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Mackey-complete spaces and power series: A topological model of Differential Linear Logic

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We want a smooth and quantitative model of Intuitionistic Differential Linear Logic.
Models of Differential Linear Logic

Those are models of Linear Logic ...

Cartesian closed category

Spaces

Monoidal closed category

... with a biproduct structure, and a codereliction operator :

\[ \bar{d} : A \rightarrow !A \]

and some coherence conditions...

Ehrhard, A semantical introduction to differential linear logic. 2011
Fiore, Differential structure in models of multiplicative biadditive intuitionistic
linear logic. TLCA 2007
Plan

Cartesian closed category → Mackey-complete topological vector spaces → Monoidal closed category
Plan

Smooth maps
Power series

Mackey-complete spaces

Linear bounded maps
Smoothness

- The first models of Differential Linear Logic were discrete, operations being quantified on bases of vector spaces (Köthe spaces, Finiteness spaces).
- However, differentiation is historically of a continuous nature. We want to be able to match this intuition in a model of Differential Linear Logic.

Kriegl and Michor, The convenient setting of global analysis, 1997
Blute, Ehrhard and Tasson, A convenient differential category, 2010
Challenges

We (also) wanted a cartesian closed category of differentiable functions.

\[ \mathcal{D}(E \times F, G) \neq \mathcal{D}(E, \mathcal{D}(F, G)) \]

\[ C^\infty(E \times F, G) \neq C^\infty(E, C^\infty(F, G)) \]

We need a good definition of smoothness

We also need tools to handle power series.

\[ f = \sum_n f_n \text{ converging} \]

We need some notion of completeness as a way to obtain convergence
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Taboo

A space of (non necessarily linear) functions between to finite dimensional spaces is not finite dimensional.

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If we try to norm the spaces of (non necessarily linear) functions, then we have a problem.

- We want to use power series or analytic functions.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can’t be bounded on an unbounded space (Liouville’s Theorem).
- Thus functions must depart from an open ball, but arrive in a closed ball. Thus they do not compose.
- This is why Coherent Banach spaces don’t work.
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We can’t restrict ourselves to normed spaces.
Bounded sets and linear maps
We work with Haussdorf complex topological vector spaces: complex vector spaces endowed with a Haussdorf topology making addition and scalar multiplication continuous.

A bounded set $B$ is a set such that for every open set $U$ containing $0$, there is a scalar $r$ such that $B \subseteq rU$.

A function is a bounded function if it maps bounded sets on bounded sets.
A complete locally-convex topological vector space is a locally-convex topological vector space in which every Cauchy net converges.
Mackey-completeness

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A Mackey-Cauchy net in $E$ is a net $(x_\gamma)_{\gamma \in \Gamma}$ such that there is a net of scalars $\lambda_{\gamma,\gamma'}$ decreasing towards 0 and a bounded set $B$ of $E$ such that:

$$\forall \gamma, \gamma' \in \Gamma, x_\gamma - x_{\gamma'} \in \lambda_{\gamma,\gamma'} B.$$
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Mackey-completeness is a very weak condition and works well with bounded sets.
A monoidal closed category

- Endow $E \otimes F$ with the Mackey-completion of the finest locally convex topology such that $E \times F \to E \otimes F$ is bounded.
- Endow the space $\mathcal{L}(E, F)$ of all linear bounded function between $E$ and $F$ with the topology of uniform convergence on bounded subsets of $E$.

One get a symmetric monoidal closed category of Mackey-complete complex tvs and linear bounded maps between them.

$$\mathcal{L}(E\hat{\otimes} F, G) \simeq \mathcal{L}(E, \mathcal{L}(F, G))$$
Smooth functions
A smooth curve $c : \mathbb{R} \rightarrow E$ is a curve infinitely many times differentiable.

A smooth function $f : E \rightarrow F$ is a function sending a smooth curve on a smooth curve.

In Banach spaces, the definition coincides with the usual one (all iterated derivatives exists and are continuous).
Bounded sets and smooth functions

Linear continuous functions are bounded, but a linear bounded function may not be continuous.

However, linear bounded functions are smooth.
Smooth functions and differentials

A smooth map is Gateau-differentiable. Let us write $\mathcal{C}^\infty(E, F)$ for the space of all smooth maps between $E$ and $F$.

**Theorem**

The differentiation operator

$$\bar{d} : \begin{cases} \mathcal{C}^\infty(E, F) & \rightarrow \mathcal{C}^\infty(E, \mathcal{L}(E, F)) \\ f & \mapsto \left( x \mapsto \left( y \mapsto \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} \right) \right) \end{cases}$$

is well-defined, linear and bounded.
Power series
Anatomy

\[ f = \sum_{n=0}^{\infty} f_n \]

\[ f(x) = \lim_{N \to \infty} \sum_{n=0}^{N} f_n(x) \]
The sum converges uniformly on bounded sets

\[ f_n \text{ is a } n\text{-monomial} : \]
there is a bounded \( n \)-linear function \( \tilde{f}_n \)
such that \( f_n(x) = \tilde{f}_n(x, \ldots, x) \)

\[ f = \sum_{n=0}^{\infty} f_n \]
Anatomy

The sum converges uniformly on bounded sets:
\[ \forall B, \forall U, \exists N, \sum_{n \geq N} f_n(B) \subset U \]

\[ f = \sum_{n=0}^{\infty} f_n \]
The sum converges uniformly on bounded sets.

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\[ f = \sum_{n=0}^{\infty} f_n \]

Prop: \( f \) is bounded.
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Prop : \( f \) is smooth
Anatomy

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Prop : \( f \) is bounded

Prop : \( f \) is smooth

Cauchy inequality : if \( f(b) \subset b' \), then \( \forall n, f_n(b) \subset b' \)
Mackey-Arens theorem

A subset $B \subset E$ is bounded iff for every $\ell \in E'$, $l(b)$ is bounded in $\mathbb{C}$.

Scalar testing for power series

Let $f : E \to F$ be a bounded function and let $f_k$ be $k$-monomials such that for every $\ell \in F'$, $\sum_k \ell \circ f_k$ converges towards $\ell \circ f$ uniformly on bounded sets of $E$. Then, $f = \sum_k f_k$ is also a power series.
The composition of a power series is a power series.

Let us write $S(E, F)$ for the space of powers series between $E$ and $F$, endowed with the topology of uniform convergence on bounded subsets of $E$.

Theorem

If $E$, $F$, and $G$ are Mackey-complete spaces, then

$$S(E \times F, G) \simeq S(E, S(F, G)).$$
Cartesian closedness

proof

Going back to the scalar case and to Fubini’s theorem: we can permute absolutely converging double series in \( \mathbb{C} \).

\[
\psi : \begin{cases} 
S(E, S(F, G)) \to S(E \times F, G) \\
\sum_n (f_n : x \mapsto \sum_m f_{n,m}^x) \mapsto \left( (x, y) \mapsto \sum_k \sum_{n+m=k} f_{n,m}^y(y) \right)
\end{cases}
\]
What if ...

... we wanted a smooth and quantitative model of Intuitionistic Differential Linear Logic.
What if ...

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- The category of Mackey-complete reflexive spaces and linear bounded map is not closed.
- We can cheat by using pairs (as in Coherent Banach spaces) or by endowing the spaces with their weak topology.
Conclusion

- **Mackey-completeness** is a minimal and very weak condition for power series to converge.
- The use of *bounded sets* and Mackey-convergence within these sets is crucial.
- The *quantitative setting* allows for cartesian closedness.
- The *topologies are simpler* than in the model of intuitionistic Differential Linear Logic with smooth maps (Blute, Ehrhard and Tasson 2010).

Thank you!