{R}) \mapsto \phi(\mathrm{D}f))$$

Cut elimination





Coweakening and co-contraction

	$S_D(E,\mathbb{R})'$	$\mathcal{C}^\infty(E,\mathbb{R})'$
c	If Kernel Theorem	Due to Seely isomorphism
$\bar{c}$	convolution $!A \otimes !_D A \to !_D A$	convolution
w	?	$\phi \mapsto \phi_{\vert\mathbb{R}}$
$\bar{w}$	?	$1 \mapsto \delta_0$

An example

Scalar solutions defined on $\mathbb{R}^n$ of

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} f = f$$

are the $z \mapsto \lambda e^{x_1 + ... + x_n}$.

$$S'(\mathbb{R}^n) \otimes S'(R^M) \simeq S'(\mathbb{R}^{n+m}).$$

$$\lambda e^{x_1 + ... + x_n} \mu e^{y_1 + ... + y_m} = \lambda \mu e^{x_1 + ... + x_n + y_1 + ... + y_m}.$$

$S(\mathbb{R} \cdot \mathbb{R})'$ verifies $w, \bar{w}$ (which corresponds to the initial condition of the differential equation) and $\bar{c}, c$.

Conclusion

The space of solutions to a linear partial differential equation form an exponential in Linear Logic

Conclusion

What we have :

- An interpretation of the linear involutive negation of LL in term of reflexive TVS.
- An interpretation of the exponential in terms of distributions.
- An interpretation of $\parr$ in term of the Schwartz epsilon product.
- The beginning of a generalization of DiLL to linear *PDE*'s.

What we could get :

- A constructive Type Theory for differential equations.
- Logical interpretations of fundamental solutions, specific spaces of distributions, or operation on distributions.

The Chu construction

A construction invented by a student of Barr, in 1979. It modelises duality in Coherent Banach spaces.

The Chu construction for topological vector spaces

We consider the category $\textsc{Chu}$ of pairs of vector spaces (E_1, E_2) and pairs of maps $(f_1 : E_1 \rightarrow F_1, f_2 : F_2 \rightarrow E_2) : (E_1, E_2) \rightarrow (F_1, F_2)$. Let us define :

- $(E_1, E_2)^\perp = (E_2, E_1)$
- $(E_1, E_2) \otimes (F_1, F_2) = (E_1 \otimes F_1, \mathcal{L}(E_2, F_1))$
- $(E_1, E_2) \multimap (F_1, F_2) = (L(E_1, F_1), E_1 \otimes F_2)$

$\textsc{Chu}$ is then a $*$-autonomous category.

Weak and Mackey, two adjoint functors to Chu

There is an functor from the category of topological vector spaces and continuous linear map to the category $\textsc{Chu}$:

$$E \mapsto (E, E'), f \mapsto (f, f^t).$$

It has two adjoints :

- $\mathcal{W}$ maps (E_1, E_2) to E endowed with the coarsiest topology such that $E' = E_2$.
- $\mathcal{M}$ maps (E_1, E_2) to E endowed with the finest topology such that $E' = E_2$.

$$\mathcal{L}(E, \mathcal{W}(F, F')) \simeq \textsc{Chu}((E, E'), (F, F')) \simeq \mathcal{L}(\mathcal{M}(E, E'), F)$$

The categories of weak spaces and Mackey spaces both form models of Differential Linear Logic, with formal power series as non-linear functions.