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Towards a Type Theory for Linear Partial Differential Equations

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Linear Logic

A decomposition of the implication

$A \Rightarrow B \simeq !A \multimap B$

Denotational semantic

We interpret formulas as sets and proofs as functions between these sets.

Denotational semantic of LL

We have a cohabitation between linear functions and non-linear functions.

Linear Logic

Classical logic

$\neg A = A \Rightarrow \perp$ and $\neg\neg A \simeq A$.

Linear Logic features an involutive linear negation :

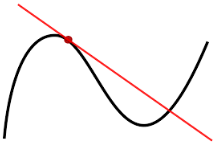
$$A^{\perp} \simeq A \multimap 1$$

$$A^{\perp\perp} \simeq A$$

Smoothness

Differentiation

Differentiating a function $f : \mathbb{R}^n \to \mathbb{R}$ at x is finding a linear approximation $d(f)(x) : v \mapsto D(f)(x)(v)$ of f near x.



Smooth functions are functions which can be differentiated everywhere in their domain and whose differentials are smooth.

Differential Linear Logic

A modification of the exponential rules of Linear Logic in order to allow differentiation.

Semantics

For each $f :!A \multimap B \simeq \mathcal{C}^\infty(A, B)$ we have $Df(0) : A \multimap B$

Syntax

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c$$

$$\frac{\vdash \Gamma, !A, !A}{\vdash \Gamma, !A}\, \bar{c} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, !A}\, \bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

co-dereliction

$$\bar{d} : x \mapsto f \mapsto Df(0)(x)$$

Why differential linear logic ?

- Differentiation was in the air since the study of Analytic functors by Girard :

$$\bar{d}(x) : \sum f_n \mapsto f_1(x)$$

- DiLL was developed after a study vectorial models of LL inspired by coherent spaces : Finiteness spaces (Ehrard 2005), Köthe spaces (Ehrhard 2002).

It leads to differential λ-calculus and applications for probabilistic programming languages.

Normal functors, power series and λ-calculus. Girard, APAL(1988)

Differential interaction nets, Ehrhard and Regnier, TCS (2006)

Smoothness of proofs

- Proofs are interpreted as graphs, relations between sets, power series on finite dimensional vector spaces, strategies between games: those are discrete objects.
- Differentiation appeals to differential geometry, manifolds, functional analysis : we want to find a denotational model of DiLL where proofs are smooth functions.

TEASING: to get to differential equations.

Plan

Denotational semantics of classical linear logic

Interpreting LL in vector spaces

Consider formulas interpreted by finite dimensional vector spaces or Banach spaces.

$$\begin{tikzcd}
\begin{array}{c}\text{Cartesian closed category :}\\ \text{Non-linear functions}\\ !A \multimap B,\ \&,\ \oplus\end{array} \arrow[r, bend left, "U"] & \begin{array}{c}\text{Monoidal Closed Category :}\\ \text{Linear Functions}\\ A \multimap B,\ \otimes,\ \parr\end{array} \arrow[l, bend left, "!"]
\end{tikzcd}$$

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The product

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Cartesian closed category :
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$!A \multimap B$, $\&$, $\oplus$

Monoidal Closed Category :
Linear Functions
$A \multimap B$, $\otimes$, $⅋$

$!$

The product

The coproduct

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The product

The coproduct

Monoidal Closed Category :

Linear Functions

$A \multimap B$, $\otimes$, $⅋$

The tensor product

The epsilon product [1]

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[1] Work with Y. Dabrowski

Interpreting LL in vector spaces

Consider formulas interpreted by finite dimensional vector spaces or Banach spaces.

U

Cartesian closed category :

Non-linear functions

$!A \multimap B$, $\&$, $\oplus$

Linear Functions

$A \multimap B$, $\otimes$, $\parr$

$A \otimes B \multimap C \simeq A \multimap (B \multimap C)$

$!$

Interpreting LL in vector spaces

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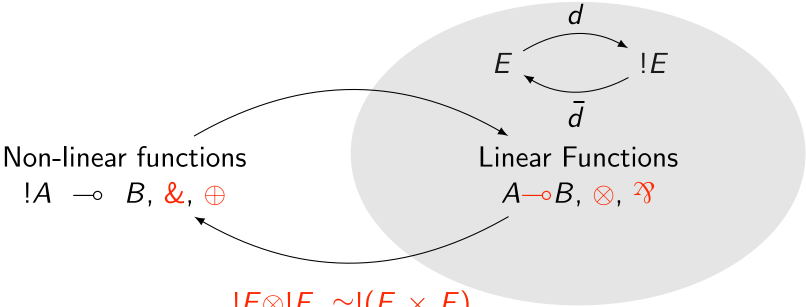
$A \otimes B \multimap C \simeq A \multimap (B \multimap C)$

$$!E \otimes !F \simeq !(E \times F)$$

Interpreting DiLL in vector spaces



$$!E \otimes !F \simeq !(E \times F)$$

$$d \circ \bar{d} = Id_E$$

$!E \otimes !F \simeq !(E \times F)$ allows to have a cartesian closed Co-Kleisli category

Interpreting DiLL in vector spaces

$$\begin{tikzcd} E \arrow[r, bend left, "d"] & !E \arrow[l, bend left, "\bar{d}"] \end{tikzcd}$$

Non-linear functions

$!A \multimap B$, $\&$, $\oplus$

Linear Functions

$A \multimap B$, $\otimes$, $⅋$

$$!E \otimes !F \simeq !(E \times F)$$
$$d \circ \bar{d} = Id_E$$

$d \circ \bar{d} = Id_E$ expresses the fact that the differential at 0 of a linear function is the same linear function.

We want to find good spaces in which we can interpret all these constructions, and an appropriate notion of smooth functions.

Challenges

We encounter several difficulties in the context of topological vector spaces :

- Finding a good topological tensor product.
- Finding a category of smooth functions which is Cartesian closed.
- Interpreting the involutive linear negation $(E^{\perp})^{\perp} \simeq E$

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Convenient differential category Blute, Ehrhard Tasson Cah. Geom. Diff. (2010)

Mackey-complete spaces and Power series, K. and Tasson, MSCS 2016.

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Weak topologies for Linear Logic, K. LMCS 2015.

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- Interpreting the involutive linear negation $(E^\perp)^\perp \simeq E$

- *A model of LL with Schwartz' epsilon product*, K. and Dabrowski, *In preparation.*
- *Distributions and Smooth Differential Linear Logic*, K., *In preparation*

The categorical semantics of an involutive linear negation

Linear Logic features an involutive linear negation :

$$A^{\perp} \simeq A \multimap 1$$

$$A^{\perp\perp} \simeq A$$

*-autonomous categories are monoidal closed categories with a distinguished object 1 such that $E \simeq (E \multimap 1) \multimap 1$ through d_A.

$$d_A : \begin{cases} E \to (E \multimap 1) \multimap 1 \\ x \mapsto ev_x : f \mapsto f(x) \end{cases}$$

$*$-autonomous categories of vector spaces

I want to explain to my math colleague what is a *-autonomous category: $\perp$ neutral for $\parr$, thus $\perp \simeq \mathbb{R}$, $A \multimap 1$ is $A' = \mathcal{L}(A, \mathbb{R})$.

$$d_A : \begin{cases} E \to E'' \\ x \mapsto ev_x : f \mapsto f(x) \end{cases}$$

should be an isomophism.

Exclamation

Well, this is a just a category of reflexive vector space.

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Exclamation

Well, this is a just a category of reflexive vector space.

Disapointment

Well, the category of reflexive topological vector space is not closed (eg: Hilbert spaces).

A model with Distributions

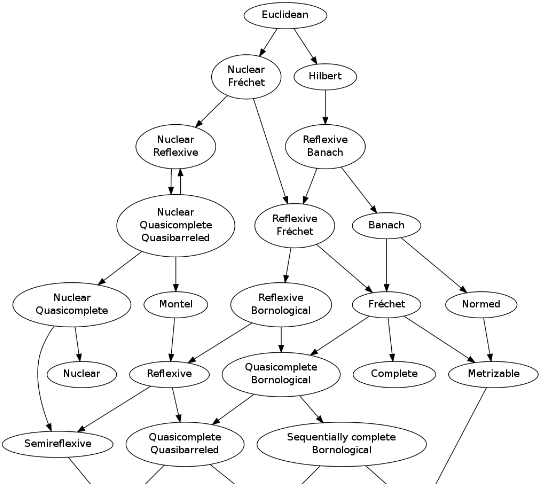
Topological vector spaces

We work with Hausdorff topological vector spaces : real or complex vector spaces endowed with a Haussdorf topology making addition and scalar multiplication continuous.

- The topology on E determines E'.
- The topology on E' determines whether $E \simeq E''$.

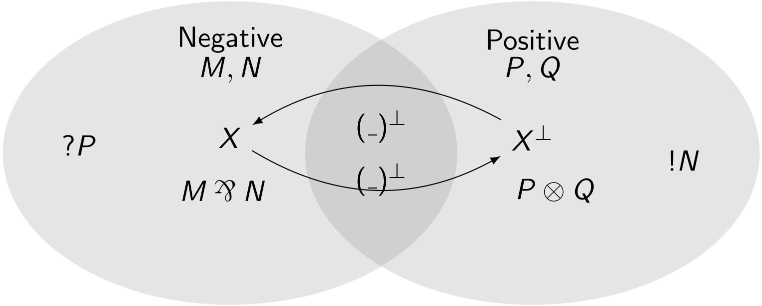
We work within the category $\textsc{TopVect}$ of topological vector spaces and continuous linear functions between them.

Topological models of DiLL



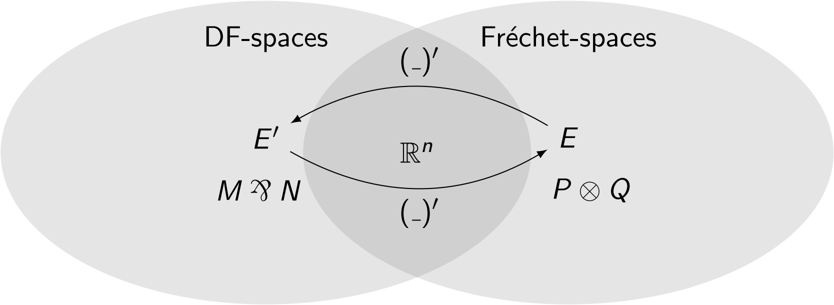
Let us take the other way around, through Nuclear Fréchet spaces.

Polarized models of LL



Fréchet and DF spaces

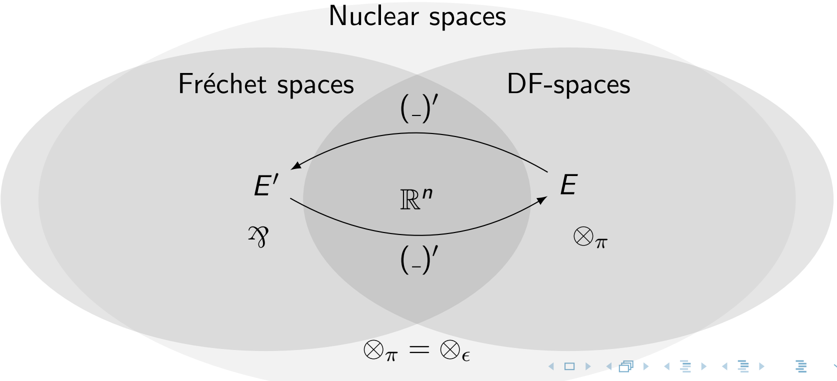
- Fréchet : metrizable complete spaces.
- (DF)-spaces : such that the dual of a Fréchet is (DF) and the dual of a (DF) is Fréchet.



Nuclear spaces

Nuclear spaces are spaces in which for which you can identify the two canonical topologies on tensor products :

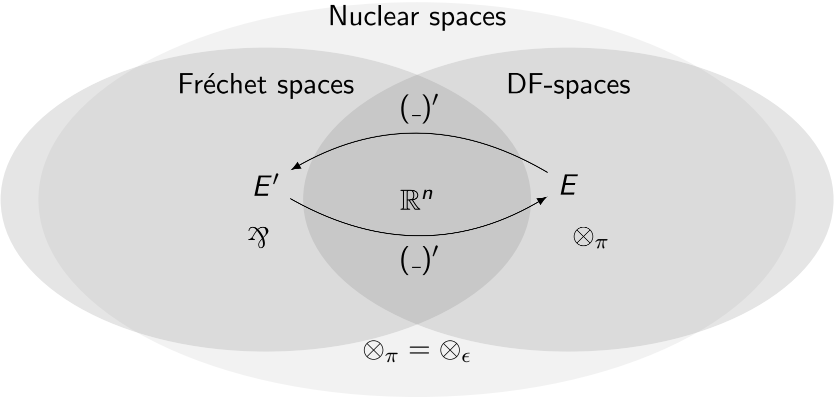
$$\forall F, E \otimes_\pi F = E \otimes_\epsilon F$$



Nuclear spaces

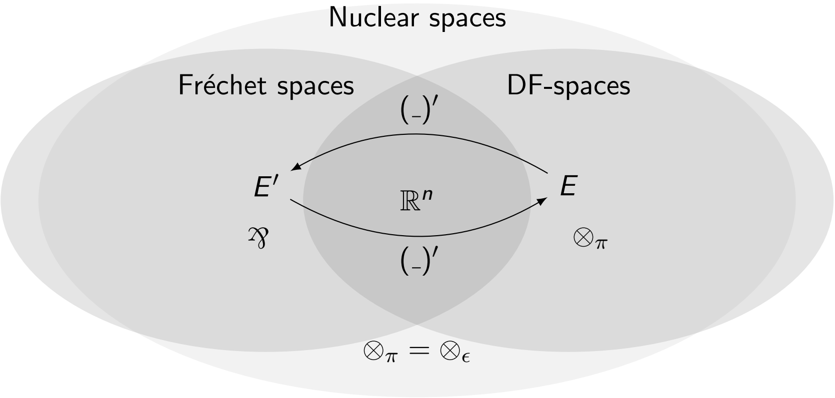
A polarized $\star$-autonomous category

A Nuclear space which is also Fréchet or (DF) is reflexive.



Nuclear spaces

We get a polarized model of MALL : involutive negation $(\_)^{\perp}$, $\otimes$, $⅋$, $\oplus$, $\times$.



Distributions and the Kernel theorems

Examples of Nuclear Fréchet spaces includes :

$$\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}),\ \mathcal{C}_c^\infty(\mathbb{R}^n,\mathbb{R}),\ \mathcal{H}(\mathbb{C},\mathbb{C}),\ ..$$

Typical Nucléar Fréchet spaces are distributions spaces Schwartz' generalized functions :

$$\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})',\ \mathcal{C}_c^\infty(\mathbb{R}^n,\mathbb{R})',\ \mathcal{H}'(\mathbb{C},\mathbb{C}),\ ..$$

The Kernel Theorems

$\mathcal{C}_c^\infty(E,\mathbb{R})' \otimes \mathcal{C}_c^\infty(F,\mathbb{R})' \simeq \mathcal{C}_c^\infty(E \times F,\mathbb{R})'$

Distributions and the Kernel theorems

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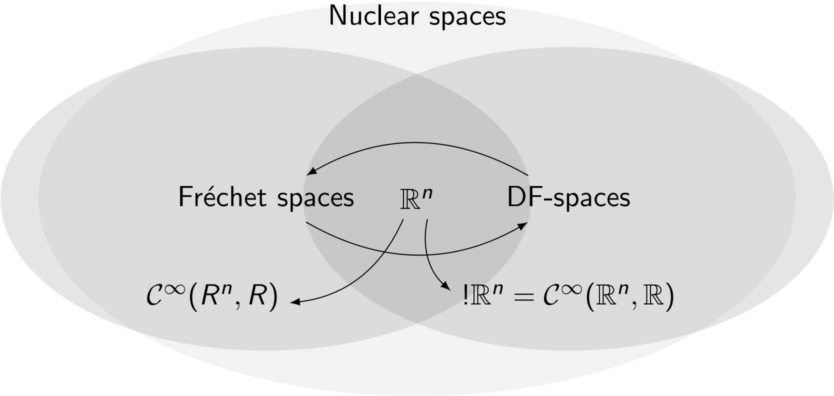
The Kernel Theorems

$\mathcal{C}^\infty(E, \mathbb{R})' \otimes \mathcal{C}^\infty(F, \mathbb{R})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{R})'$

We define $!\mathbb{R}^n = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$. Thanks to the Kernel theorems, $!$ verifies all the rules of Differential Linear Logic. However, $!R^n$ is not a finite dimensional vector space.

A Smooth differential Linear Logic



$$!\mathbb{R}^n = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})' \in \textsc{Nucl}$$

$$!\mathbb{R}^n \otimes !\mathbb{R}^m \simeq !(\mathbb{R}^{n+m})$$

Smooth DiLL

A new graded syntax

Finitary formulas : $A, B := X | A \otimes B | A \parr B | A \oplus B | A \times B.$
General formulas : $U, V := A | !A | ?A | U \otimes V | U \parr V | U \oplus V | U \times V$

For the old rules

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c$$

$$\frac{\vdash \Gamma, !A, !A}{\vdash \Gamma, !A}\, \bar{c} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, !A}\, \bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

We have obtained a smooth classical model of DiLL, to the price of Higher Order and Digging $!A \multimap !!A$.

Linear Partial Differential Equations as Exponentials

Differential Linear Logic

Is a modification of the exponential rules of Linear Logic in order to allow differentiation.

Semantics

For each $f :!A \multimap B \simeq \mathcal{C}^\infty(A, B)$ we have $Df(0) : A \multimap B$

Syntax

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash \Gamma, !A, !A}{\vdash \Gamma, !A}\, \bar{c} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, !A}\, \bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

Semantic of the dereliction

$d : E \to ?E = (!E')$ expresses the fact that $E \multimap 1 \subset !E \multimap 1$, ie :

$$\mathcal{L}(E, \mathbb{R}) \subset \mathcal{C}^\infty(E, \mathbb{R})$$

Spaces of solutions to a differential equations

A differential operator on $\mathcal{C}^\infty(\mathbb{R}^n, R)$

$$\mathrm{D} = \sum_{|\alpha| \leq n} \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdot \partial^{\alpha_n} x_n}$$

For example : $\mathrm{D}(f) = \frac{\partial^n f}{\partial x_1 \cdot \partial x_n}$.

Theorem(Schwartz)

Under some considerations on D, the space $S_{\mathrm{D}}(E, \mathbb{R})'$ of distributions solutions to $\mathrm{D}(f) = f$ is a Nuclear Fréchet space of functions.

Thus $S_{\mathrm{D}}(E, \mathbb{R})'$ is an exponential.

A new exponential

Spaces of Smooth functions	Exponentials
$\mathcal{C}^\infty(E,\mathbb{R})$	$\mathcal{C}^{\infty\prime}(E,\mathbb{R})$
$S_{\mathrm{D}}(E,\mathbb{R})$	$S'_{\mathrm{D}}(E,\mathbb{R})$
$E' \simeq \mathcal{L}(E,\mathbb{R})$	$E'' \simeq E$

Linear functions are exactly those in $\mathcal{C}^\infty(E,\mathbb{R})$ such that for all x : $f(x) = D(f)(0)(x)$.

$$\forall x, ev_x(f) = ev_x(\bar{d})(f).$$

Dereliction and co-dereliction for D

For linear functions

$$\bar{d} : E \to \mathcal{C}^\infty(E,\mathbb{R})', x \mapsto (f \mapsto D(f)(x)).$$

$$d : \mathcal{C}^\infty(E,\mathbb{R})' \to S'(E,\mathbb{R}), \phi \mapsto \phi_{\mathcal{L}(E,\mathbb{R})}$$

For solutions of $\mathrm{D}f = f$

$$\bar{d}_\mathrm{D} : E \to \mathcal{C}^\infty(E,\mathbb{R})', x \mapsto (f \mapsto \mathrm{D}(f)(x)).$$

$$d_\mathrm{D} : \mathcal{C}^\infty(E,\mathbb{R})' \to S'(E,\mathbb{R}), \phi \mapsto \phi_{S_\mathrm{D}(E,\mathbb{R})}$$

The map $\bar{d}_\mathrm{D}$ represents the equation to solve, wile d_D represents the fact that we are for looking solutions in $\mathcal{C}^\infty(E,\mathbb{R})$.

Exponentials and invariants

Spaces of Smooth functions	Exponentials	Equations
$\mathcal{C}^\infty(E, \mathbb{R})$	$\mathcal{C}^\infty(E, \mathbb{R})$	
$S_{\mathrm{D}}(E, \mathbb{R})$	$S'_{\mathrm{D}}(E, \mathbb{R})$	
$E' \simeq \mathcal{L}(E, \mathbb{R})$	$E'' \simeq E$	$d \circ \bar{d} = Id$

Exponentials and invariants

Spaces of Smooth functions	Exponentials	PDE
$\mathcal{C}^\infty(E,\mathbb{R})$	$\mathcal{C}^\infty(E,\mathbb{R})'$	
$S_{\mathrm{D}}(E,\mathbb{R})$	$S'_{\mathrm{D}}(E,\mathbb{R})$	$\begin{tikzcd} E \arrow[r, "\bar{d}_{\mathrm{D}}"] \arrow[dr, "ev_E"'] & !E \arrow[d, "d_{\mathrm{D}}"] \\ & S'(E,\mathbb{R}) \end{tikzcd}$
$E' \simeq \mathcal{L}(E,\mathbb{R})$	$E'' \simeq E$	$\begin{tikzcd} E \arrow[r, "\bar{d}"] \arrow[dr, dash, "ev_E"'] & !E \arrow[d, "d"] \\ & E'' \end{tikzcd}$

The logic of linears PDE's

Rules

$$\frac{\vdash \Gamma, !E}{\vdash \Gamma, E^{\perp\perp}}\, d \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

$$\frac{\vdash \Gamma, !E}{\vdash \Gamma, S'(E,\mathbb{R})}\, d_{\mathrm{D}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}_{\mathrm{D}}$$

Cut elimination

$$\begin{tikzcd} E \arrow[r, "\bar{d}_{\mathrm{D}}"] \arrow[dr, "ev_E"'] & !E \arrow[d, "d_{\mathrm{D}}"] \\ & S'(E,\mathbb{R}) \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "\bar{d}"] \arrow[dr, dash, "ev_E"'] & !E \arrow[d, "d"] \\ & E'' \end{tikzcd}$$

Solutions of a linear PDE also verify w and $\bar{w}$. If verifying a Kernel isomorphisms they would also verify c and $\bar{c}$.

An example

Scalar solutions defined on $\mathbb{R}^n$ of

$$\frac{\partial^n}{\partial x_1 \ldots \partial x_n} f = f$$

are the $z \mapsto \lambda e^{x_1 + \ldots + x_n}$.

$$S'(\mathbb{R}^n) \otimes S'(R^M) \simeq S'(\mathbb{R}^{n+m}).$$

$$\lambda e^{x_1 + \ldots + x_n} \mu e^{y_1 + \ldots + y_m} = \lambda \mu e^{x_1 + \ldots + x_n + y_1 + \ldots + y_m}.$$

$S(\mathbb{R}{\cdot}\mathbb{R})'$ verifies $w, \bar{w}$ (which corresponds to the initial condition of the differential equation) and $\bar{c}, c$.

Conclusion

The space of solutions to a linear partial differential equation form an exponential in Linear Logic

Conclusion

What you get :

- An interpretation of the linear involutive negation of LL in term of reflexive topological spaces.
- An interpretation of the exponential in terms of distributions.
- An interpretation of $\parr$ in term of the Schwartz epsilon product.
- A generalization of DiLL to linear *PDE*.

What you could see :

- A constructive Type Theory for differential equations.
- An interpretation of the exponential in terms of Fourier's transformation.

Thank you.