TLLA, Oxford, September 2017

Smooth Linear Logic and Linear Partial Differential Equations

Marie Kerjean

IRIF, Université Paris Diderot
kerjean@irif.fr

What do we want

We want a model of classical Differential Linear Logic, where proofs are interpreted by smooth functions.

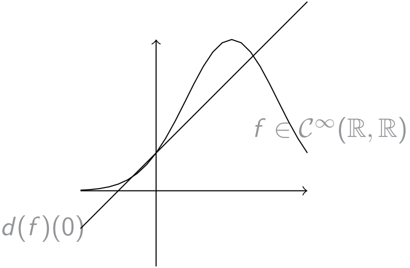
What do we get

Almost that, but we can solve Linear Partial Differential equations in it.

Smoothness

Differentiation

Differentiating a function $f : \mathbb{R}^n \to \mathbb{R}$ at x is finding a linear approximation $d(f)(x) : v \mapsto d(f)(x)(v)$ of f near x.



Smooth functions are functions which can be differentiated everywhere in their domain and whose differentials are smooth.

Differentiating proofs

- Differentiation was in the air since the study of Analytic functors by Girard :

$$\bar{d}(x) : \sum f_n \mapsto f_1(x)$$

- DiLL was developed after a study of vectorial models of LL inspired by coherent spaces : Finiteness spaces (Ehrard 2005), Köthe spaces (Ehrhard 2002).

Normal functors, power series and λ-calculus. Girard, APAL(1988)

Differential interaction nets, Ehrhard and Regnier, TCS (2006)

Differential Linear Logic

The rules of DiLL are those of MALL and :

co-dereliction

$$\bar{d} : x \mapsto f \mapsto df(0)(x)$$

Smoothness of proofs

- Traditionally proofs are interpreted as graphs, relations between sets, power series on finite dimensional vector spaces, strategies between games: those are discrete objects.

- Differentiation appeals to differential geometry, manifolds, functional analysis : we want to find a denotational model of DiLL where proofs are general smooth functions.

The categorical semantic of Differential Linear Logic

Linearity and Smoothness

We work with vector spaces with some notion of continuity on them : for example, normed spaces, or complete normed spaces (Banach spaces).

What's required

- A (monoidal closed) $*$-autonomous category : $E \simeq (E^{\perp})^{\perp}$
- A comonad ! verifying : $!E \otimes !F \simeq !(E \times F)$
- A bialgebra structure $(!E, w, c, \bar{w}, \bar{c})$
- A good notion of differentiation $\bar{d}$ such that $\bar{d} \circ d = Id$
- And coherence conditions

Spaces of linear and smooth functions

The linear dual

$A^{\perp}$ is the linear dual of A, interpreted by $\mathcal{L}(A, \mathbb{R}) = A'$. We want reflexive vector spaces : $A'' \simeq A$.

We want non-linear proof to be interpreted by smooth functions :

$$\mathcal{L}(!E, F) \simeq \mathcal{C}^{\infty}(E, F).$$

The exponential is the dual of the space of smooth scalar functions

$$!E \simeq (!E)'' \simeq \mathcal{L}(!E, \mathbb{R})' \simeq \mathcal{C}^{\infty}(E, \mathbb{R})'$$

A typical inhabitant of $!E$ is $ev_x : f \mapsto f(x)$.

An exponential for smooth functions

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

$$\dim \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m) = \infty.$$

An exponential for smooth functions

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

$$\dim \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m) = \infty.$$

We can't restrict ourselves to finite dimensional spaces.

The tentative to have a normed space of analytic functions fails (Coherent Banach spaces).

- We want to use functions.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can't be bounded on an unbounded space (Liouville's Theorem).
- Thus functions must depart from an open ball, but arrive in a closed ball. Thus they do not compose.
- This is why Coherent Banach spaces don't work.

An exponential for smooth functions

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

$$\dim \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m) = \infty.$$

We can't restrict ourselves to finite dimensional spaces.

The tentative to have a normed space of analytic functions fails (Coherent Banach spaces).

- We want to use functions.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can't be bounded on an unbounded space (Liouville's Theorem).
- Thus functions must depart from an open ball, but arrive in a closed ball. Thus they do not compose.
- This is why Coherent Banach spaces don't work.

We can't restrict ourselves to normed spaces.

A model with Distributions

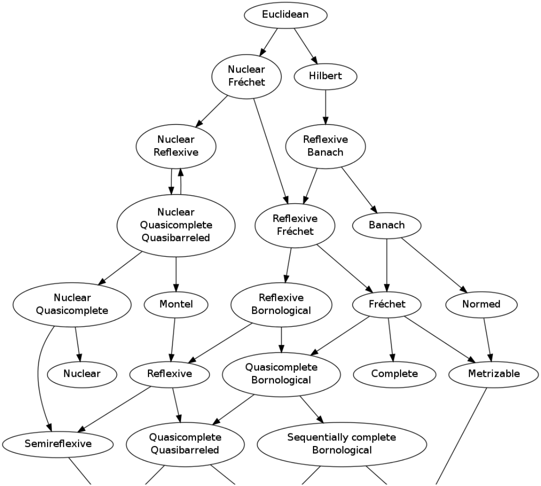
Topological vector spaces

We work with Hausdorff topological vector spaces : real or complex vector spaces endowed with a Hausdorff topology making addition and scalar multiplication continuous.

- The topology on E determines E'.
- The topology on E' determines whether $E \simeq E''$.

We work within the category TopVect of topological vector spaces and continuous linear functions between them.

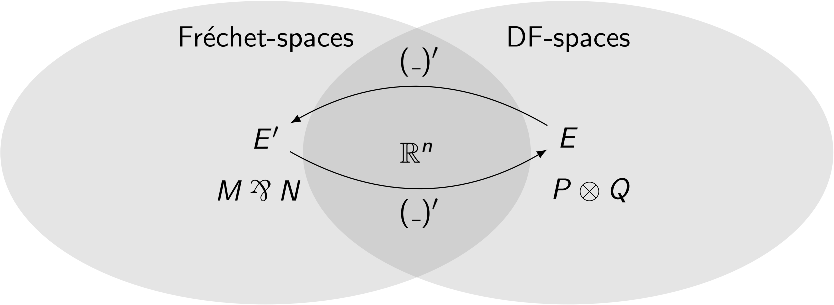
Topological models of DiLL



Let us take the other way around, through Nuclear Fréchet spaces.

Fréchet and DF spaces

- Fréchet : metrizable complete spaces.
- (DF)-spaces : such that the dual of a Fréchet is (DF) and the dual of a (DF) is Fréchet.

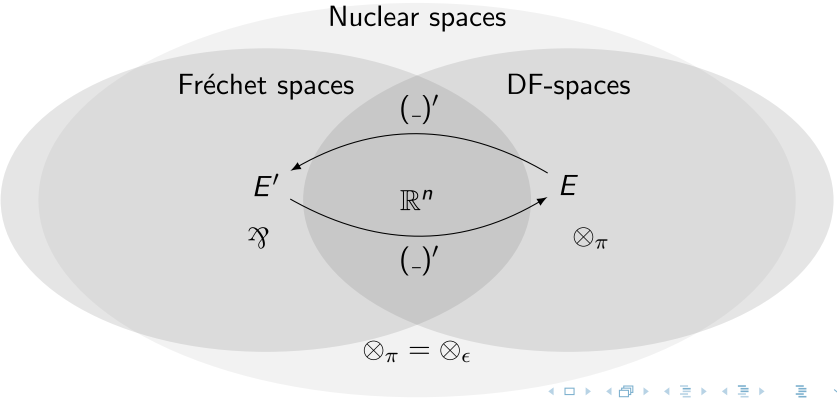


These spaces are in general not reflexive.

Nuclear spaces

Nuclear spaces are spaces in which one can identify the two canonical topologies on tensor products :

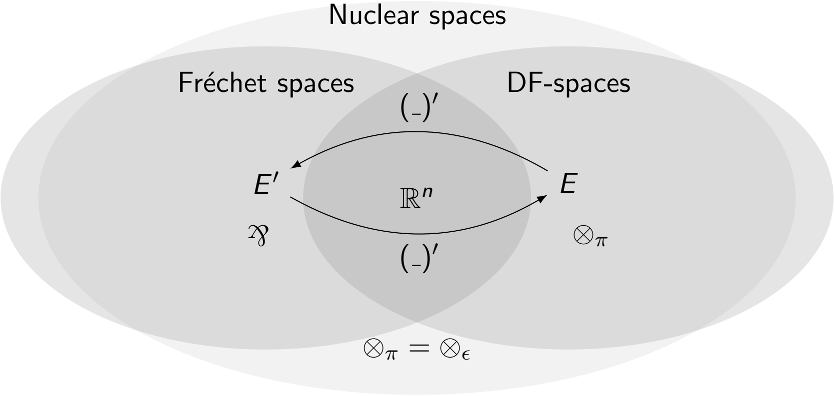
$$\forall F, E \otimes_\pi F = E \otimes_\epsilon F$$



Nuclear spaces

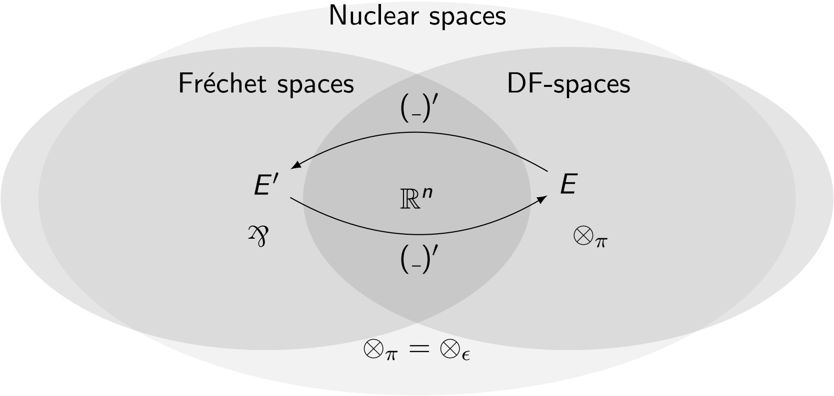
A polarized $\star$-autonomous category

A Nuclear space which is also Fréchet or dual of a Fréchet is reflexive.



Nuclear spaces

We get a polarized model of MALL : involutive negation $(\_)^{\perp}$, $\otimes$, $\parr$, $\oplus$, $\times$.



Distributions and the Kernel theorems

A typical Nuclear Fréchet space is the space of smooth functions on $\mathbb{R}^n$: $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$.

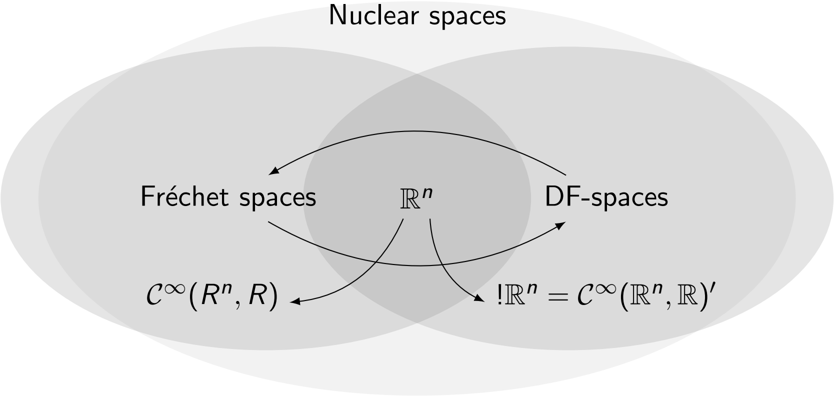
A typical Nuclear DF spaces is Schwartz' space of distributions with compact support : $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$.

The Kernel Theorems

$\mathcal{C}^\infty(E, \mathbb{R})' \otimes \mathcal{C}^\infty(F, \mathbb{R})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{R})'$

$$!\mathbb{R}^n = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'.$$

A model of Smooth differential Linear Logic



A Smooth differential Linear Logic

A graded semantic

Finite dimensional vector spaces:

$$R^n, R^m := \mathbb{R} | R^n \otimes R^m | R^n ⅋ R^m | R^n \oplus R^m | R^n \times R^m.$$

Nuclear spaces :

$$U, V := R^n | !R^n | ?R^n | U \otimes V | U ⅋ V | U \oplus V | U \times V.$$

$$!\mathbb{R}^n = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})' \in \textsc{Nucl}$$

$$!\mathbb{R}^n \otimes !\mathbb{R}^m \simeq !(\mathbb{R}^{n+m})$$

We have obtained a smooth classical model of DiLL, to the price of Digging $!A \multimap !!A$.

Smooth DiLL, a failed exponential

A new graded syntax

Finitary formulas : $A, B := X | A \otimes B | A \parr B | A \oplus B | A \times B.$
General formulas : $U, V := A | !A | ?A | U \otimes V | U \parr V | U \oplus V | U \times V$

For the old rules

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash}{\vdash !A}\, \bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

The categorical semantic of smooth DiLL is the one of LL, but where ! is a monoidal functor and d and $\bar{d}$ are to be defined independently.

Linear Partial Differential Equations as Exponentials

Work in progress

Linear functions as solutions to an equation

$$f \in \mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R}) \text{ is linear} \quad \begin{array}{l} \text{iff } \forall x, f(x) = D(f)(0)(x) \\ \text{iff } f = \bar{d}(f) \\ \text{iff } \exists g \in \mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R}), f = \bar{d}g \end{array}$$

Another definition for $\bar{d}$

A linear partial differential operator D acts on $\mathcal{C}^{\infty}(\mathbb{R}^n, R)$:

$$D(f)(x) = \sum_{|\alpha| \leq n} a_{\alpha}(x) \frac{\partial^{\alpha} f}{\partial x^{\alpha}}.$$

Another exponential is possible

Definition

$$!_D A = (D(\mathcal{C}^\infty(A,\mathbb{R})))'$$

that is the space of linear functions acting on functions $f = Dg$, for $g \in \mathcal{C}^\infty(A,\mathbb{R})$, when $A \subset \mathbb{R}^n$ for some n.

$$\bar{d}_D : !_D A \to !A; \phi \mapsto (f \mapsto \phi(D(f)))$$

$$d_D : !A \to !_D A; \phi \mapsto \phi_{|D(\mathcal{C}^\infty(A)}$$

Functions	E'	$D(\mathcal{C}^\infty(A))$	$\mathcal{C}^\infty(A)$
$!$	$E'' \simeq E$	$!_D A = D(\mathcal{C}^\infty(A))'$	$!A = \mathcal{C}^\infty(A)'$
d	$\phi \mapsto \phi_{\|(A)'}$	$\phi \mapsto \phi_{\|D(\mathcal{C}^\infty(A))}$	
$\bar{d}$	$x \mapsto (f \mapsto d(f)(0)(x))$	$\phi \mapsto (f \mapsto \phi(D(f)))$	

Recall : The structural morphisms on $!E$

- The codereliction $\bar{d}_E : E \rightarrow !E = \mathcal{C}^\infty(E,\mathbb{R})'$ encodes the differential operator.
- In a $\star$-autonomous category $d_E : E \rightarrow ?E$ encode the fact that linear functions are smooth.

- $c :!E \rightarrow !E \otimes !E \rightarrow$ is deduced from the Seely isomorphism and maps $ev_x \otimes ev_x$ to ev_x.
- $\bar{c} !E \otimes !E \rightarrow !E$ is the convolution $\star$ between two distributions
- $w :!E \rightarrow \mathbb{R}$ maps ev_x to 1.
- $\bar{w} : \mathbb{R} \rightarrow !E$ maps 1 to $ev_0 : f \mapsto f(0)$, the neutral for $\star$.

$!_D$

Consider D a LPDO with constant coefficients :

$$D = \sum_{\alpha, |\alpha| \leq n} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}.$$

Existence of a fundamental solution

For such D there is $E_0 \in \mathcal{C}^\infty(A)'$ such that $DE_0 = ev_0$.

D commutes with convolution

If $f \in D(\mathcal{C}^\infty(A))$ and $g \in \mathcal{C}^\infty(A)$, then $f * g \in D(\mathcal{C}^\infty(A))$.

$?A^\perp$	E'	$D(\mathcal{C}^\infty(A, \mathbb{R}))$	$\mathcal{C}^\infty(A, \mathbb{R})$
$!A$	$E'' \simeq E$	$D(\mathcal{C}^\infty(A, \mathbb{R}))'$	$\mathcal{C}^\infty(A, \mathbb{R})'$
$\bar{c}$		$* : !A \otimes !_D A \to !_D A$	$* : !A \otimes !A \to !A$
$\bar{w}$		$1 \mapsto E_0$	$1 \mapsto ev_0$

and a co-algebra structure

Intermediates rules for D

work in progress

Syntax

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ d$$

$$\frac{\vdash \Gamma, !A, !A}{\vdash \Gamma, !A}\ \bar{c} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, !A}\ \bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \bar{d}$$

Syntax for $!_D$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\ w \qquad \frac{\vdash \Gamma, ?A, ?_D A}{\vdash \Gamma, ?_D A}\ c \qquad \frac{\vdash \Gamma, ?_D A}{\vdash \Gamma, ?A}\ d_D$$

$$\frac{\vdash}{\vdash !_D A}\ \bar{w}_D \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !_D A}{\vdash \Gamma, \Delta, !_D A}\ \bar{c}_D \qquad \frac{\vdash \Gamma, !_D A}{\vdash \Gamma, !A}\ \bar{d}$$

Solving Linear PDE's with constant coefficient

$\bar{w}$ is the fundamental solution

E_0 is the fundamental solution, such that $DE_0 = ev_0$. Its existence is guaranteed when D has constant coefficients.

Solving Linear PDE through $\bar{w}$ and $\bar{c}$

If $f \in \mathcal{C}^\infty(A)$, then $D(E_0 * f) = f$.

Solving Linear PDE's with constant coefficient

$\bar{w}$ is the fundamental solution

E_0 is the fundamental solution, such that $DE_0 = ev_0$. Its existence is guaranteed when D has constant coefficients.

Solving Linear PDE through $\bar{w}$ and $\bar{c}$

If $f \in \mathcal{C}^\infty(A)$, then $D(E_0 * f) = f$.
If $f \in E'$, then $d(ev_0 * f) = f$.

The algebraic equation is the one of the resolution of the differential equation.

Conclusion

What we have :

- An interpretation of the linear involutive negation of LL in term of reflexive TVS.
- An interpretation of the exponential in terms of distributions.
- The first hints for a generalization of DiLL to linear *PDE*'s .

What we could get :

- A constructive Type Theory for differential equations.
- Logical interpretations of fundamental solutions, specific spaces of distributions, Fourier transformations or operation on distributions.

Bibliography

- *Convenient differential category* Blute, Ehrhard Tasson Cah. Geom. Diff. (2010)
- *Weak topologies for Linear Logic*, K. LMCS 2015.
- *Mackey-complete spaces and Power series*, K. and Tasson, MSCS 2016.
- *A model of LL with Schwartz' epsilon product*, K. and Dabrowski, *In preparation*.
- *Distributions and Smooth Differential Linear Logic*, K., *In preparation*.