QSLC workshop, Marseille 2016

Mackey-complete spaces and power series : A topological model of Differential Linear Logic

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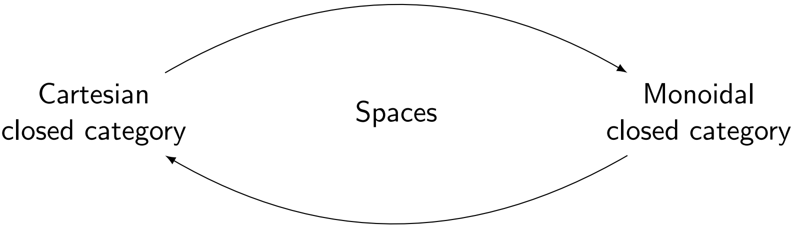
September 3, 2016

Slogan

We want a smooth and quantitative model of Intuitionistic Differential Linear Logic.

Models of Differential Linear Logic

Those are models of Linear Logic ...



... with a biproduct structure, and a codereliction operator :

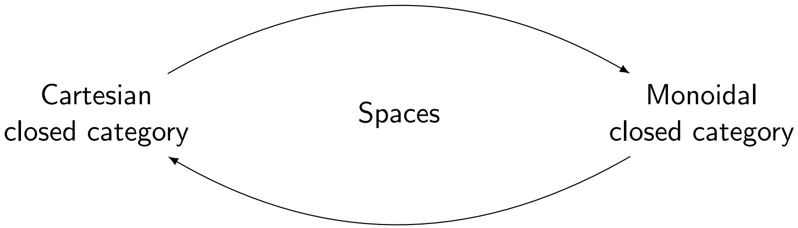
$$\bar{d} : A \rightarrow !A$$

and some coherence conditions...

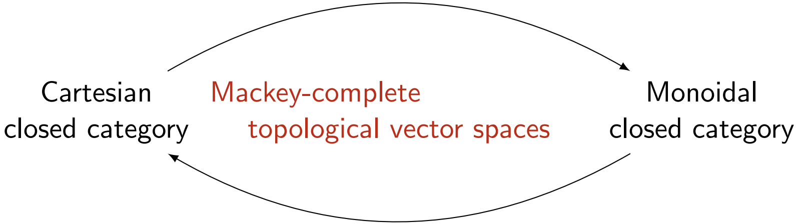
Ehrhard, A semantical introduction to differential linear logic. 2011

Fiore, Differential structure in models of multiplicative biadditive intuitionistic linear logic. TLCA 2007

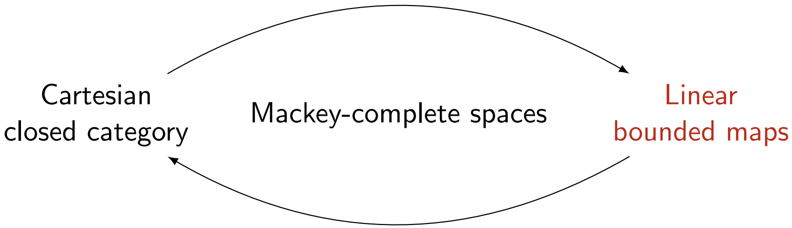
Plan


Cartesian
closed category
Spaces
Monoidal
closed category

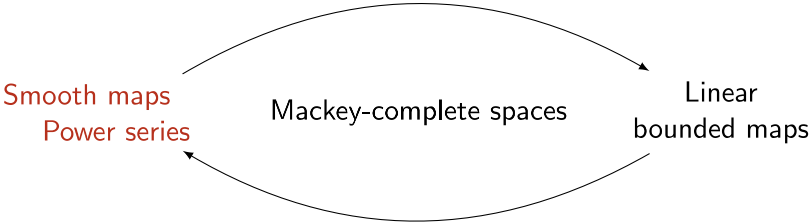
Plan


Cartesian
closed category
Mackey-complete
topological vector spaces
Monoidal
closed category

Plan

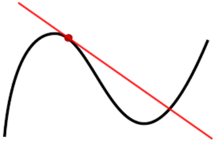

Cartesian
closed category
Mackey-complete spaces
Linear
bounded maps

Plan


Smooth maps
Power series
Mackey-complete spaces
Linear
bounded maps

Smoothness

- The first models of Differential Linear Logic were discrete, operations being quantified on bases of vector spaces (Köthe spaces, Finiteness spaces).
- However, differentiation is historically of a continuous nature. We want to be able to match this intuition in a model of Differential Linear Logic.



Kriegl and Michor, The convenient setting of global analysis, 1997

Blute, Ehrhard and Tasson, A convenient differential category, 2010

Challenges

We (also) wanted a cartesian closed category of differentiable functions.

$$\mathcal{D}(E \times F, G) \neq \mathcal{D}(E, \mathcal{D}(F, G)$$

$$\mathcal{C}^\infty(E \times F, G) \neq \mathcal{C}^\infty(E, \mathcal{C}^\infty(F, G))$$

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$$f = \underbrace{\sum_n}_{\text{converging}} f_n$$

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We need some notion of completeness as a way to obtain convergence

Taboo

A space of (non necessarily linear) functions between to finite dimensional spaces is not finite dimensional.

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- We want to use power series or analytic functions.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can't be bounded on an unbounded space (Liouville's Theorem).
- Thus functions must depart from an open ball, but arrive in a closed ball. Thus they do not compose.
- This is why Coherent Banach spaces don't work.

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We can't restrict ourselves to normed spaces.

Bounded sets and linear maps

Topological vector spaces

We work with Haussdorf complex topological vector spaces : complex vector spaces endowed with a Haussdorf topology making addition and scalar multiplication continuous.

A bounded set B is a set such that for every open set U containing 0, there is a scalar r such that $B \subseteq rU$.

A function is a bounded function if it maps bounded sets on bounded sets.

Mackey-completeness

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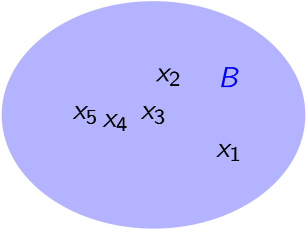
A Mackey-Cauchy net in E is a net $(x_\gamma)_{\gamma\in\Gamma}$ such that there is a net of scalars $\lambda_{\gamma,\gamma'}$ decreasing towards 0 and a bounded set B of E such that:

$$\forall \gamma, \gamma' \in \Gamma, x_\gamma - x_{\gamma'} \in \lambda_{\gamma,\gamma'} B.$$



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Mackey-completeness is a very weak condition and works well with bounded sets.

A monoidal closed category

- Endow $E \otimes F$ with the Mackey-completion of the finest locally convex topology such that $E \times F \to E \otimes F$ is bounded.
- Endow the space $\mathcal{L}(E, F)$ of all linear bounded function between E and F with the topology of uniform convergence on bounded subsets of E.

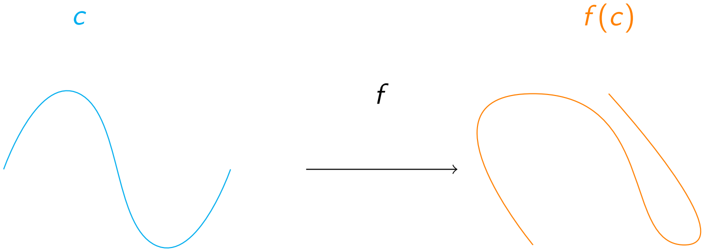
One get a symmetric monoidal closed category of Mackey-complete complex tvs and linear bounded maps between them.

$$\mathcal{L}(E \hat{\otimes} F, G) \simeq \mathcal{L}(E, \mathcal{L}(F, G))$$

Smooth functions

Smooth maps à la Frölicher,Kriegl and Michor

A smooth curve $c : \mathbb{R} \to E$ is a curve infinitely many times differentiable.

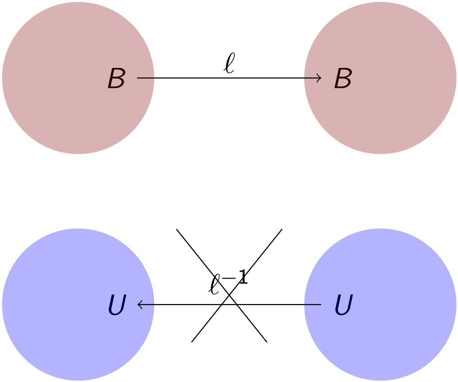


A smooth function $f : E \to F$ is a function sending a smooth curve on a smooth curve.

In Banach spaces, the definition coincides with the usual one (all iterated derivatives exists and are continuous).

Bounded sets and smooth functions

Linear continuous functions are bounded, but a linear bounded function may not be continuous.



However, linear bounded functions are smooth.

Smooth functions and differentials

A smooth map is Gateau-differentiable. Let us write $\mathcal{C}^\infty(E, F)$ for the space of all smooth maps between E and F.

Theorem

The differentiation operator

$$\bar{d} : \begin{cases} \mathcal{C}^\infty(E, F) \to \mathcal{C}^\infty(E, \mathcal{L}(E, F)) \\ \qquad f \mapsto \left(x \mapsto \left(y \mapsto \lim_{t \to 0} \dfrac{f(x + ty) - f(x)}{t} \right) \right) \end{cases}$$

is well-defined, linear and bounded.

Power series

Anatomy

$$f = \sum_{n=0}^{\infty} f_n$$

$$f(x) = \lim_{N \to \infty} \sum_{n=0}^{N} f_n(x)$$

Anatomy

The sum converges uniformly on bounded sets

f_n is a *n-monomial* :

there is a bounded n-linear function $\tilde{f}_n$ such that $f_n(x) = \tilde{f}_n(x, \ldots, x)$

$$f = \sum_{n=0}^{\infty} f_n$$

Anatomy

The sum converges uniformly on bounded sets
$\forall B, \forall U, \exists N, \sum_{n \geq N} f_n(B) \subset U$

f_n is a n-monomial

$$f = \sum_{n=0}^{\infty} f_n$$

Anatomy

The sum converges
uniformly on bounded sets

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Cauchy inequality : if $f(b) \subset b'$, then $\forall n, f_n(b) \subset b'$

Back to the scalars

Mackey-Arens theorem

A subset $B \subset E$ is bounded iff for every $\ell \in E'$, $l(b)$ is bounded in $\mathbb{C}$.

Scalar testing for power series

Let $f : E \to F$ be a bounded function and let f_k be k-monomials such that for every $\ell \in F'$, $\sum_k \ell \circ f_k$ converges towards $\ell \circ f$ uniformly on bounded sets of E. Then, $f = \sum_k f_k$ is also a power series.

A cartesian closed category

Theorem : A category ...

The composition of a power series is a power series.

Let us write $S(E, F)$ for the space of powers series between E and F, endowed with the topology of uniform convergence on bounded subsets of E.

Theorem

If E, F, and G are Mackey-complete spaces, then

$$S(E \times F, G) \simeq S(E, S(F, G)).$$

Cartesian closedness

proof

Going back to the scalar case and to Fubini's theorem :
we can permute absolutely converging double series in $\mathbb{C}$.

$$\psi : \begin{cases} S(E, S(F,G)) \to S(E \times F, G) \\ \sum_n (f_n : x \mapsto \sum_m f^x_{n,m}) \mapsto \left((x,y) \mapsto \sum_k \sum_{n+m=k} f^x_{n,m}(y) \right) \end{cases} .$$

What if ...

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- The category of Mackey-complete reflexive spaces and linear bounded map is not closed.
- We can cheat by the using pairs (as in Coherent Banach spaces) or by endowing the spaces with their weak topology.

Conclusion

- Mackey-completeness is a minimal and very weak condition for power series to converge.
- The use of bounded sets and Mackey-convergence within these sets is crucial.
- The quantitative setting allows for cartesian closedness.
- The topologies are simpler than in the model of intuitionistic Differential Linear Logic with smooth maps (Blute, Ehrhard and Tasson 2010).

Thank you !