A prismoid framework for languages with resources

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Abstract

Inspired by the Multiplicative Exponential fragment of Linear Logic, we define a framework called the **prismoid of resources** where each vertex is a language which refines the λ-calculus by using a different choice to make explicit or implicit (meta-level) the definition of the contraction, weakening, and substitution operations. For all the calculi in the prismoid we show simulation of β-reduction, confluence, preservation of β-strong normalisation and strong normalisation for typed terms. Full composition also holds for all the calculi of the prismoid handling explicit substitutions. The whole development of the prismoid is done by making the set of resources a parameter of the formalism, so that all the properties for each vertex are obtained as a particular case of the general abstract proofs.

1. Introduction

Linear Logic [Gir87] has significantly contributed in many fields of computer science, particularly because it provides a logical tool to formalise the notion of control of resources by means of weakening, contraction and dereliction. The Multiplicative Exponential fragment of Linear Logic, called MELL, is able to encode Intuitionistic as well as Classical Logic, either by means of sequent trees or Proof-Nets [Gir87]. MELL Proof-Nets give a succinct representation of proofs by eliminating irrelevant syntactical details appearing in sequent calculi. The cut-elimination process of Proof-Nets has been widely studied by means of the Geometry of Interaction, giving rise to optimal implementations of functional programming [Lam90, GAL92, DR93, AG98].

Many different [vO01, DG01, DCKP03, KL07, Kes07, FMS05] cut elimination systems for λ-calculus, known as *explicit substitution* (ES) calculi, were explained in terms of, or were inspired by, the fine notion of reduction associated to MELL Proof-Nets. All of them integrate special operators for the control of resources, thus allowing more refined cut-elimination procedures, but not necessarily the same.

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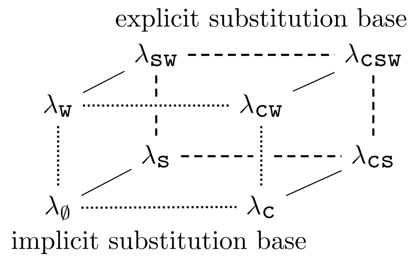
In this paper we develop an homogeneous framework, called the *prismoid of resources*, which provides eight languages – the vertexes of the prismoid – dedicated to the control of resources for the λ-calculus, together with different transformation functions – the arrows of the prismoid – between these languages.

More precisely, each vertex of the prismoid is a specialised λ-calculus defined by a set of well-formed terms and a set of axioms and reduction rules as well. Each calculus is parametrised by a set of *sorts* wich are of two kinds: resources $\mathtt{w}$ (**w**eakening) and $\mathtt{c}$ (**c**ontraction), and cut-elimination operation $\mathtt{s}$ (**s**ubstitution). If a sort in the set $\{\mathtt{c}, \mathtt{s}, \mathtt{w}\}$ belongs to a given calculus, then the treatment of the corresponding operations to deal with this sort is completely explicit in this calculus, i.e. is given by syntax and rules belonging to the language itself. The eight calculi of the prismoid correspond to 2^3 different ways to combine the sorts $\{\mathtt{c}, \mathtt{s}, \mathtt{w}\}$ by means of explicit (Ex) or implicit (Im) (meta-level) operations:

	Resource $\mathtt{c}$	Resource $\mathtt{s}$	Resource $\mathtt{w}$
$\lambda_\emptyset$	Im	Im	Im
$\lambda_\mathtt{c}$	Ex	Im	Im
$\lambda_\mathtt{s}$	Im	Ex	Im
$\lambda_\mathtt{w}$	Im	Im	Ex
$\lambda_\mathtt{cs}$	Ex	Ex	Im
$\lambda_\mathtt{cw}$	Ex	Im	Ex
$\lambda_\mathtt{sw}$	Im	Ex	Ex
$\lambda_\mathtt{csw}$	Ex	Ex	Ex

Thus for example, the $\lambda_\mathtt{cs}$-calculus has only explicit control of contraction and substitution, the λ-calculus (called here $\lambda_\emptyset$-calculus), has no explicit control at all, and the $\lambda_\mathtt{csw}$-calculus – a slight variation of $\lambda\mathtt{lxr}$ [KL07] – has explicit control of everything.

For every subset of sorts $\mathcal{B} \subseteq \{\mathtt{c}, \mathtt{s}, \mathtt{w}\}$, the corresponding $\mathcal{B}$-calculus of the prismoid implements λ-calculus in the sense that β-reduction can be simulated by $\mathcal{B}$-reduction. It is also possible to take off some explicit information from a given $\mathcal{B}$-calculus in order to project $\mathcal{B}$-reduction into a less refined relation. More precisely, for every $\mathcal{A} \subseteq \{\mathtt{c}, \mathtt{w}\}$, $\mathcal{A}$-reduction (resp. $\mathcal{A} \cup \mathtt{s}$-reduction) is projected into β-reduction (resp. $\mathtt{s}$-reduction). This asymmetry between languages with and without sort $\mathtt{s}$ are reflected in the prismoid by means of two conceptually different *bases*. The base $\mathfrak{B}_I$ contains all the calculi without explicit substitutions, namely $\{\lambda_\emptyset, \lambda_\mathtt{c}, \lambda_\mathtt{w}, \lambda_\mathtt{cw}\}$, and the base $\mathfrak{B}_E$ only contains those with explicit substitutions, i.e. $\{\lambda_\mathtt{s}, \lambda_\mathtt{cs}, \lambda_\mathtt{sw}, \lambda_\mathtt{csw}\}$.



For all the calculi of the prismoid we study a set of properties which guarantee that they are well-behaved, namely, simulation of β-reduction, confluence, preservation of β-strong normalisation (PSN) and strong normalisation (SN) for simply typed terms. Thus in particular, none of the calculi suffers from Mellies' counter-example [Mel95]. Full composition, stating that explicit substitution is able to implement the underlying notion of higher-order substitution, is also shown for all calculi with sort $\mathtt{s}$, i.e. those included in the explicit substitution base. Each property is stated and proved by making the set of sorts a *parameter*, so that the properties for each vertex of the prismoid turn out to be a particular case of some general abstract proof, which may hold for the whole prismoid or just for only one base.

Related Work: Different calculi with explicit resources were inspired by MELL Proof-Nets. The calculus in [Kes07] encodes MELL reductions by using explicit substitutions, while [FMS05] encodes only those that are *closed* and uses also director strings technology. The calculus in [vO01] refines β-reduction by adding only explicit control for weakening and contraction (but not for linear substitution), while [DCKP03] encodes into MELL Proof-Nets the λ_{ws}-calculus [DG01] which refines β-reduction with explicit weakening and substitution (but not with contraction). The $\lambda\mathtt{lxr}$-calculus [KL07] has explicit control of everything and a slight variation of it is one of the languages of the prismoid presented in this paper.

While explicit substitution is usually [ACCL91, KR95, BBLRD96] defined by means of the propagation of an operator through the structure of terms, the behaviour of calculi of the prismoid incorporates also a mechanism to decrease the multiplicity of variables that are affected by substitutions. This notion is close in spirit to MELL Proof-Nets, and shares common ideas with calculi acting at a distance [Mil07, dB87, Ned92, SP94, KLN05, Ó Conchúir06, AK10]. However, none of the previous formalisms handles weakening and contraction as explicit operators.

This paper is an extended and revised version of [KR09].

Road Map: Section 2 introduces syntax and operational semantics of the prismoid. Section 3 explores how to enrich the λ-calculus by adding more explicit control of resources, while Section 4 deals with the dual operation which forgets information given by explicit weakening and contraction. Section 5 is devoted to PSN and confluence on untyped terms. Finally, typed terms are introduced in Section 6 together with a SN proof for them. We conclude and give future directions of work in Section 7.

2. Terms and Rules of the Prismoid

2.1. Terms

We assume a denumerable set of variable symbols $x, y, z, \ldots$. Lists and sets of variables are denoted by capital Greek letters $\Gamma, \Delta, \Pi, \ldots$. We write $\Gamma; y$ for $\Gamma \cup \{y\}$ when $y \notin \Gamma$. We use $\Gamma \setminus \Delta$ for **set difference** and $\Gamma \setminus\!\!\setminus \Delta$ for **obligation set difference** which is equal to set difference when $\Delta \subseteq \Gamma$ but undefined otherwise.

Terms are given by the grammar:

$$t, u ::= x \mid \lambda x.t \mid tu \mid t[x/u] \mid \mathcal{W}_x(t) \mid \mathcal{C}_x^{y|z}(t)$$

The terms x, $\lambda x.t$, tu, $t[x/u]$, $\mathcal{W}_x(t)$ and $\mathcal{C}_x^{y|z}(t)$ are respectively called **term variable**, **abstraction**, **application**, **closure**, **weakening** and **contraction**.

The **size** of the term t is denoted by $\mathtt{size}(t)$. **Free** and **bound** variables of t, respectively written $\mathtt{fv}(t)$ and $\mathtt{bv}(t)$, are defined as usual: $\lambda x.u$ and $u[x/v]$ bind x in u, $\mathcal{C}_x^{y|z}(u)$ binds y and z in u, x is free in $\mathcal{C}_x^{y|z}(u)$ and in $\mathcal{W}_x(t)$.

We use the following **abbreviations**: $t_1 t_2 \ldots t_n$ means $((t_1 t_2) \ldots) t_n$, $t[\overline{x}/\overline{v}]$ means $t[x_1/v_1] \ldots [x_n/v_n]$ when n is clear from the context. A closure $t[\overline{x}/\overline{u}]$ has **independent substitutions** $[\overline{x}/\overline{u}]$ iff $x_i \cap \mathtt{fv}(u_j) = \emptyset$ for all i, j. For example the substitutions are independent in $x[x/y][x/z]$, but not in $x[x/y][y/z]$.

Given three lists of *distinct* variables $\Gamma = x_1, \ldots, x_n$, $\Delta = y_1, \ldots, y_n$ and $\Pi = z_1, \ldots, z_n$ of the same length, the notations $\mathcal{W}_\Gamma(t)$ and $\mathcal{C}_\Gamma^{\Delta|\Pi}(t)$ mean, respectively, $\mathcal{W}_{x_1}(\ldots \mathcal{W}_{x_n}(t))$ and $\mathcal{C}_{x_1}^{y_1|z_1}(\ldots \mathcal{C}_{x_n}^{y_n|z_n}(t))$. These notations will extend naturally to sets of variables of same size thanks to the equivalence relation in Figure 2. The particular cases $\mathcal{C}_\emptyset^{\emptyset|\emptyset}(t)$ and $\mathcal{W}_\emptyset(t)$ mean simply t.

Given lists $\Gamma = x_1, \ldots, x_n$ and $\Delta = y_1, \ldots, y_n$ of *distinct* variables, the **renaming** of Γ by Δ in t, written $R_\Delta^\Gamma(t)$, is the capture-avoiding simultaneous substitution of y_i for every free occurrence of x_i in t. For example $R_{y_1 y_2}^{x_1 x_2}(\mathcal{C}_{x_1}^{y|z}(x_2 yz)) = \mathcal{C}_{y_1}^{y|z}(y_2 yz)$.

Alpha-conversion is the (standard) congruence generated by *renaming* of bound variables. For example, $\lambda x_1.x_1 \mathcal{C}_x^{y_1|z_1}(y_1 z_1) \equiv_\alpha \lambda x_2.x_2 \mathcal{C}_x^{y_2|z_2}(y_2 z_2)$. All the operations defined along the paper are considered modulo alpha-conversion so that in particular capture of variables is not possible.

The set of **positive free variables** in a term t, written $\mathtt{fv}^+(t)$, denotes the free variables of t which represent a term variable at the end of some (possibly empty) contraction chain. Formally,

$$
\begin{array}{lcll}
\mathtt{fv}^+(y) & := & \{y\} & \\
\mathtt{fv}^+(\lambda y.u) & := & \mathtt{fv}^+(u) \setminus \{y\} & \\
\mathtt{fv}^+(u\ v) & := & \mathtt{fv}^+(u) \cup \mathtt{fv}^+(v) & \\
\mathtt{fv}^+(\mathcal{W}_y(u)) & := & \mathtt{fv}^+(u) & \\
\mathtt{fv}^+(u[y/v]) & := & (\mathtt{fv}^+(u) \setminus \{y\}) \cup \mathtt{fv}^+(v) & \\
\mathtt{fv}^+(\mathcal{C}_y^{z|w}(u)) & := & (\mathtt{fv}^+(u) \setminus \{z, w\}) \cup \{y\} & \text{if } z \in \mathtt{fv}^+(u) \text{ or } w \in \mathtt{fv}^+(u) \\
\mathtt{fv}^+(\mathcal{C}_y^{z|w}(u)) & := & \mathtt{fv}^+(u) & \text{otherwise}
\end{array}
$$

For instance, x is a positive free variable in $\mathcal{C}_x^{x_1|x_2}(\mathcal{W}_{x_1}(y)\ x_2)$ because there is a chain from the contraction $\mathcal{C}_x^{x_1|x_2}(\_)$ to the term variable x_2. Moreover, x is also positive in $\mathcal{C}_x^{x_1|x_2}(\mathcal{C}_{x_1}^{y|z}(z))$ because there is a chain from x to the term variable z. However x is not positive in $\mathcal{C}_x^{x_1|x_2}(\mathcal{C}_{x_1}^{x_3|x_4}(y))$ because there is no chain starting at x and ending on a term variable.

The **number of occurrences** of the free variable (resp. positive free variable) x in the term t is written $|t|_x$ (resp. $|t|_x^+$). We extend this definition to sets by $|t|_\Gamma^+ = \Sigma_{x\in\Gamma}|t|_x^+$. Thus for example, given $t = \mathcal{W}_{x_1}(xx)\ \mathcal{W}_x(y)\ \mathcal{C}_z^{z_1|z_2}(z_2)$, we have $x, y, z \in \mathtt{fv}^+(t)$ with $|t|_x^+ = 2$, $|t|_y^+ = |t|_z^+ = 1$ but $x_1 \notin \mathtt{fv}^+(t)$.

Given a list of *distinct* variables $x_1 \ldots x_n$, which are all fresh in t, we write $t_{[x:=x_1\ldots x_n]}$, for the capture-avoiding **non-deterministic replacement** of $n \geq 1$ *positive* occurrences of x in t by the variables $x_1 \ldots x_n$. Thus for example, $(\mathcal{W}_x(t)\ x\ x)_{[x:=y_1y_2]}$ denotes $\mathcal{W}_x(t)\ y_1\ y_2$ or $\mathcal{W}_x(t)\ y_2\ y_1$. In the same way, $(\mathcal{W}_x(t)\ x\ x)_{[x:=y]}$ denotes either $\mathcal{W}_x(t)\ y\ x$ or $\mathcal{W}_x(t)\ x\ y$, but neither $\mathcal{W}_y(t)\ x\ x$ nor $\mathcal{W}_x(t)\ y\ y$.

Now, let us consider a set of **resources** $\mathcal{R} = \{\mathtt{c}, \mathtt{w}\}$ and a set of **sorts** $\mathcal{S} = \mathcal{R} \cup \{\mathtt{s}\}$. For every subset $\mathcal{B} \subseteq \mathcal{S}$, we define a calculus $\lambda_\mathcal{B}$ in the **prismoid of resources** which is equipped with a set of **well-formed** terms, denoted $\mathcal{T}_\mathcal{B}$ and defined in Section 2.2, together with a **reduction relation**, denoted $\rightarrow_\mathcal{B}$ and defined in Section 2.3.

Each calculus $\lambda_\mathcal{B}$ belongs to a **base** : the explicit substitution base $\mathfrak{B}_E$ which contains all the calculi having at least sort $\mathtt{s}$ and the implicit substitution base $\mathfrak{B}_I$ containing all the other calculi.

2.2. Well-Formed terms

A term t belongs to the set of **well-formed** terms $\mathcal{T}_\mathcal{B}$ iff $\exists\ \Gamma$ s.t. $\Gamma \Vdash_\mathcal{B} t$ is derivable in the system given by the rules appearing in Figure 1. A term $t \in \mathcal{T}_\mathcal{B}$ is also called a $\mathcal{B}$**-term**. From now on we only consider well-formed terms.

$$\frac{}{x \Vdash_\mathcal{B} x} \qquad \frac{\Gamma \Vdash_\mathcal{B} u \quad \Delta \Vdash_\mathcal{B} v}{\Gamma \uplus_\mathcal{B} \Delta \Vdash_\mathcal{B} uv} \qquad \frac{\Gamma \Vdash_\mathcal{B} u}{\Gamma \setminus\!\!\setminus_\mathcal{B} x \Vdash_\mathcal{B} \lambda x.u} \qquad \frac{\Gamma \Vdash_\mathcal{B} u}{\Gamma; x \Vdash_\mathcal{B} \mathcal{W}_x(u)}\ (\mathtt{w} \in \mathcal{B})$$

$$\frac{\Gamma \Vdash_\mathcal{B} v \quad \Delta \Vdash_\mathcal{B} u}{\Gamma \uplus_\mathcal{B} (\Delta \setminus\!\!\setminus_\mathcal{B} x) \Vdash_\mathcal{B} u[x/v]}\ (\mathtt{s} \in \mathcal{B}) \qquad \frac{\Gamma \Vdash_\mathcal{B} u}{x; (\Gamma \setminus\!\!\setminus_\mathcal{B} \{y, z\}) \Vdash_\mathcal{B} \mathcal{C}_x^{y|z}(u)}\ (\mathtt{c} \in \mathcal{B})$$

Figure 1: Well-formed terms of the prismoid

In the previous rules, the symbol ; is used to denote disjoint union. Also, $\uplus_\mathcal{B}$ means standard union if $\mathtt{c} \notin \mathcal{B}$ and disjoint union if $\mathtt{c} \in \mathcal{B}$. Similarly, $\Gamma \setminus\!\!\setminus_\mathcal{B} \Delta$ is used for $\Gamma \setminus \Delta$ if $\mathtt{w} \notin \mathcal{B}$ and for $\Gamma \setminus\!\!\setminus \Delta$ if $\mathtt{w} \in \mathcal{B}$.

Notice that variables, applications and abstractions belong to all calculi of the prismoid while weakening, contraction and substitutions only appear in calculi having the corresponding sort. If t is a $\mathcal{B}$-term, then $\mathtt{w} \in \mathcal{B}$ implies that bound variables of t cannot be useless, and $\mathtt{c} \in \mathcal{B}$ implies that no free variable of t has more than one free occurrence. Thus for example the term $\lambda z.x\ y$ belongs to the calculus $\lambda_{\mathcal{B}}$ only if $\mathtt{w} \notin \mathcal{B}$ (thus it belongs to $\lambda_{\emptyset}$, $\lambda_{\mathtt{C}}$, $\lambda_{\mathtt{S}}$, $\lambda_{\mathtt{CS}}$), and $(xz)[z/yx]$ belongs to $\lambda_{\mathcal{B}}$ only if $\mathtt{s} \in \mathcal{B}$ and $\mathtt{c} \notin \mathcal{B}$ (thus it belongs to $\lambda_{\mathtt{S}}$ and $\lambda_{\mathtt{SW}}$). A useful property is that $\Gamma \Vdash_{\mathcal{B}} t$ implies $\Gamma = \mathtt{fv}(t)$.

We introduce the following measure $\mathtt{o}_x(t)$ which counts free occurrences of x in t by taking care of duplications if the variable is contracted. The **number of contracted occurrences** of the free variable x in the well-formed term t, written $\mathtt{o}_x(t)$, is defined modulo alpha-conversion so that bound variables of t are assumed to be disjoint from x. Formally,

$$\begin{array}{lcll}
\mathtt{o}_x(x) & := & 1 & \\
\mathtt{o}_x(y) & := & 0 & \\
\mathtt{o}_x(\lambda y.t) & := & \mathtt{o}_x(t) & \\
\mathtt{o}_x(tu) & := & \mathtt{o}_x(t) + \mathtt{o}_x(u) & \\
\mathtt{o}_x(t[y/u]) & := & \mathtt{o}_x(t) + \mathtt{o}_x(u) & \\
\mathtt{o}_x(\mathcal{W}_y(t)) & := & \left\{ \begin{array}{ll} 1 & \text{if } x = y \\ \mathtt{o}_x(t) & \text{if } x \neq y \end{array} \right. \\
\mathtt{o}_x(\mathcal{C}_y^{y_1|y_2}(t)) & := & \left\{ \begin{array}{ll} 1 + \mathtt{o}_{y_1}(t) + \mathtt{o}_{y_2}(t) & \text{if } x = y \\ \mathtt{o}_x(t) & \text{if } x \neq y \end{array} \right.
\end{array}$$

We extend this definition to sets by $\mathtt{o}_\Gamma(t) := \sum_{x \in \Gamma} \mathtt{o}_x(t)$.

Before introducing the notion of substitution, we need an extra function which cleans-up useless resources. Indeed, given a $\mathcal{B}$-term t and a set of variables Γ, the **deletion** function $\mathtt{del}_\Gamma(t)$ removes from t all the occurrences of variables in Γ that are useless, *i.e.* that are free but not positive in t. This operation is defined modulo alpha-conversion so that bound variables of t are always assumed to be disjoint from Γ.

$$\begin{array}{lcll}
\mathtt{del}_\Gamma(y) & := & y & \\
\mathtt{del}_\Gamma(u\ v) & := & \mathtt{del}_\Gamma(u)\ \mathtt{del}_\Gamma(v) & \\
\mathtt{del}_\Gamma(\lambda y.u) & := & \lambda y.\mathtt{del}_\Gamma(u) & \\
\mathtt{del}_\Gamma(u[y/v]) & := & \mathtt{del}_\Gamma(u)[y/\mathtt{del}_\Gamma(v)] & \\
\mathtt{del}_\Gamma(\mathcal{W}_x(u)) & := & \left\{ \begin{array}{ll} u & \text{if } x \in \Gamma \\ \mathcal{W}_x(\mathtt{del}_\Gamma(u)) & \text{if } x \notin \Gamma \end{array} \right. \\
\mathtt{del}_\Gamma(\mathcal{C}_x^{y|z}(u)) & := & \left\{ \begin{array}{ll} \mathtt{del}_{\Gamma \setminus x \cup \{y,z\}}(u) & \text{if } x \in \Gamma \ \&\ x \notin \mathtt{fv}^+(\mathcal{C}_x^{y|z}(u)) \\ \mathcal{C}_x^{y|z}(\mathtt{del}_\Gamma(u)) & \text{otherwise} \end{array} \right.
\end{array}$$

For example, $\mathtt{del}_x(\mathcal{W}_x(a)\ x) = a\ x$ and $\mathtt{del}_x(\mathcal{C}_x^{x_1|x_2}(y)\ x) = y\ x$. This operation does not increase the size of terms. Moreover, if $x \in \mathtt{fv}(t) \setminus \mathtt{fv}^+(t)$, then $\mathtt{size}(\mathtt{del}_x(t)) < \mathtt{size}(t)$. Also, $\mathtt{del}_\Gamma(t) = t$ if $\mathtt{fv}(t) \cap \Gamma = \emptyset$.

Lemma 1 (Preservation of Well-Formed Terms by Deletion). *If* $\Gamma \Vdash_{\mathcal{B}} t$ *and* $\Delta \subseteq \Gamma$ *then* $(\Gamma \setminus\!\!\setminus_{\mathcal{B}} (\Delta \setminus \mathtt{fv}(\mathtt{del}_\Delta(t)))) \Vdash_{\mathcal{B}} \mathtt{del}_\Delta(t)$*, which simplifies to* $\Gamma \setminus\!\!\setminus_{\mathcal{B}} \Delta \Vdash_{\mathcal{B}} \mathtt{del}_\Delta(t)$ *if* $|t|^+_\Delta = 0$.

Proof. By induction on $\mathtt{size}(t)$.

For instance, cleaning-up useless x in the term $x\ \mathcal{W}_x(y)$ gives $\{x,y\} \setminus\!\!\setminus_{\mathtt{W}} (x \setminus \{x,y\}) \Vdash_{\mathtt{W}} \mathtt{del}_x(x\ \mathcal{W}_x(y))$ that is $x, y \Vdash_{\mathtt{W}} x\ y$.

To introduce the reduction rules of the prismoid we need a meta-level notion of substitution, defined on alpha-equivalence classes, which is at the same time the one implemented by the explicit control of resources. A **well-formed substitution** is a pair of the form $\{x/u\}$, where the term u, called the **body** of the substitution, is a well-formed term. More precisely, if $u \in \mathcal{T}_{\mathcal{B}}$, the substitution is also called a **$\mathcal{B}$-substitution.**

The **application of a $\mathcal{B}$-substitution** $\{x/u\}$ to a $\mathcal{B}$-term t (called the **target** of the substitution), written $t\{x/u\}$, is defined as follows:

- If $|t|^+_x = 0$, then
  - If $|t|_x = 0$ or $\mathtt{w} \notin \mathcal{B}$ then $t\{x/u\} := \mathtt{del}_x(t)$.
  - Otherwise, $t\{x/u\} := \mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{del}_x(t))$.
- If $|t|^+_x \geq 2$, then $t\{x/u\} := t_{[x:=y]}\{y/u\}\{x/u\}$.
- If $|t|^+_x = 1$, $t\{x/u\} := \mathtt{del}_x(t)\{\!\{x/u\}\!\}$ where $t\{\!\{x/u\}\!\}$ is defined by induction on t as follows:

$$
\begin{array}{lcll}
x\{\!\{x/u\}\!\} & := & u & \\
y\{\!\{x/u\}\!\} & := & y & x \neq y \\
(s\ v)\{\!\{x/u\}\!\} & := & s\{\!\{x/u\}\!\}\ v\{\!\{x/u\}\!\} & \\
(\lambda y.v)\{\!\{x/u\}\!\} & := & \lambda y.v\{\!\{x/u\}\!\} & x \neq y \ \&\ y \notin \mathtt{fv}(u) \\
s[y/v]\{\!\{x/u\}\!\} & := & s\{\!\{x/u\}\!\}[y/v\{\!\{x/u\}\!\}] & x \neq y \ \&\ y \notin \mathtt{fv}(u) \\
\mathcal{W}_y(v)\{\!\{x/u\}\!\} & := & \mathcal{W}_{y\setminus\mathtt{fv}(u)}(v\{\!\{x/u\}\!\}) & x \neq y \\
\mathcal{C}^{y_1|y_2}_y(v)\{\!\{x/u\}\!\} & := & \mathcal{C}^{y_1|y_2}_y(v\{\!\{x/u\}\!\}) & \begin{cases} x \neq y \\ y_1, y_2, y \notin \mathtt{fv}(u) \end{cases} \\
\mathcal{C}^{x_1|x_2}_x(v)\{\!\{x/u\}\!\} & := & \mathcal{C}^{\Delta|\Pi}_{\Gamma}(v\{x_1/R^{\Gamma}_{\Delta}(u)\}\{x_2/R^{\Gamma}_{\Pi}(u)\}) & \begin{cases} \Gamma := \mathtt{fv}(u) \\ \Delta, \Pi \text{ are fresh} \end{cases}
\end{array}
$$

For instance, $(\mathcal{W}_x(a)\ \mathcal{W}_x(b))\{x/y\} = \mathcal{W}_y(a\ b)$ and $(\mathcal{C}^{x_1|x_2}_x(a)\ x)\{x/b\} = a\ b$.

This definition looks complex, this is because it is covering all the calculi of the prismoid by a unique homogeneous specification. The restriction of this operation to particular subsets of resources results in simplified notions of substitutions. As a typical example, the previous definition can be shown to be equivalent to the well-known notion of higher-order substitution on $\emptyset$-terms [Bar84] given by:

$$
\begin{array}{lcll}
x\{x/u\} & := & u & \\
y\{x/u\} & := & y & x \neq y \\
(\lambda y.v)\{x/u\} & := & \lambda y.v\{x/u\} & x \neq y \ \&\ y \notin \mathtt{fv}(u) \\
(s\ v)\{x/u\} & := & s\{x/u\}\ v\{x/u\} &
\end{array}
$$

Substitution definition also simplifies to the following one for $\mathtt{c}$-terms:

$$
\begin{array}{llll}
x\{x/u\} & := & u & \\
y\{x/u\} & := & y & x \neq y \\
(\lambda y.v)\{x/u\} & := & \lambda y.v\{x/u\} & x \neq y \ \&\ y \notin \mathtt{fv}(u) \\
(s\ v)\{x/u\} & := & s\{x/u\}\ v\{x/u\} & \\
\mathcal{C}_y^{y_1|y_2}(t)\{x/u\} & := & \mathcal{C}_y^{y_1|y_2}(t\{x/u\}) & \begin{cases} x \neq y \\ y, y_1, y_2 \notin fv(u) \end{cases} \\
\mathcal{C}_x^{y_1|y_2}(t)\{x/u\} & := & \mathcal{C}_\Gamma^{\Delta|\Pi}(t\{y_1/R_\Delta^\Gamma(u)\}\{y_2/R_\Pi^\Gamma(u)\}) & \begin{cases} x \in \mathtt{fv}^+(\mathcal{C}_x^{y_1|y_2}(t)) \\ \Gamma := \mathtt{fv}(u) \\ \Delta, \Pi \text{ are fresh} \end{cases} \\
\mathcal{C}_x^{y_1|y_2}(t)\{x/u\} & := & \mathtt{del}_x(\mathcal{C}_x^{y_1|y_2}(t)) & x \notin \mathtt{fv}^+\mathcal{C}_x^{y_1|y_2}(t)
\end{array}
$$

Lemma 2. *Definitions of $t\{x/u\}$ and $t\{\!\{x/u\}\!\}$ are well-founded.*

Proof. By induction on $\langle \mathtt{o}_x(t), \mathtt{size}(t)\rangle$.

Lemma 3. *Let $t \in \mathcal{T}_{\mathcal{B}}$ s.t. $|t|_x^+ \geq 1$. Then substitution verifies the following equalities:*

$$
\begin{array}{llll}
x\{x/u\} & = & u & \\
y\{x/u\} & = & y & x \neq y \\
(\lambda y.v)\{x/u\} & = & \lambda y.v\{x/u\} & x \neq y \\
(s\ v)\{x/u\} & = & s\{x/u\}\ v\{x/u\} & \\
s[y/v]\{x/u\} & = & s\{x/u\}[y/v\{x/u\}] & x \neq y \\
\mathcal{W}_y(t)\{x/u\} & = & \mathcal{W}_y(t\{x/u\}) & x \neq y \ \&\ y \notin \mathtt{fv}(u) \\
\mathcal{W}_y(t)\{x/u\} & = & t\{x/u\} & x \neq y \ \&\ y \in \mathtt{fv}(u) \\
\mathcal{C}_y^{y_1|y_2}(t)\{x/u\} & = & \mathcal{C}_y^{y_1|y_2}(t\{x/u\}) & x \neq y \ \&\ y \notin \mathtt{fv}(u) \\
\mathcal{C}_x^{x_1|x_2}(t)\{x/u\} & = & \mathcal{C}_\Gamma^{\Delta|\Pi}(t\{x_1/R_\Delta^\Gamma(u)\}\{x_2/R_\Pi^\Gamma(u)\}) & \begin{cases} \Gamma = \mathtt{fv}(u) \\ \Delta, \Pi \textit{ are fresh} \end{cases}
\end{array}
$$

Proof. By substitution definition.

Lemma 4. *Let $t \in \mathcal{T}_{\mathcal{B}}$. The function $\mathtt{del}()$ enjoys the following properties :*

1. $x \notin \mathtt{fv}(\mathtt{del}_x(t))$ *if* $x \notin \mathtt{fv}^+(t)$.
2. $\mathtt{del}_x(\mathtt{del}_y(t)) = \mathtt{del}_y(\mathtt{del}_x(t))$.
3. $\mathtt{del}_x(t\{y/v\}) = \mathtt{del}_x(t)\{y/v\}$ *if* $x \notin \mathtt{fv}(v)$.
4. $\mathtt{del}_x(t\{\!\{y/v\}\!\}) = \mathtt{del}_x(t)\{\!\{y/v\}\!\}$ *if* $x \notin \mathtt{fv}(v)$.
5. $\mathtt{del}_x(t)\{\!\{x/v\}\!\} = \mathtt{del}_x(t)$ *if* $x \notin \mathtt{fv}^+(t)$.
6. $\mathtt{del}_x(t) = t$ *if* $|t|_x = |t|_x^+$.
7. $t\{x/u\}\{y/u\} = \mathtt{del}_{x,y}(t)\{\!\{x/u\}\!\}\{\!\{y/u\}\!\}$ *if* $|t|_x^+ \geq 1$ *or* $|t|_y^+ \geq 1$.

Proof. By induction on $\mathtt{size}(t)$.

For instance, $\mathtt{del}_x(\mathcal{W}_y(\mathcal{W}_x(z))\{y/w\}) = \mathtt{del}_x(\mathcal{W}_w(\mathcal{W}_x(z))) = \mathcal{W}_w(z) = \mathcal{W}_y(z)\{y/w\} = \mathtt{del}_x(\mathcal{W}_y(\mathcal{W}_x(z)))\{y/w\}$ illustrates the third case.

2.3. Rewriting rules and equations

We now introduce the **reduction system** of the prismoid. In the last column of Figure 2 we use the notation $\mathcal{A}^+$ (resp. $\mathcal{A}^-$) to specify that the equation/rule belongs to the calculus $\lambda_{\mathcal{B}}$ iff $\mathcal{A} \subseteq \mathcal{B}$ (resp. $\mathcal{A} \cap \mathcal{B} = \emptyset$). Thus, each calculus $\lambda_{\mathcal{B}}$ contains only a strict subset of the reduction rules and equations in Figure 2.

All the equations and rules can be understood by means of MELL Proof-Nets reduction (see for example [KL07]). The reduction rules can be split into four groups: the first one fires implicit/explicit substitution, the second one implements substitution by decrementing multiplicity of variables and/or performing propagation, the third one pulls weakening operators as close to the top as possible and the fourth one pushes contractions as deep as possible. Alpha-conversion guarantees that no capture of variables occurs during reduction. The use of positive conditions (conditions involving positive free variables) in some of the rules will become clear when discussing projection at the end of Section 4.

The notations $\Rightarrow_{\mathcal{R}}$, $\equiv_{\mathcal{E}}$ and $\rightarrow_{\mathcal{R}\cup\mathcal{E}}$, mean, respectively, the rewriting (resp. equivalence and rewriting modulo) relation generated by the rules $\mathcal{R}$ (resp. equations $\mathcal{E}$ and rules $\mathcal{R}$ modulo equations $\mathcal{E}$). Similarly, $\Rightarrow_{\mathcal{B}}$, $\equiv_{\mathcal{B}}$ and $\rightarrow_{\mathcal{B}}$ mean, respectively, the rewriting (resp. equivalence and rewriting modulo) relation generated by the rules (resp. the equations and rules modulo equations) of the calculus $\lambda_{\mathcal{B}}$. Thus for example the reduction relation $\rightarrow_{\emptyset}$ is only generated by the β-rule exactly as in λ-calculus. Another example is $\rightarrow_{\mathtt{C}}$ which can be written $\rightarrow_{\{\beta,\mathtt{CL},\mathtt{CA_L},\mathtt{CA_R},\mathtt{CGc}\}\cup\{\mathtt{CC}_{\mathcal{A}},\mathtt{C}_{\mathcal{C}},\mathtt{CC}_{\mathcal{C}}\}}$. Sometimes we mix both notations to denote particular subrelations, thus for example $\rightarrow_{\mathtt{C}\setminus\beta}$ means $\rightarrow_{\{\mathtt{CL},\mathtt{CA_L},\mathtt{CA_R},\mathtt{CGc}\}\cup\{\mathtt{CC}_{\mathcal{A}},\mathtt{C}_{\mathcal{C}},\mathtt{CC}_{\mathcal{C}}\}}$. We give in the appendix an independent specification for each calculus of the prismoid.

Among the eight calculi of the prismoid we can distinguish the $\lambda_{\emptyset}$-calculus, known as λ-calculus, which is defined by means of the $\rightarrow_{\emptyset}$-reduction relation on $\emptyset$-terms. Another language of the prismoid is the $\lambda_{\mathtt{CSW}}$-calculus, a variation of $\lambda\mathtt{lxr}$ [KL07], defined by means of the $\rightarrow_{\{\mathtt{C},\mathtt{S},\mathtt{W}\}}$-reduction relation on $\{\mathtt{c},\mathtt{s},\mathtt{w}\}$-terms. A last example is the $\lambda_{\mathtt{W}}$-calculus given by means of $\rightarrow_{\mathtt{W}}$-reduction, that is, $\rightarrow_{\{\beta,\mathtt{LW},\mathtt{AW_l},\mathtt{AW_r}\}\cup\{\mathtt{WW}_{\mathcal{C}}\}}$.

A $\mathcal{B}$-term t is in $\mathcal{B}$**-normal form** is there is no u s.t. $t \rightarrow_{\mathcal{B}} u$. A $\mathcal{B}$-term t is said to be $\mathcal{B}$**-strongly normalising**, written $t \in \mathcal{SN}_{\mathcal{B}}$, iff there is no infinite $\mathcal{B}$-reduction sequence starting at t.

In order to show that well-formed terms are stable by reduction we first need the following property.

Lemma 5 (Preservation of Well-Formed Terms by Substitution). *Let* $\Gamma \Vdash_{\mathcal{B}} t$ *and* $\Delta \Vdash_{\mathcal{B}} u$ *and* $x \notin \Delta$. *If* ($x \in \mathtt{fv}^+(t)$ *or* $\mathtt{w} \in \mathcal{B}$) *and* $(\Gamma \setminus\!\!\setminus_{\mathcal{B}} x) \uplus_{\mathcal{B}} \Delta$ *is defined, then* $(\Gamma \setminus\!\!\setminus_{\mathcal{B}} x) \uplus_{\mathcal{B}} \Delta \Vdash_{\mathcal{B}} t\{x/u\}$. *Otherwise,* $\Gamma \setminus\!\!\setminus_{\mathcal{B}} x \Vdash_{\mathcal{B}} t\{x/u\}$.

Proof. By induction on $\langle \mathtt{o}_x(t), \mathtt{size}(t)\rangle$.

- If $|t|^+_x = 0$ and ($|t|_x = 0$ or $\mathtt{w} \notin \mathcal{B}$) then we are done by Lemma 1.

Equations :					
$(\mathtt{CC}_{\mathcal{A}})$	$\mathcal{C}^{x\|z}_{w}(\mathcal{C}^{y\|p}_{x}(t))$	$\equiv$	$\mathcal{C}^{x\|y}_{w}(\mathcal{C}^{z\|p}_{x}(t))$		$\mathtt{c}^{+}$
$(\mathtt{C}_{\mathcal{C}})$	$\mathcal{C}^{y\|z}_{x}(t)$	$\equiv$	$\mathcal{C}^{z\|y}_{x}(t)$		$\mathtt{c}^{+}$
$(\mathtt{CC}_{\mathcal{C}})$	$\mathcal{C}^{b\|c}_{a}(\mathcal{C}^{y\|z}_{x}(t))$	$\equiv$	$\mathcal{C}^{y\|z}_{x}(\mathcal{C}^{b\|c}_{a}(t))$	$x \neq b, c$ & $a \neq y, z$	$\mathtt{c}^{+}$
$(\mathtt{WW}_{\mathcal{C}})$	$\mathcal{W}_{x}(\mathcal{W}_{y}(t))$	$\equiv$	$\mathcal{W}_{y}(\mathcal{W}_{x}(t))$		$\mathtt{w}^{+}$
$(\mathtt{SS}_{\mathcal{C}})$	$t[x/u][y/v]$	$\equiv$	$t[y/v][x/u]$	$y \notin \mathtt{fv}(u)$ & $x \notin \mathtt{fv}(v)$	$\mathtt{s}^{+}$
Rules :					
(β)	$(\lambda x.t)\ u$	$\to$	$t\{x/u\}$		$\mathtt{s}^{-}$
$(\mathtt{B})$	$(\lambda x.t)\ u$	$\to$	$t[x/u]$		$\mathtt{s}^{+}$
$(\mathtt{V})$	$x[x/u]$	$\to$	u		$\mathtt{s}^{+}$
$(\mathtt{SGc})$	$t[x/u]$	$\to$	t	$x \notin \mathtt{fv}(t)$	$\mathtt{s}^{+}$ & $\mathtt{w}^{-}$
$(\mathtt{SDup})$	$t[x/u]$	$\to$	$t_{[x:=y]}[x/u][y/u]$	$\|t\|^{+}_{x} > 1$ & y fresh	$\mathtt{s}^{+}$ & $\mathtt{c}^{-}$
$(\mathtt{SL})$	$(\lambda y.t)[x/u]$	$\to$	$\lambda y.t[x/u]$		$\mathtt{s}^{+}$
$(\mathtt{SA}_{\mathtt{L}})$	$(t\ v)[x/u]$	$\to$	$t[x/u]\ v$	$x \notin \mathtt{fv}(v)$	$\mathtt{s}^{+}$
$(\mathtt{SA}_{\mathtt{R}})$	$(t\ v)[x/u]$	$\to$	$t\ v[x/u]$	$x \notin \mathtt{fv}(t)$	$\mathtt{s}^{+}$
$(\mathtt{SS})$	$t[x/u][y/v]$	$\to$	$t[x/u[y/v]]$	$y \in \mathtt{fv}^{+}(u) \setminus \mathtt{fv}(t)$	$\mathtt{s}^{+}$
$(\mathtt{SW}_{1})$	$\mathcal{W}_{x}(t)[x/u]$	$\to$	$\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(t)$		$(\mathtt{sw})^{+}$
$(\mathtt{SW}_{2})$	$\mathcal{W}_{y}(t)[x/u]$	$\to$	$\mathcal{W}_{y\setminus\mathtt{fv}(u)}(t[x/u])$	$x \neq y$	$(\mathtt{sw})^{+}$
$(\mathtt{LW})$	$\lambda x.\mathcal{W}_{y}(t)$	$\to$	$\mathcal{W}_{y}(\lambda x.t)$	$x \neq y$	$\mathtt{w}^{+}$
$(\mathtt{AW}_{\mathtt{l}})$	$\mathcal{W}_{y}(u)\ v$	$\to$	$\mathcal{W}_{y\setminus\mathtt{fv}(v)}(u\ v)$		$\mathtt{w}^{+}$
$(\mathtt{AW}_{\mathtt{r}})$	$u\ \mathcal{W}_{y}(v)$	$\to$	$\mathcal{W}_{y\setminus\mathtt{fv}(u)}(u\ v)$		$\mathtt{w}^{+}$
$(\mathtt{SW})$	$t[x/\mathcal{W}_{y}(u)]$	$\to$	$\mathcal{W}_{y\setminus\mathtt{fv}(t)}(t[x/u])$		$(\mathtt{sw})^{+}$
$(\mathtt{SCa})$	$\mathcal{C}^{y\|z}_{x}(t)[x/u]$	$\to$	$\mathcal{C}^{\Delta\|\Pi}_{\Gamma}(t[y/R^{\Gamma}_{\Delta}(u)][z/R^{\Gamma}_{\Pi}(u)])$	$\begin{cases} y, z \in \mathtt{fv}^{+}(t) \\ \Gamma := \mathtt{fv}(u) \\ \Delta, \Pi \text{ are fresh} \end{cases}$	$(\mathtt{cs})^{+}$
$(\mathtt{CL})$	$\mathcal{C}^{y\|z}_{w}(\lambda x.t)$	$\to$	$\lambda x.\mathcal{C}^{y\|z}_{w}(t)$		$\mathtt{c}^{+}$
$(\mathtt{CA}_{\mathtt{L}})$	$\mathcal{C}^{y\|z}_{w}(t\ u)$	$\to$	$\mathcal{C}^{y\|z}_{w}(t)\ u$	$y, z \notin \mathtt{fv}(u)$	$\mathtt{c}^{+}$
$(\mathtt{CA}_{\mathtt{R}})$	$\mathcal{C}^{y\|z}_{w}(t\ u)$	$\to$	$t\ \mathcal{C}^{y\|z}_{w}(u)$	$y, z \notin \mathtt{fv}(t)$	$\mathtt{c}^{+}$
$(\mathtt{CS})$	$\mathcal{C}^{y\|z}_{w}(t[x/u])$	$\to$	$t[x/\mathcal{C}^{y\|z}_{w}(u)]$	$y, z \in \mathtt{fv}^{+}(u)$	$(\mathtt{cs})^{+}$
$(\mathtt{SCb})$	$\mathcal{C}^{y\|z}_{w}(t)[x/u]$	$\to$	$\mathcal{C}^{y\|z}_{w}(t[x/u])$	$x \neq w$ & $y, z \notin \mathtt{fv}(u)$	$(\mathtt{cs})^{+}$
$(\mathtt{CW}_{1})$	$\mathcal{C}^{y\|z}_{w}(\mathcal{W}_{y}(t))$	$\to$	$R^{z}_{w}(t)$		$(\mathtt{cw})^{+}$
$(\mathtt{CW}_{2})$	$\mathcal{C}^{y\|z}_{w}(\mathcal{W}_{x}(t))$	$\to$	$\mathcal{W}_{x}(\mathcal{C}^{y\|z}_{w}(t))$	$x \neq y, z$	$(\mathtt{cw})^{+}$
$(\mathtt{CGc})$	$\mathcal{C}^{y\|z}_{w}(t)$	$\to$	$R^{z}_{w}(t)$	$y \notin \mathtt{fv}(t)$	$\mathtt{c}^{+}$ & $\mathtt{w}^{-}$

Figure 2: The reduction rules and equations of the prismoid

- If $|t|^{+}_{x} = 0$ and $|t|_{x} \neq 0$ and $\mathtt{w} \in \mathcal{B}$ then $t\{x/u\} = \mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{del}_{x}(t))$. By hypothesis $\Gamma \Vdash_{\mathcal{B}} t$ and by Lemma 1, $\Gamma \setminus\!\!\setminus_{\mathcal{B}} x \Vdash_{\mathcal{B}} \mathtt{del}_{x}(t)$. By definition, $\Gamma \setminus\!\!\setminus_{\mathcal{B}} x; (\Delta \setminus \Gamma) \Vdash_{\mathcal{B}} \mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{del}_{x}(t))$. If $\mathtt{c} \in \mathcal{B}$, then $\Gamma \cap \Delta = \emptyset$ so that the left part of the last statement is exactly $\Gamma \setminus\!\!\setminus_{\mathcal{B}} x \uplus_{\mathcal{B}} \Delta$ and thus we are done. Otherwise $\mathtt{c} \notin \mathcal{B}$, then we trivially conclude since $\Gamma \setminus\!\!\setminus_{\mathcal{B}} x; (\Delta \setminus \Gamma) = \Gamma \setminus\!\!\setminus_{\mathcal{B}} x \uplus_{\mathcal{B}} \Delta$.

- If $|t|_x^+ = n+1$ with $n \geq 1$ then we have $\uplus_{\mathcal{B}} = \cup$ and :

$$
\dfrac{\dfrac{\dfrac{\vdots}{\dfrac{\dfrac{\dfrac{\dfrac{[hyp]}{\Gamma \Vdash_{\mathcal{B}} t}}{\Gamma, x_1, \ldots, x_n \Vdash_{\mathcal{B}} t_{[x:=x_1\ldots x_n]}}\ x_1 \ldots x_n \text{ fresh}}{\Gamma, x_2, \ldots, x_n \cup \Delta \Vdash_{\mathcal{B}} t_{[x:=x_1\ldots x_n]}\{x_1/u\}}\ i.h.}{}\ i.h.}}{\Gamma \cup \Delta \cup \ldots \cup \Delta \Vdash_{\mathcal{B}} t_{[x:=x_1\ldots x_n]}\{x_1/u\}\ldots\{x_n/u\}}\ i.h.}{(\Gamma \cup \Delta \cup \ldots \cup \Delta) \setminus\!\!\setminus_{\mathcal{B}} x \cup \Delta \Vdash_{\mathcal{B}} t_{[x:=x_1\ldots x_n]}\{x_1/u\}\ldots\{x_n/u\}\{x/u\}}\ i.h.
$$

We conclude since the last set of variables is equal to $(\Gamma \setminus\!\!\setminus_{\mathcal{B}} x) \cup \Delta$ with $\Gamma \setminus\!\!\setminus_{\mathcal{B}} x$ well defined since $|t|_x^+ = n+1$. We can use the i.h. in the first three cases since $\mathtt{o}_{x_i}(t_{[x:=x_1\ldots x_n]}\{x_1/u\}\ldots\{x_{i-1}/u\}) < \mathtt{o}_x(t)$ and in the last case because $\mathtt{o}_x(t_{[x:=x_1\ldots x_n]}\{x_1/u\}\ldots\{x_n/u\}) < \mathtt{o}_x(t)$.

- Now we analyse all interesting cases where $|t|_x^+ = 1$:
  - $t = x$, then $\Gamma = x$ and $t\{x/u\} = u$ so that $\Delta \Vdash_{\mathcal{B}} t\{x/u\}$ by hypothesis.
  - $t = \lambda y.t'$, so that $y \neq x$ by α-conversion. We have $\Gamma = \Gamma' \setminus\!\!\setminus_{\mathcal{B}}\ y$ (so that $\Gamma' \Vdash_{\mathcal{B}} t'$), thus $(\lambda y.t')\{x/u\} = \lambda y.\mathtt{del}_x(t')\{\!\{x/u\}\!\} = \lambda y.t'\{x/u\}$ and

$$
\dfrac{\dfrac{\dfrac{[hyp]}{\Gamma' \Vdash_{\mathcal{B}} t'}}{(\Gamma' \setminus\!\!\setminus_{\mathcal{B}} x) \uplus_{\mathcal{B}} \Delta \Vdash_{\mathcal{B}} t'\{x/u\}}\ i.h.\ (\mathtt{o}_x(t') = \mathtt{o}_x(t)\ \&\ \mathtt{size}(t') < \mathtt{size}(t))}{(\Gamma' \setminus\!\!\setminus_{\mathcal{B}} x \uplus_{\mathcal{B}} \Delta) \setminus\!\!\setminus_{\mathcal{B}}\ y \Vdash_{\mathcal{B}} \lambda y.t'\{x/u\}}
$$

    We conclude since $(\Gamma' \setminus\!\!\setminus_{\mathcal{B}} x \uplus_{\mathcal{B}} \Delta) \setminus\!\!\setminus_{\mathcal{B}}\ y = \Gamma \setminus\!\!\setminus_{\mathcal{B}} x \uplus_{\mathcal{B}} \Delta$ as desired.
  - $t = v\ w$. We have $\Gamma = \Gamma_v \uplus_{\mathcal{B}} \Gamma_w$, $\Gamma_v \Vdash_{\mathcal{B}} v$ and $\Gamma_w \Vdash_{\mathcal{B}} w$. Suppose $|v|_x^+ = 1$ (the case where $|w|_x^+ = 1$ is symmetric). Thus $(v\ w)\{x/u\} = \mathtt{del}_x(v)\{\!\{x/u\}\!\}\ \mathtt{del}_x(w)\{\!\{x/u\}\!\} = v\{x/u\}\ w$ and :

$$
\dfrac{\dfrac{\dfrac{[hyp]}{\Gamma_v \Vdash_{\mathcal{B}} v}}{\Gamma_v \setminus\!\!\setminus_{\mathcal{B}} x \uplus \Delta \Vdash_{\mathcal{B}} v\{x/u\}}\ i.h. \qquad \dfrac{[hyp]}{\Gamma_w \Vdash_{\mathcal{B}} w}}{(\Gamma_v \setminus\!\!\setminus_{\mathcal{B}} x \uplus_{\mathcal{B}} \Gamma_w) \uplus_{\mathcal{B}} \Delta \Vdash_{\mathcal{B}} v\{x/u\}\ w}
$$

We can conclude since $\Gamma_v \setminus x \uplus_{\mathcal{B}} \Gamma_w = \Gamma \setminus x$

- $t = \mathcal{C}_x^{y|z}(t')$. By hypothesis we have $x; \Gamma' \setminus\!\!\setminus_{\mathcal{B}} \{y,z\} \Vdash_{\mathcal{B}} \mathcal{C}_x^{y|z}(t')$ (so that $\Gamma' \Vdash_{\mathcal{B}} t'$) with $\Gamma = x; \Gamma' \setminus\!\!\setminus_{\mathcal{B}} \{y,z\}$. Definition of substitution gives $t\{x/u\} = \mathtt{del}_x(\mathcal{C}_x^{y|z}(t'))\{\!\{x/u\}\!\} = \mathcal{C}_x^{y|z}(t')\{\!\{x/u\}\!\} = \mathcal{C}_\Delta^{\Delta'|\Delta''}(t'\{y/u'\}\{z/u''\})$, where $\Delta = \mathtt{fv}(u)$,
  If $\mathtt{o}_y(t') > 0$ and $\mathtt{o}_z(t') > 0$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[hyp]\\ \vdots \\ \Gamma' \Vdash_{\mathcal{B}} t'\end{array}}{\Gamma' \setminus\!\!\setminus_{\mathcal{B}} y \uplus_{\mathcal{B}} \Delta' \Vdash_{\mathcal{B}} t'\{y/u'\}}\ i.h.}{\Gamma' \setminus\!\!\setminus_{\mathcal{B}} \{y,z\} \uplus_{\mathcal{B}} \Delta' \uplus_{\mathcal{B}} \Delta'' \Vdash_{\mathcal{B}} t'\{y/u'\}\{z/u''\}}\ i.h.}{\Gamma' \setminus\!\!\setminus_{\mathcal{B}} \{y,z\} \uplus_{\mathcal{B}} \Delta \Vdash_{\mathcal{B}} \mathcal{C}_\Delta^{\Delta'|\Delta''}(t'\{y/u'\}\{z/u''\})}$$

  The first (resp. second) application of the i.h. is valid since $\mathtt{o}_y(t') < \mathtt{o}_x(t)$ (resp. $\mathtt{o}_z(t'\{y/u'\}) < \mathtt{o}_x(t)$). We can conclude since $\Gamma \setminus\!\!\setminus_{\mathcal{B}} x = \Gamma' \setminus\!\!\setminus_{\mathcal{B}} \{y,z\}$.
  Finally, suppose $\mathtt{o}_y(t') = 0$ and $\mathtt{w} \notin \mathcal{B}$ (otherwise, the proof is similar to another detailed case). Then,

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[hyp]\\ \vdots \\ \Gamma' \Vdash_{\mathcal{B}} t'\end{array}}{\Gamma' \setminus\!\!\setminus_{\mathcal{B}} y \Vdash_{\mathcal{B}} t'\{y/u'\}}\ i.h.}{\Gamma' \setminus\!\!\setminus_{\mathcal{B}} \{y,z\} \uplus_{\mathcal{B}} \Delta'' \Vdash_{\mathcal{B}} t'\{y/u'\}\{z/u''\}}\ i.h.}{(\Gamma' \setminus\!\!\setminus_{\mathcal{B}} \{y,z\} \uplus_{\mathcal{B}} \Delta') \setminus\!\!\setminus_{\mathcal{B}} \Delta' \setminus\!\!\setminus_{\mathcal{B}} \Delta''; \Delta \Vdash_{\mathcal{B}} \mathcal{C}_\Delta^{\Delta'|\Delta''}(t'\{y/u'\}\{z/u''\})}$$

  We can conclude since $(\Gamma' \setminus\!\!\setminus_{\mathcal{B}} \{y,z\} \uplus_{\mathcal{B}} \Delta') \setminus\!\!\setminus_{\mathcal{B}} \Delta' \setminus\!\!\setminus_{\mathcal{B}} \Delta''; \Delta$ is exactly $\Gamma' \setminus\!\!\setminus_{\mathcal{B}} \{y,z\} \uplus_{\mathcal{B}} \Delta$ ($\setminus\!\!\setminus_{\mathcal{B}} = \setminus$ since $\mathtt{w} \notin \mathcal{B}$ and $; = \uplus_{\mathcal{B}}$ since $\mathtt{c} \in \mathcal{B}$.
- The case $t = w[y/v]$ is similar to lambda and application together.

For instance, suppose $x \Vdash_{\mathtt{C}} \mathcal{C}_x^{x_1|x_2}(x_1\ x_2)$ and $y \Vdash_{\mathtt{C}} y$. In this case, we have $\setminus\!\!\setminus_{\mathtt{C}} = \setminus$ and $\uplus_{\mathcal{B}}$ is the disjoint union. $(x \setminus\!\!\setminus x) \uplus_{\mathtt{C}} y = y$ is defined and $\mathcal{C}_x^{x_1|x_2}(x_1\ x_2)\{x/y\} = \mathcal{C}_y^{y_1|y_2}(y_1\ y_2)$ so that $y \Vdash_{\mathtt{C}} \mathcal{C}_y^{y_1|y_2}(y_1\ y_2)$.

As expected, substitution enjoys the following property.

Lemma 6 (Substitution Permutation). *Let* $t, u, v \in \mathcal{T}_{\mathcal{B}}$ *s.t.* $x \notin \mathtt{fv}(v)$ *and* $y \notin \mathtt{fv}(u)$. *Then:*

1. $t\{x/u\}\{y/v\} \equiv_{\mathcal{B}} t\{y/v\}\{x/u\}$
2. $t\{\!\{x/u\}\!\}\{\!\{y/v\}\!\} \equiv_{\mathcal{B}} t\{\!\{y/v\}\!\}\{\!\{x/u\}\!\}$

Proof. We prove both statements simultaneously by induction on the tuple $\langle \mathtt{o}_{\{x,y\}}(t), \mathtt{size}(t) \rangle$.

1. • First, we treat cases where $|\mathtt{fv}^+(t)|_x \geq 2$ or $|\mathtt{fv}^+(t)|_y \geq 2$. Let us suppose $|\mathtt{fv}^+(t)|_x \geq 2$ and $|\mathtt{fv}^+(t)|_y \geq 2$, the other cases being similar. Then $|\mathtt{fv}^+(t)|_x = n+1$ and $|\mathtt{fv}^+(t)|_y = m+1$ so that:

$$
\begin{array}{ll}
 & t\{x/u\}\{y/v\} \\
= & t_{[x:=x_1...x_n]}\overline{\{x_n/u\}}\{x/u\}_{[y:=y_1...y_m]}\{y_1/v\}...\{y_n/v\}\{y/v\} \\
 & \qquad \text{where } \overline{\{x_n/u\}} = \{x_1/u\}...\{x_n/u\} \\
= & t_{[x:=x_1...x_n]_{[y:=y_1...y_m]}}\overline{\{x_n/u\}}\{x/u\}\{y_1/v\}...\{y_n/v\}\{y/v\} \\
\equiv_{\mathcal{B}} (i.h.) & t_{[x:=x_1...x_n]_{[y:=y_1...y_m]}}\{y_1/v\}...\{y_n/v\}\{y/v\}\overline{\{x_n/u\}}\{x/u\} \\
= & t\{y/v\}\{x/u\}
\end{array}
$$

• If $|t|^+_x = 0$ and ($|\mathtt{fv}(t)|_x = 0$ or $\mathtt{w} \notin \mathcal{B}$) then

$$t\{x/u\}\{y/v\} = \mathtt{del}_x(t)\{y/v\} =_{L.\ 4:3} \mathtt{del}_x(t\{y/v\}) = t\{y/v\}\{x/u\}$$

• If $|t|^+_x = 0$ and $|\mathtt{fv}(t)|_x \neq 0$ and $\mathtt{w} \in \mathcal{B}$ then

$$t\{x/u\}\{y/v\} = \mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{del}_x(t))\{y/v\}$$

There are two interesting cases :

– $|t|^+_y = 0$ and $|\mathtt{fv}(t)|_y > 0$

$$
\begin{array}{ll}
 & t\{x/u\}\{y/v\} = \\
= & \mathcal{W}_{\mathtt{fv}(v)\setminus\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{del}_y(\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{del}_x(t)))) \\
= & \mathcal{W}_{\mathtt{fv}(v)\setminus\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{del}_y(\mathtt{del}_x(t)))) \\
=_{L.\ 4:2} & \mathcal{W}_{\mathtt{fv}(v)\setminus\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{del}_x(\mathtt{del}_y(t)))) \\
= & \mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(v)\setminus\mathtt{fv}(t)}(\mathtt{del}_x(\mathcal{W}_{\mathtt{fv}(v)\setminus\mathtt{fv}(t)}(\mathtt{del}_y(t)))) \\
= & \mathcal{W}_{\mathtt{fv}(v)\setminus\mathtt{fv}(t)}(\mathtt{del}_y(t))\{x/u\} \\
= & t\{y/v\}\{x/u\}
\end{array}
$$

– $|t|^+_y = 1$

$$
\begin{array}{ll}
 & t\{x/u\}\{y/v\} \\
= & \mathtt{del}_y(\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{del}_x(t)))\{\!\{y/v\}\!\} \\
= & \mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)\setminus\mathtt{fv}(v)}(\mathtt{del}_y(\mathtt{del}_x(t))\{\!\{y/v\}\!\}) \\
=_{L.\ 4:2} & \mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(v)\setminus\mathtt{fv}(t)}(\mathtt{del}_x(\mathtt{del}_y(t)\{\!\{y/v\}\!\})) \\
= & \mathtt{del}_y(t)\{\!\{y/v\}\!\}\{x/u\} \\
= & t\{y/v\}\{x/u\}
\end{array}
$$

• We now consider the case where $|t|^+_x = |t|^+_y = 1$. We proceed by case analysis on t.

– The case $t = z$ is impossible by hypothesis.

– $t = \lambda w.t'$.

$$
\begin{array}{ll}
 & (\lambda w.t')\{x/u\}\{y/v\} \\
= & \mathtt{del}_x(\lambda w.t')\{\!\{x/u\}\!\}\{y/v\} \\
= & (\lambda w.\mathtt{del}_x(t')\{\!\{x/u\}\!\})\{y/v\} \\
=_{L.\ 4:4} & \lambda w.\mathtt{del}_y(\mathtt{del}_x(t'))\{\!\{x/u\}\!\}\{\!\{y/v\}\!\} \\
\equiv_{\mathcal{B}}\ (i.h.) & \lambda w.\mathtt{del}_y(\mathtt{del}_x(t'))\{\!\{y/v\}\!\}\{\!\{x/u\}\!\} \\
=_{L.\ 4:2} & \lambda w.\mathtt{del}_x(\mathtt{del}_y(t'))\{\!\{y/v\}\!\}\{\!\{x/u\}\!\} \\
=_{L.\ 4:4} & \lambda w.\mathtt{del}_x(\mathtt{del}_y(t')\{\!\{y/v\}\!\})\{\!\{x/u\}\!\} \\
= & (\lambda w.t')\{y/v\}\{x/u\}
\end{array}
$$

- $t = w\ w'$.

$$
\begin{array}{ll}
 & t\{x/u\}\{y/v\} \\
= & w\{x/u\}\{y/v\}\ w'\{x/u\}\{y/v\} \\
\equiv_{\mathcal{B}}\ (i.h.) & w\{y/v\}\{x/u\}\ w'\{y/v\}\{x/u\} \\
= & t\{y/v\}\{x/u\}
\end{array}
$$

- The case $t = s[z/w]$ is similar to the application case.
- The case $t = \mathcal{W}_x(t')$ is impossible by hypothesis.
- The case $t = \mathcal{W}_z(t')$ with $z \neq x, y$ is straightforward by induction.
- $t = \mathcal{C}_a^{b|c}(t')$. We only consider the case where $a = x$

$$
\begin{array}{ll}
 & t\{x/u\}\{y/v\} \\
= & \mathcal{C}_\Gamma^{\Delta|\Pi}(t'\{b/R_\Delta^\Gamma(u)\}\{c/R_\Pi^\Gamma(u)\})\{y/v\} \\
= & \mathcal{C}_\Gamma^{\Delta|\Pi}(t'\{b/R_\Delta^\Gamma(u)\}\{c/R_\Pi^\Gamma(u)\}\{y/v\}) \\
\equiv_{\mathcal{B}}\ (i.h.) & \mathcal{C}_\Gamma^{\Delta|\Pi}(t'\{y/v\}\{b/R_\Delta^\Gamma(u)\}\{c/R_\Pi^\Gamma(u)\}) \\
= & \mathcal{C}_a^{b|c}(t'\{y/v\})\{x/u\} \\
= & \mathcal{C}_a^{b|c}(\mathtt{del}_y(t')\{\!\{y/v\}\!\})\{x/u\} \\
= & t\{y/v\}\{x/u\}
\end{array}
$$

2. This statement can be proved in a similar way.

Lemma 7 (Preservation of Well-Formed Terms by Reduction). *If $\Gamma \Vdash_{\mathcal{B}} t$ and $t \to_{\mathcal{B}} u$, then $\exists\ \Delta \subseteq \Gamma$ s.t. $\Delta \Vdash_{\mathcal{B}} u$. Moreover if $\mathtt{w} \in \mathcal{B}$, $\Delta = \Gamma$.*

Proof. By induction on $\mathtt{size}(t)$ using Lemma 5.

Lemma 8. *Let $t \in \mathcal{T}_{\mathcal{B}}$ and $\Gamma \subseteq \mathtt{fv}(t)$ s.t. $|t|_\Gamma^+ = 0$. Then $t \to_{\mathcal{B}}^* \mathtt{del}_\Gamma(t)$ if $\mathtt{w} \notin \mathcal{B}$, and $t \to_{\mathcal{B}}^* \mathcal{W}_\Gamma(\mathtt{del}_\Gamma(t))$, if $\mathtt{w} \in \mathcal{B}$.*

Proof. By induction on $\mathtt{size}(t)$.

For instance $\mathcal{C}_x^{y|z}(w) \to_{\mathtt{CGc}} w = \mathtt{del}_x(\mathcal{C}_x^{y|z}(w))$ and $\mathcal{W}_y(z)\ \mathcal{W}_z(a) \to_{\mathtt{AW_l}} \to_{\mathtt{AW_r}} \mathcal{W}_y(z\ \mathcal{W}_z(a)) = \mathcal{W}_y(\mathtt{del}_y(\mathcal{W}_y(z)\ \mathcal{W}_z(a)))$.

Lemma 9 (Full Composition). *Let $t[\overline{y}/\overline{v}] \in \mathcal{T}_{\mathcal{B}}$ be a term having independent substitutions $[\overline{y}/\overline{v}]$. Then $t[\overline{y}/\overline{v}] \to^*_{\mathcal{B}} t\{\overline{y}/\overline{v}\}$.*

Proof. By induction on $\langle \mathtt{o}_{\overline{y}}(t), \mathtt{size}(t)\rangle$, where $\mathtt{o}_{\overline{y}}(t) = \Sigma_{i\in\{1\dots n\}}\mathtt{o}_{y_i}(t)$. Let $[\overline{y}/\overline{v}] = [x/u][\overline{x}/\overline{u}]$. We first show $t[x/u] \to^*_{\mathcal{B}} t\{x/u\}$, so that $t\{x/u\}[\overline{x}/\overline{u}] \to^*_{\mathcal{B}} t\{x/u\}\{\overline{x}/\overline{u}\} = t\{\overline{y}/\overline{v}\}$ by the i.h. since independence of $[\overline{y}/\overline{v}]$ imply $\mathtt{o}_{\overline{x}}(t\{x/u\}) < \mathtt{o}_{\overline{y}}(t)$.

- If $x \notin \mathtt{fv}(t)$, then $t[x/u] \to_{\mathtt{SGc}} t = t\{x/u\}$.
- If $|t|^+_x = n + 1 \geq 2$, then we can apply n times the rule $\mathtt{SDup}$ in such a way that each reduction step only replaces one occurrence of the truly free variable x of t. This gives the following, where we can apply the i.h. since the substitutions are independent:

$$\begin{array}{ll}
t[x/u] & \to_{\mathtt{SDup}} \\
t_{[x:=z_n]}[x/u][z_n/u] & \to_{\mathtt{SDup}} \\
\vdots & \\
t_{[x:=z_1\dots z_n]}[x/u][z_1/u]\dots[z_n/u] & \equiv_{\mathtt{SS}_{\mathcal{C}}} \\
t_{[x:=z_1\dots z_n]}[z_1/u]\dots[z_n/u][x/u] & \to^*_{\mathcal{B}} (i.h.) \\
t_{[x:=z_1\dots z_n]}\{z_1/u\}\dots\{z_n/u\}\{x/u\} & = t\{x/u\}
\end{array}$$

- If $|t|^+_x = 0$ and $|\mathtt{fv}(t)|_x > 0$, we consider the case where $\mathtt{w} \in \mathcal{B}$, as the one where $\mathtt{w} \notin \mathcal{B}$ is similar to the case where $x \notin \mathtt{fv}(t)$:

$$\begin{array}{ll}
t[x/u] & \to^*_{L.\ 8} \\
\mathcal{W}_x(\mathtt{del}_x(t))[x/u] & \to_{\mathtt{SW}_1} \\
\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(\mathtt{del}_x(t))}(\mathtt{del}_x(t)) & = \\
\mathcal{W}_{\mathtt{fv}(u)\setminus(\mathtt{fv}(t)\setminus\{x\})}(\mathtt{del}_x(t)) & = (x \notin \mathtt{fv}(u)) \\
\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{del}_x(t)) & = t\{x/u\}
\end{array}$$

- Now, consider the case where $|t|^+_x = 1$. We proceed by case analysis on t:
  - $t = x$. Then $x[x/u] \to_{\mathtt{V}} u = t\{x/u\}$.
  - $t = \lambda y.t'$. Then $t[x/u] \to_{\mathtt{SL}} \lambda y.t'[x/u] \to^*_{\mathcal{B}} (i.h.) \lambda y.t'\{x/u\} = t\{x/u\}$.
  - $t = v\ w$.

    If $x \in \mathtt{fv}^+(v)$ (so that $x \notin \mathtt{fv}^+(w)$) and $x \in \mathtt{fv}(w)$:

$$\begin{array}{ll}
(v\ w)[x/u] & \to^*_{\mathcal{B}}\ (L.\ 8) \\
(v\ \mathcal{W}_x(\mathtt{del}_x(w)))[x/u] & \to_{\mathtt{AW_r}} \\
(v\ \mathtt{del}_x(w))[x/u] & \to_{\mathtt{SA_L}} \\
(v[x/u]\ \mathtt{del}_x(w)) & \to^*_{\mathcal{B}}\ (i.h.) \\
(v\{x/u\}\ \mathtt{del}_x(w)) & = \\
(\mathtt{del}_x(v)\{\!\{x/u\}\!\}\ \mathtt{del}_x(w)) & =_{L.\ 4:5} \\
\mathtt{del}_x(v)\{\!\{x/u\}\!\}\ \mathtt{del}_x(w)\{\!\{x/u\}\!\} & = (v\ w)\{x/u\}
\end{array}$$

If $x \in \mathtt{fv}^+(v)$ (so that $x \notin \mathtt{fv}^+(w)$) and $x \notin \mathtt{fv}(w)$:

$$\begin{array}{ll} (v\ w)[x/u] & \to_{\mathtt{SA_L}} \\ v[x/u]\ w & \to^*_{\mathcal{B}} \ (i.h.) \\ v\{x/u\}\ w & = \\ \mathtt{del}_x(v)\{\!\{x/u\}\!\}\ w & = (v\ w)\{x/u\} \end{array}$$

If $x \in \mathtt{fv}^+(w)$, then the proof is similar but uses rules $\mathtt{AW_1}$ and $\mathtt{SA_R}$.

- $t = v[y/w]$. Similar to the previous case using $\mathtt{SW}$ and $\mathtt{SS}_{\mathcal{C}}$ in the first case; $\mathtt{SW_2}$ and $\mathtt{SS}$ in the second case.
- $t = \mathcal{W}_y(v)$.
  The case $y = x$ is impossible by hypothesis so that $y \neq x$ and we have:

$$\begin{array}{ll} \mathcal{W}_y(v)[x/u] & \to_{\mathtt{SW_2}} \\ \mathcal{W}_{y\setminus\mathtt{fv}(u)}(v[x/u]) & \to^*_{\mathcal{B}} \ (i.h.) \\ \mathcal{W}_{y\setminus\mathtt{fv}(u)}(v\{x/u\}) & = \\ \mathcal{W}_y(\mathtt{del}_x(v)\{\!\{x/u\}\!\}) & = \\ \mathcal{W}_y(\mathtt{del}_x(v))\{\!\{x/u\}\!\} & = \mathcal{W}_y(v)\{x/u\} \end{array}$$

- $t = \mathcal{C}_y^{y_1|y_2}(v)$. We consider the case where $y = x$, the other one is straightforward. Let $\Gamma = \mathtt{fv}(u)$. Then,

$$\begin{array}{ll} \mathcal{C}_x^{y_1|y_2}(v)[x/u] & \to_{\mathtt{SCa}} \\ \mathcal{C}_\Gamma^{\Delta|\Pi}(v[y_1/R^\Gamma_\Delta(u)][y_2/R^\Gamma_\Pi(u)]) & \to^*_{\mathcal{B}} \ (i.h.) \\ \mathcal{C}_\Gamma^{\Delta|\Pi}(v\{y_1/R^\Gamma_\Delta(u)\}\{y_2/R^\Gamma_\Pi(u)\}) & = \\ \mathcal{C}_x^{y_1|y_2}(v)\{x/u\} & \end{array}$$

For instance, if $\Gamma = \mathtt{fv}(u)$, Π, Δ are fresh, $u_1 = R^\Gamma_\Delta(u)$ and $u_2 = R^\Gamma_\Pi(u)$, then

$$\begin{array}{ll} \mathcal{C}_y^{y_1|y_2}(\mathcal{W}_{y_1}(\mathcal{W}_x(y_2)))[x/v][y/u] & \equiv \mathtt{SS}_{\mathcal{C}} \\ \mathcal{C}_y^{y_1|y_2}(\mathcal{W}_{y_1}(\mathcal{W}_x(y_2)))[y/v][x/v] & \to_{\mathtt{SCa}} \\ \mathcal{C}_\Gamma^{\Delta|\Pi}(\mathcal{W}_{y_1}(\mathcal{W}_x(y_2))[y_1/u_1][y_2/u_2])[x/v] & \to_{\mathtt{SW_1}} \\ \mathcal{C}_\Gamma^{\Delta|\Pi}(\mathcal{W}_{\mathtt{fv}(u_1)}(\mathcal{W}_x(y_2))[y_2/u_2])[x/v] & \to_{\mathtt{SW_2}} \\ \mathcal{C}_\Gamma^{\Delta|\Pi}(\mathcal{W}_{\mathtt{fv}(u_1)}(\mathcal{W}_x(y_2)[y_2/u_2]))[x/v] & \to_{\mathtt{SW_2}} \\ \mathcal{C}_\Gamma^{\Delta|\Pi}(\mathcal{W}_{\mathtt{fv}(u_1)}(\mathcal{W}_x(y_2[y_2/u_2])))[x/v] & \to_{\mathtt{V}} \\ \mathcal{C}_\Gamma^{\Delta|\Pi}(\mathcal{W}_{\mathtt{fv}(u_1)}(\mathcal{W}_x(u_2)))[x/v] & \end{array}$$

This is correct since:

$$\begin{array}{ll} \mathcal{C}_y^{y_1|y_2}(\mathcal{W}_{y_1}(\mathcal{W}_x(y_2)))[x/v]\{y/u\} & = \\ \mathcal{C}_y^{y_1|y_2}(\mathcal{W}_{y_1}(\mathcal{W}_x(y_2)))[x/v]\{\!\{y/u\}\!\} & = \\ \mathcal{C}_\Gamma^{\Delta|\Pi}(\mathcal{W}_{y_1}(\mathcal{W}_x(y_2))\{y_1/u_1\}\{y_2/u_2\})[x/v] & = \\ \mathcal{C}_\Gamma^{\Delta|\Pi}(\mathcal{W}_{\mathtt{fv}(u_1)}(\mathcal{W}_x(u_2)))[x/v] & \end{array}$$

3. Adding Resources

This section is devoted to the simulation of the $\lambda_\emptyset$-calculus into richer calculi having more resources. We consider the function $\mathtt{AR}_{\mathcal{A}}(\_) : \mathcal{T}_\emptyset \mapsto \mathcal{T}_{\mathcal{A}}$ for $\mathcal{A} \subseteq \mathcal{R}$ which enriches a $\lambda_\emptyset$-term in order to fulfill the constraints needed to be an $\mathcal{A}$-term. Adding is done not only on a static level (the terms) but also on a dynamic level (the reduction).

$$
\begin{array}{lll}
\mathtt{AR}_{\mathcal{A}}(x) & := x & \\
\mathtt{AR}_{\mathcal{A}}(\lambda x.t) & := \lambda x.\mathcal{W}_x(\mathtt{AR}_{\mathcal{A}}(t)) & \mathtt{w} \in \mathcal{A} \ \&\ x \notin \mathtt{fv}(t) \\
\mathtt{AR}_{\mathcal{A}}(\lambda x.t) & := \lambda x.\mathtt{AR}_{\mathcal{A}}(t) & \text{otherwise} \\
\mathtt{AR}_{\mathcal{A}}(t\ u) & := \mathcal{C}_{\Gamma}^{\Delta|\Pi}(R_{\Delta}^{\Gamma}(\mathtt{AR}_{\mathcal{A}}(t))R_{\Pi}^{\Gamma}(\mathtt{AR}_{\mathcal{A}}(u))) & \begin{cases} \mathtt{c} \in \mathcal{A} \ \&\ \Gamma := \mathtt{fv}(t) \cap \mathtt{fv}(u) \\ \Delta, \Pi \text{ are fresh} \end{cases} \\
\mathtt{AR}_{\mathcal{A}}(t\ u) & := \mathtt{AR}_{\mathcal{A}}(t)\ \mathtt{AR}_{\mathcal{A}}(u) & \text{otherwise}
\end{array}
$$

For example, adding resource $\mathtt{c}$ (resp. $\mathtt{w}$) to $t = \lambda x.yy$ gives $\lambda x.\mathcal{C}_y^{y_1|y_2}(y_1 y_2)$ (resp. $\lambda x.\mathcal{W}_x(yy)$), while adding both of them gives $\lambda x.\mathcal{W}_x(\mathcal{C}_y^{y_1|y_2}(y_1 y_2))$.

Lemma 10. *Let $t \in \mathcal{T}_\emptyset$, then we have*

1. $\mathtt{fv}(t) = \mathtt{fv}(\mathtt{AR}_{\mathcal{A}}(t)) = \mathtt{fv}^+(\mathtt{AR}_{\mathcal{A}}(t))$.
2. $\mathtt{del}_\Gamma(\mathtt{AR}_{\mathcal{A}}(t)) = \mathtt{AR}_{\mathcal{A}}(t)$.

Proof. By induction on $\mathtt{size}(t)$.

Point 1 says that $\mathtt{AR}_{\mathcal{A}}()$ only adds useful (i.e. positive) variables; thus deleting any non positive free variable in $\mathtt{AR}_{\mathcal{A}}(t)$ will leave the term unchanged as stated by Point 2.

We now establish the relation between $\mathtt{AR}_{\mathcal{A}}()$ and well-formed substitution; this is a technical key lemma of the paper.

Lemma 11. *Let $t, u \in \mathcal{T}_\emptyset$ and $\mathcal{A} \subseteq \mathcal{R}$. Then*

- *If $\mathtt{c} \notin \mathcal{A}$ then $\mathtt{AR}_{\mathcal{A}}(t)\{x/\mathtt{AR}_{\mathcal{A}}(u)\} = \mathtt{AR}_{\mathcal{A}}(t\{x/u\})$.*
- *If $\mathtt{c} \in \mathcal{A}$ then $\mathcal{C}_{\Gamma}^{\Delta|\Pi}(R_{\Delta}^{\Gamma}(\mathtt{AR}_{\mathcal{A}}(t))\{x/R_{\Pi}^{\Gamma}(\mathtt{AR}_{\mathcal{A}}(u))\}) \rightarrow^*_{\mathcal{A}} \mathtt{AR}_{\mathcal{A}}(t\{x/u\})$ where $\Gamma = (\mathtt{fv}(t) \setminus x) \cap \mathtt{fv}(u)$ and Δ, Π are fresh sets of variables.*

Proof. By induction on $\mathtt{size}(t)$, using the simplified definition of substitution for $\emptyset$-terms in Section 2.2. By Lemma 10:1, x cannot be a free variable of t which is not positive so that we can use the simplification notion of substitution given by Lemma 3. The case $\mathtt{c} \notin \mathcal{A}$ can be easily done by i.h. so we only consider $\mathtt{c} \in \mathcal{A}$.

First suppose $x \notin \mathtt{fv}(t)$. Then,

$$
\begin{array}{ll}
\mathcal{C}_{\Gamma}^{\Delta|\Pi}(R_{\Delta}^{\Gamma}(\mathtt{AR}_{\mathcal{A}}(t))\{x/R_{\Pi}^{\Gamma}(\mathtt{AR}_{\mathcal{A}}(u))\}) & = \\
\mathcal{C}_{\Gamma}^{\Delta|\Pi}(R_{\Delta}^{\Gamma}(\mathtt{AR}_{\mathcal{A}}(t))) & \rightarrow_{\mathtt{CGc}} \\
R_{\Gamma}^{\Delta}(R_{\Delta}^{\Gamma}(\mathtt{AR}_{\mathcal{A}}(t))) & = \\
\mathtt{AR}_{\mathcal{A}}(t) & = \\
\mathtt{AR}_{\mathcal{A}}(t\{x/u\}) &
\end{array}
$$

Otherwise, $x \in \mathtt{fv}(t)$ (and in particular, $x \in \mathtt{fv}^+(t)$ by Lemma 10:1). We consider different cases.

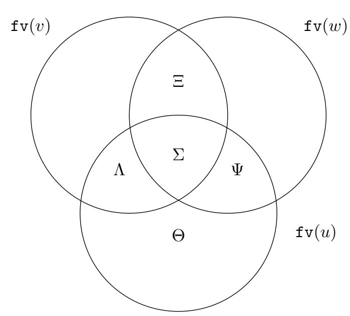
- The case $t = x$ is similar to the case where $\mathtt{c} \notin \mathcal{A}$.
- $t = \lambda y.t'$.
  - $y \notin \mathtt{fv}(t')$ and $\mathtt{w} \in \mathcal{A}$.

$$\begin{array}{ll}
\mathcal{C}_\Gamma^{\Delta|\Pi}((R_\Delta^\Gamma(\mathtt{AR}_\mathcal{A}(\lambda y.t')))\{x/R_\Pi^\Gamma(\mathtt{AR}_\mathcal{A}(u))\}) & = \\
\mathcal{C}_\Gamma^{\Delta|\Pi}((\lambda y.\mathcal{W}_y(R_\Delta^\Gamma(\mathtt{AR}_\mathcal{A}(t'))))\{x/R_\Pi^\Gamma(\mathtt{AR}_\mathcal{A}(u))\}) & = \\
\mathcal{C}_\Gamma^{\Delta|\Pi}(\lambda y.\mathcal{W}_y(R_\Delta^\Gamma(\mathtt{AR}_\mathcal{A}(t'))\{x/R_\Pi^\Gamma(\mathtt{AR}_\mathcal{A}(u))\})) & \to_{\mathtt{CL}} \\
\lambda y.\mathcal{C}_\Gamma^{\Delta|\Pi}(\mathcal{W}_y(R_\Delta^\Gamma(\mathtt{AR}_\mathcal{A}(t'))\{x/R_\Pi^\Gamma(\mathtt{AR}_\mathcal{A}(u))\})) & \to_{\mathtt{CW}_2} \\
\lambda y.\mathcal{W}_y(\mathcal{C}_\Gamma^{\Delta|\Pi}(R_\Delta^\Gamma(\mathtt{AR}_\mathcal{A}(t'))\{x/R_\Pi^\Gamma(\mathtt{AR}_\mathcal{A}(u))\})) & \to_\mathcal{A}^* \ (i.h.) \\
\lambda y.\mathcal{W}_y(\mathtt{AR}_\mathcal{A}(t'\{x/u\})) & = \\
\mathtt{AR}_\mathcal{A}(\lambda y.t'\{x/u\}) & = \\
\mathtt{AR}_\mathcal{A}((\lambda y.t')\{x/u\}) &
\end{array}$$

  - Otherwise

$$\begin{array}{ll}
\mathcal{C}_\Gamma^{\Delta|\Pi}((R_\Delta^\Gamma(\mathtt{AR}_\mathcal{A}(\lambda y.t')))\{x/R_\Pi^\Gamma(\mathtt{AR}_\mathcal{A}(u))\}) & = \\
\mathcal{C}_\Gamma^{\Delta|\Pi}(\lambda y.(R_\Delta^\Gamma(\mathtt{AR}_\mathcal{A}(t')))\{x/R_\Pi^\Gamma(\mathtt{AR}_\mathcal{A}(u))\}) & \to_{\mathtt{CL}} \\
\lambda y.\mathcal{C}_\Gamma^{\Delta|\Pi}(R_\Delta^\Gamma(\mathtt{AR}_\mathcal{A}(t'))\{x/R_\Pi^\Gamma(\mathtt{AR}_\mathcal{A}(u))\}) & \to_\mathcal{A}^* \ (i.h.) \\
\lambda y.\mathtt{AR}_\mathcal{A}(t\{x/u\}) & = \\
\mathtt{AR}_\mathcal{A}((\lambda y.t')\{x/u\}) &
\end{array}$$

- $t = v\ w$. Then by α-equivalence we can suppose $x \notin \mathtt{fv}(u)$. Let us consider the following names for the sets of free variables of the terms under consideration.



Note that $\Phi = \mathtt{fv}(t) \cap \mathtt{fv}(u)$ is a permutation of Σ, Λ, Ψ.
Also note that $\mathtt{fv}(v) \cap \mathtt{fv}(w)$ is a permutation of Σ, Ξ and hence

$$\mathtt{AR}_\mathcal{A}(t) \equiv \mathcal{C}_{\Sigma,\Xi}^{\Sigma_3,\Xi_3|\Sigma_4,\Xi_4}(R_{\Sigma_3,\Xi_3}^{\Sigma,\Xi}(\mathtt{AR}_\mathcal{A}(v))R_{\Sigma_4,\Xi_4}^{\Sigma,\Xi}(\mathtt{AR}_\mathcal{A}(w)))$$

We then have:

$$\begin{array}{ll} & \mathcal{C}^{\Sigma_1,\Lambda_1,\Psi_1|\Sigma_2,\Lambda_2,\Psi_2}_{\Sigma,\Lambda,\Psi}(R^{\Sigma,\Lambda,\Psi}_{\Sigma_1,\Lambda_1,\Psi_1}(\mathtt{AR}_{\mathcal{A}}(t))\{x/R^{\Sigma,\Lambda,\Psi}_{\Sigma_2,\Lambda_2,\Psi_2}(\mathtt{AR}_{\mathcal{A}}(u))\}) \\ = & \mathcal{C}^{\Sigma_1,\Lambda_1,\Psi_1|\Sigma_2,\Lambda_2,\Psi_2}_{\Sigma,\Lambda,\Psi}(\mathcal{C}^{\Sigma_3,\Xi_3|\Sigma_4,\Xi_4}_{\Sigma_1,\Xi}(v'w')\{x/R^{\Sigma,\Lambda,\Psi}_{\Sigma_2,\Lambda_2,\Psi_2}(\mathtt{AR}_{\mathcal{A}}(u))\}) \\ = & H \end{array}$$

where $v' = R^{\Lambda,\Sigma,\Xi}_{\Lambda_1,\Sigma_3,\Xi_3}(\mathtt{AR}_{\mathcal{A}}(v))$ and $w' = R^{\Psi,\Sigma,\Xi}_{\Psi_1,\Sigma_4,\Xi_4}(\mathtt{AR}_{\mathcal{A}}(w))$.

– If $x \in \mathtt{fv}(v) \cap \mathtt{fv}(w)$, then x is in Ξ (since $x \notin \mathtt{fv}(u)$), so Ξ is a permutation of $\Xi'; x$ for some list Ξ'. Hence $\mathcal{C}^{\Sigma_3,\Xi_3|\Sigma_4,\Xi_4}_{\Sigma_1,\Xi}()$ is equivalent by $\mathtt{CC}_{\mathcal{C}}$ to $\mathcal{C}^{\Sigma_3,\Xi_3'|\Sigma_4,\Xi_4'}_{\Sigma_1,\Xi'}(\mathcal{C}^{x_3|x_4}_x())$, where $\Xi_3'; x_3$ and $\Xi_4'; x_4$ are the corresponding permutations of Ξ_3 and Ξ_4, respectively. Noticing that $\mathtt{fv}(u)$ is a permutation of $\Theta, \Sigma, \Lambda, \Psi$, so that

$$H \quad \equiv_{\mathtt{CC}_{\mathcal{C}}} \mathcal{C}^{\Sigma_1,\Lambda_1,\Psi_1|\Sigma_2,\Lambda_2,\Psi_2}_{\Sigma,\Lambda,\Psi}(\mathcal{C}^{\Sigma_3,\Xi_3'|\Sigma_4,\Xi_4'}_{\Sigma_1,\Xi'}(\mathcal{C}^{x_3|x_4}_x(v'w'))\{S\})$$

where

$$S = x/R^{\Sigma,\Lambda,\Psi}_{\Sigma_2,\Lambda_2,\Psi_2}(\mathtt{AR}_{\mathcal{A}}(u))$$

Performing substitution S gives :

$$\mathcal{C}^{\Sigma_1,\Lambda_1,\Psi_1|\Sigma_2,\Lambda_2,\Psi_2}_{\Sigma,\Lambda,\Psi}(\mathcal{C}^{\Sigma_3,\Xi_3'|\Sigma_4,\Xi_4'}_{\Sigma_1,\Xi'}(\mathcal{C}^{\Theta_5,\Sigma_5,\Lambda_5,\Psi_5|\Theta_6,\Sigma_6,\Lambda_6,\Psi_6}_{\Theta,\Sigma_2,\Lambda_2,\Psi_2}(H_1)))$$

where H_1 is equal to:

$$(v'w')\{x_3/R^{\Theta,\Sigma,\Lambda,\Psi}_{\Theta_5,\Sigma_5,\Lambda_5,\Psi_5}(\mathtt{AR}_{\mathcal{A}}(u))\}\{x_4/R^{\Theta,\Sigma,\Lambda,\Psi}_{\Theta_6,\Sigma_6,\Lambda_6,\Psi_6}(\mathtt{AR}_{\mathcal{A}}(u))\}$$

$$= v'\{x_3/R^{\Theta,\Sigma,\Lambda,\Psi}_{\Theta_5,\Sigma_5,\Lambda_5,\Psi_5}(\mathtt{AR}_{\mathcal{A}}(u))\}\ w'\{x_4/R^{\Theta,\Sigma,\Lambda,\Psi}_{\Theta_6,\Sigma_6,\Lambda_6,\Psi_6}(\mathtt{AR}_{\mathcal{A}}(u))\}$$

Now we rearrange the contractions:

$$\begin{array}{ll} & \mathcal{C}^{\Sigma_1,\Lambda_1,\Psi_1|\Sigma_2,\Lambda_2,\Psi_2}_{\Sigma,\Lambda,\Psi}(\mathcal{C}^{\Sigma_3,\Xi_3'|\Sigma_4,\Xi_4'}_{\Sigma_1,\Xi'}(H_2)) \\ & \qquad \text{where } H_2 := \mathcal{C}^{\Theta_5,\Sigma_5,\Lambda_5,\Psi_5|\Theta_6,\Sigma_6,\Lambda_6,\Psi_6}_{\Theta,\Sigma_2,\Lambda_2,\Psi_2}(H_1) \\ \equiv_{\mathtt{CC}_{\mathcal{C}}} & \mathcal{C}^{\Theta_5|\Theta_6}_{\Theta}(\mathcal{C}^{\Xi_3'|\Xi_4'}_{\Xi'}(\mathcal{C}^{\Lambda_1|\Lambda_2}_{\Lambda}(\mathcal{C}^{\Lambda_5|\Lambda_6}_{\Lambda_2}(\mathcal{C}^{\Psi_1|\Psi_2}_{\Psi}(\mathcal{C}^{\Psi_5|\Psi_6}_{\Psi_2}(H_3)))))) \\ & \qquad \text{where } H_3 := \mathcal{C}^{\Sigma_1|\Sigma_2}_{\Sigma}(\mathcal{C}^{\Sigma_3|\Sigma_4}_{\Sigma_1}(\mathcal{C}^{\Sigma_5|\Sigma_6}_{\Sigma_2}(H_1))) \\ \equiv_{\mathtt{CC}_{\mathcal{A}}} & \mathcal{C}^{\Theta_5|\Theta_6}_{\Theta}(\mathcal{C}^{\Xi_3'|\Xi_4'}_{\Xi'}(\mathcal{C}^{\Lambda_2|\Lambda_6}_{\Lambda}(\mathcal{C}^{\Lambda_1|\Lambda_5}_{\Lambda_2}(\mathcal{C}^{\Psi_5|\Psi_2}_{\Psi}(\mathcal{C}^{\Psi_1|\Psi_6}_{\Psi_2}(H_4)))))) \\ & \qquad \text{where } H_4 := \mathcal{C}^{\Sigma_1|\Sigma_2}_{\Sigma}(\mathcal{C}^{\Sigma_3|\Sigma_5}_{\Sigma_1}(\mathcal{C}^{\Sigma_4|\Sigma_6}_{\Sigma_2}(H_1))) \\ \equiv_{\mathtt{CC}_{\mathcal{C}}} & \mathcal{C}^{\Theta_5,\Xi_3',\Lambda_2,\Psi_5,\Sigma_1|\Theta_6,\Xi_4',\Lambda_6,\Psi_2,\Sigma_2}_{\Theta,\Xi',\Lambda,\Psi,\Sigma}(\mathcal{C}^{\Lambda_1,\Sigma_3|\Lambda_5,\Sigma_5}_{\Lambda_2,\Sigma_1}(H_5)) \\ & \qquad \text{where } H_5 := \mathcal{C}^{\Psi_1,\Sigma_4|\Psi_6,\Sigma_6}_{\Psi_2,\Sigma_2}(H_1) \end{array}$$

This term can be reduced by $\mathtt{CA_L}$ and then by $\mathtt{CA_R}$ to

$$H' := \mathcal{C}^{\Theta_5,\Xi_3',\Lambda_2,\Psi_5,\Sigma_1|\Theta_6,\Xi_4',\Lambda_6,\Psi_2,\Sigma_2}_{\Theta,\Xi',\Lambda,\Psi,\Sigma}(PQ)$$

$$
\begin{array}{ll}
P & := \mathcal{C}^{\Lambda_1,\Sigma_3|\Lambda_5,\Sigma_5}_{\Lambda_2,\Sigma_1}(v'\{x_3/R^{\Theta,\Sigma,\Lambda,\Psi}_{\Theta_5,\Sigma_5,\Lambda_5,\Psi_5}(\mathtt{AR}_{\mathcal{A}}(u))\}) \\
 & = R^{\Theta,\Xi',\Lambda,\Psi,\Sigma}_{\Theta_5,\Xi'_3,\Lambda_2,\Psi_5,\Sigma_1}(\mathcal{C}^{\Lambda_1,\Sigma_3|\Lambda_5,\Sigma_5}_{\Lambda,\Sigma}(R^{\Lambda,\Sigma}_{\Lambda_1,\Sigma_3}(R^{x}_{x_3}(\mathtt{AR}_{\mathcal{A}}(v))))\{S_P\})
\end{array}
$$

and where

$$S_P = x_3/R^{\Sigma,\Lambda}_{\Sigma_5,\Lambda_5}(\mathtt{AR}_{\mathcal{A}}(u))$$

$$
\begin{array}{ll}
Q & := \mathcal{C}^{\Psi_1,\Sigma_4|\Psi_6,\Sigma_6}_{\Psi_2,\Sigma_2}(w'\{x_4/R^{\Theta,\Sigma,\Lambda,\Psi}_{\Theta_6,\Sigma_6,\Lambda_6,\Psi_6}(\mathtt{AR}_{\mathcal{A}}(u))\}) \\
 & = R^{\Theta,\Xi',\Lambda,\Psi,\Sigma}_{\Theta_6,\Xi'_4,\Lambda_6,\Psi_2,\Sigma_2}(\mathcal{C}^{\Psi_1,\Sigma_4|\Psi_6,\Sigma_6}_{\Psi,\Sigma}(R^{\Psi,\Sigma}_{\Psi_1,\Sigma_4}(R^{x}_{x_4}(\mathtt{AR}_{\mathcal{A}}(w))))\{S_Q\})
\end{array}
$$

and where

$$S_Q = x_4/R^{\Sigma,\Psi}_{\Sigma_6,\Psi_6}(\mathtt{AR}_{\mathcal{A}}(u))$$

We can now apply the i.h. to both subterms and we get:

$$
\begin{array}{ll}
P & \to^*_{\mathcal{A}} P' = R^{\Theta,\Xi',\Lambda,\Psi,\Sigma}_{\Theta_5,\Xi'_3,\Lambda_2,\Psi_5,\Sigma_1}(\mathtt{AR}_{\mathcal{A}}(v\{x/u\})) \\
Q & \to^*_{\mathcal{A}} Q' := R^{\Theta,\Xi',\Lambda,\Psi,\Sigma}_{\Theta_6,\Xi'_4,\Lambda_6,\Psi_2,\Sigma_2}(\mathtt{AR}_{\mathcal{A}}(w\{x/u\}))
\end{array}
$$

So H' reduces to

$$\mathcal{C}^{\Theta_5,\Xi'_3,\Lambda_2,\Psi_5,\Sigma_1|\Theta_6,\Xi'_4,\Lambda_6,\Psi_2,\Sigma_2}_{\Theta,\Xi',\Lambda,\Psi,\Sigma}(P'Q')$$

which is $\mathtt{AR}_{\mathcal{A}}(v\{x/u\}w\{x/u\}) = \mathtt{AR}_{\mathcal{A}}((v\ w)\{x/u\})$.

– If $x \in \mathtt{fv}(v)$ et $x \notin \mathtt{fv}(w)$, the term H can be transformed to:

$$
\begin{array}{ll}
 & \mathcal{C}^{\Sigma_1,\Lambda_1,\Psi_1|\Sigma_2,\Lambda_2,\Psi_2}_{\Sigma,\Lambda,\Psi}(\mathcal{C}^{\Sigma_3,\Xi_3|\Sigma_4,\Xi_4}_{\Sigma_1,\Xi}((v'\ w')\{x/S_x\})) \\
 & \qquad \text{with } S_x = R^{\Sigma,\Lambda,\Psi}_{\Sigma_2,\Lambda_2,\Psi_2}(\mathtt{AR}_{\mathcal{A}}(u)) \\
= & \mathcal{C}^{\Sigma_1,\Lambda_1,\Psi_1|\Sigma_2,\Lambda_2,\Psi_2}_{\Sigma,\Lambda,\Psi}(\mathcal{C}^{\Sigma_3,\Xi_3|\Sigma_4,\Xi_4}_{\Sigma_1,\Xi}(v'\{x/S_x\}\ w')) \\
\equiv_{\mathtt{CC}_{\mathcal{A}},\mathtt{CC}_{\mathcal{C}}} & \mathcal{C}^{\Sigma_1,\Psi_2,\Xi_3|\Sigma_4,\Psi_1,\Xi_4}_{\Sigma,\Psi,\Xi}(\mathcal{C}^{\Sigma_3,\Lambda_1|\Sigma_2,\Lambda_2}_{\Sigma_1,\Lambda}(v'\{x/S_x\}\ w')) \\
\to_{\mathtt{CA_L}} & \mathcal{C}^{\Sigma_1,\Psi_2,\Xi_3|\Sigma_4,\Psi_1,\Xi_4}_{\Sigma,\Psi,\Xi}(\mathcal{C}^{\Sigma_3,\Lambda_1|\Sigma_2,\Lambda_2}_{\Sigma_1,\Lambda}(v'\{x/S_x\})\ w') \\
= & \mathcal{C}^{\Sigma_1,\Psi_2,\Xi_3|\Sigma_4,\Psi_1,\Xi_4}_{\Sigma,\Psi,\Xi}(R^{\Sigma,\Psi,\Xi}_{\Sigma_1,\Psi_2,\Xi_3}(V)\ R^{\Sigma,\Psi,\Xi}_{\Sigma_4,\Psi_1,\Xi_4}(\mathtt{AR}_{\mathcal{A}}(w))) \\
= & H'
\end{array}
$$

where

$$V := \mathcal{C}^{\Sigma_3,\Lambda_1|\Sigma_2,\Lambda_2}_{\Sigma,\Lambda}(R^{\Lambda,\Sigma}_{\Lambda_1,\Sigma_3}(\mathtt{AR}_{\mathcal{A}}(v))\{x/R^{\Sigma,\Lambda}_{\Sigma_2,\Lambda_2}(\mathtt{AR}_{\mathcal{A}}(u))\})$$

which reduces by the i.h. to $\mathtt{AR}_{\mathcal{A}}(v\{x/u\})$. Hence,

$$H' \to^*_{\mathcal{A}} \mathcal{C}^{\Sigma_1,\Psi_2,\Xi_3|\Sigma_4,\Psi_1,\Xi_4}_{\Sigma,\Psi,\Xi}(R^{\Sigma,\Psi,\Xi}_{\Sigma_1,\Psi_2,\Xi_3}(\mathtt{AR}_{\mathcal{A}}(v\{x/u\}))R^{\Sigma,\Psi,\Xi}_{\Sigma_4,\Psi_1,\Xi_4}(\mathtt{AR}_{\mathcal{A}}(w)))$$

which is exactly $\mathtt{AR}_{\mathcal{A}}(v\{x/u\}w) = \mathtt{AR}_{\mathcal{A}}((vw)\{x/u\})$.

– If $x \in \mathtt{fv}(v)$ et $x \notin \mathtt{fv}(w)$ the proof is symmetric.

– The case $x \notin \mathtt{fv}(v)$ and $x \notin \mathtt{fv}(w)$ cannot happen since we assumed $x \in \mathtt{fv}(t)$.

For instance if $\mathtt{c} \in \mathcal{A}$, $t = (z\ x)\ z$ and $u = z$, then:

$$\begin{array}{ll}
\mathcal{C}_z^{z_3|z_4}(R_{z_3}^z(\mathtt{AR}_\mathcal{A}((z\ x)\ z))\{x/z_4\}) & = \\
\mathcal{C}_z^{z_3|z_4}(\mathcal{C}_{z_3}^{z_1|z_2}((z_1\ x)\ z_2)\{x/z_4\}) & = \\
\mathcal{C}_z^{z_3|z_4}(\mathcal{C}_{z_3}^{z_1|z_2}((z_1\ z_4)\ z_2)) & \equiv \\
\mathcal{C}_z^{z_3|z_2}(\mathcal{C}_{z_3}^{z_1|z_4}((z_1\ z_4)\ z_2)) & \to_{\mathtt{CA_L}} \\
\mathcal{C}_z^{z_3|z_2}(\mathcal{C}_{z_3}^{z_1|z_4}(z_1\ z_4)\ z_2) & = \\
\mathcal{C}_z^{z_3|z_2}(R_{z_3}^z(\mathtt{AR}_\mathcal{A}(z\ z))\ z_2) & = \\
\mathtt{AR}_\mathcal{A}((z\ z)\ z)) &
\end{array}$$

Theorem 1 (Simulation (i)). *Let $t \in \mathcal{T}_\emptyset$ such that $t \to_\emptyset t'$. Let $\mathcal{A} \subseteq \mathcal{R}$.*

- *If $\mathtt{w} \in \mathcal{A}$, then $\mathtt{AR}_\mathcal{A}(t) \to_\mathcal{A}^+ \mathcal{W}_{\mathtt{fv}(t)\setminus\mathtt{fv}(t')}(\mathtt{AR}_\mathcal{A}(t'))$.*
- *If $\mathtt{w} \notin \mathcal{A}$, then $\mathtt{AR}_\mathcal{A}(t) \to_\mathcal{A}^+ \mathtt{AR}_\mathcal{A}(t')$.*

Proof. By induction on the reduction relation $\to_\beta$ using Lemma 11.

- The root case $t = (\lambda x.t_1)\ u \to_\beta t_1\{x/u\} = t'$ is done using Lemmas 10 and 11.
- If $\lambda x.u \Rightarrow_\beta \lambda x.u'$ with $u \Rightarrow_\beta u'$, then we only consider the case $\mathtt{w} \in \mathcal{A}$ as the other ones are straightforward.

  – If $x \notin \mathtt{fv}(u)$, then

$$\begin{array}{lll}
\mathtt{AR}_\mathcal{A}(\lambda x.u) & = & \lambda x.\mathcal{W}_x(\mathtt{AR}_\mathcal{A}(u)) \\
 & \to_{\mathcal{A}\ (i.h.)}^+ & \lambda x.\mathcal{W}_x(\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(u')}(\mathtt{AR}_\mathcal{A}(u'))) \\
 & = & \lambda x.\mathcal{W}_x(\mathcal{W}_{\mathtt{fv}(\lambda x.u)\setminus\mathtt{fv}(\lambda x.u')}(\mathtt{AR}_\mathcal{A}(u'))) \\
 & \equiv_{\mathtt{WW_C}} & \lambda x.\mathcal{W}_{\mathtt{fv}(\lambda x.u)\setminus\mathtt{fv}(\lambda x.u')}(\mathcal{W}_x(\mathtt{AR}_\mathcal{A}(u'))) \\
 & \to_{\mathtt{LW}}^* & \mathcal{W}_{\mathtt{fv}(\lambda x.u)\setminus\mathtt{fv}(\lambda x.u')}(\lambda x.\mathcal{W}_x(\mathtt{AR}_\mathcal{A}(u')))
\end{array}$$

  – If $x \in \mathtt{fv}(u)$, then

$$\begin{array}{lll}
\mathtt{AR}_\mathcal{A}(\lambda x.u) & = & \lambda x.\mathtt{AR}_\mathcal{A}(u) \\
 & \to_{\mathcal{A}\ (i.h.)}^+ & \lambda x.\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(u')}(\mathtt{AR}_\mathcal{A}(u')) \\
 & = & \lambda x.\mathcal{W}_{\mathtt{fv}(\lambda x.u)\setminus\mathtt{fv}(u')}(\mathcal{W}_{x\setminus\mathtt{fv}(u')}(\mathtt{AR}_\mathcal{A}(u'))) \\
 & = & \lambda x.\mathcal{W}_{\mathtt{fv}(\lambda x.u)\setminus\mathtt{fv}(\lambda x.u')}(\mathcal{W}_{x\setminus\mathtt{fv}(u')}(\mathtt{AR}_\mathcal{A}(u'))) \\
 & \to_{\mathtt{LW}}^* & \mathcal{W}_{\mathtt{fv}(\lambda x.u)\setminus\mathtt{fv}(\lambda x.u')}(\lambda x.\mathcal{W}_{x\setminus\mathtt{fv}(u')}(\mathtt{AR}_\mathcal{A}(u')))
\end{array}$$

- If $uv \Rightarrow_\beta u'v$ with $u \Rightarrow_\beta u'$, we only consider the case where $\mathtt{c} \in \mathcal{A}$ as the other is straightforward.

Let consider the following names:

$$\begin{array}{rcl} \Sigma & = & \mathtt{fv}(u') \cap \mathtt{fv}(v) \\ \Lambda & = & \mathtt{fv}(u') \setminus (\mathtt{fv}(u') \cap \mathtt{fv}(v)) \\ \Psi & = & (\mathtt{fv}(u) \cap \mathtt{fv}(v)) \setminus \mathtt{fv}(u') \\ \Xi & = & (\mathtt{fv}(u) \setminus \mathtt{fv}(u')) \setminus \mathtt{fv}(v) \end{array}$$

Note in particular that $\mathtt{fv}(u) \cap \mathtt{fv}(v)$ is a permutation of Σ, Ψ. Correspondingly, let Σ_l, Ψ_l and Σ_r, Ψ_r be fresh variables.

We have:

$$\begin{array}{ll} & \mathtt{AR}_{\mathcal{A}}(u\ v) \\ \equiv & \mathcal{C}^{\Sigma_l,\Psi_l|\Sigma_r,\Psi_r}_{\Sigma,\Psi}(R^{\Sigma,\Psi}_{\Sigma_l,\Psi_l}(\mathtt{AR}_{\mathcal{A}}(u))\ R^{\Sigma,\Psi}_{\Sigma_r,\Psi_r}(\mathtt{AR}_{\mathcal{A}}(v))) \\ \to^+_{\mathcal{A}\ (i.h.)} & \mathcal{C}^{\Sigma_l,\Psi_l|\Sigma_r,\Psi_r}_{\Sigma,\Psi}(R^{\Sigma,\Psi}_{\Sigma_l,\Psi_l}(\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(u')}(\mathtt{AR}_{\mathcal{A}}(u')))\ R^{\Sigma,\Psi}_{\Sigma_r,\Psi_r}(\mathtt{AR}_{\mathcal{A}}(v))) \\ \equiv_{\mathtt{WW}_{\mathcal{C}}} & \mathcal{C}^{\Sigma_l,\Psi_l|\Sigma_r,\Psi_r}_{\Sigma,\Psi}(R^{\Sigma,\Psi}_{\Sigma_l,\Psi_l}(\mathcal{W}_{\Xi,\Psi}(\mathtt{AR}_{\mathcal{A}}(u')))R^{\Sigma,\Psi}_{\Sigma_r,\Psi_r}(\mathtt{AR}_{\mathcal{A}}(v))) \\ = & \mathcal{C}^{\Sigma_l,\Psi_l|\Sigma_r,\Psi_r}_{\Sigma,\Psi}(\mathcal{W}_{\Xi}(\mathcal{W}_{\Psi_l}(R^{\Sigma}_{\Sigma_l}(\mathtt{AR}_{\mathcal{A}}(u'))))R^{\Sigma,\Psi}_{\Sigma_r,\Psi_r}(\mathtt{AR}_{\mathcal{A}}(v))) \\ \to^*_{\mathtt{AW}_1} & \mathcal{C}^{\Sigma_l,\Psi_l|\Sigma_r,\Psi_r}_{\Sigma,\Psi}(\mathcal{W}_{\Xi\setminus R^{\Sigma,\Psi}_{\Sigma_r,\Psi_r}(\mathtt{fv}(v))}(t')) \\ & \text{where } t' = \mathcal{W}_{\Psi_l\setminus R^{\Sigma,\Psi}_{\Sigma_r,\Psi_r}(\mathtt{fv}(v))}(R^{\Sigma}_{\Sigma_l}(\mathtt{AR}_{\mathcal{A}}(u'))R^{\Sigma,\Psi}_{\Sigma_r,\Psi_r}(\mathtt{AR}_{\mathcal{A}}(v))) \\ = & \mathcal{C}^{\Sigma_l,\Psi_l|\Sigma_r,\Psi_r}_{\Sigma,\Psi}(\mathcal{W}_{\Xi}(\mathcal{W}_{\Psi_l}(R^{\Sigma}_{\Sigma_l}(\mathtt{AR}_{\mathcal{A}}(u'))R^{\Sigma,\Psi}_{\Sigma_r,\Psi_r}(\mathtt{AR}_{\mathcal{A}}(v))))) \\ \to^*_{\mathtt{CW}_2} & \mathcal{W}_{\Xi}(\mathcal{C}^{\Sigma_l,\Psi_l|\Sigma_r,\Psi_r}_{\Sigma,\Psi}(\mathcal{W}_{\Psi_l}(R^{\Sigma}_{\Sigma_l}(\mathtt{AR}_{\mathcal{A}}(u'))R^{\Sigma,\Psi}_{\Sigma_r,\Psi_r}(\mathtt{AR}_{\mathcal{A}}(v))))) \\ \to^*_{\mathtt{CW}_1} & \mathcal{W}_{\Xi}(\mathcal{C}^{\Sigma_l|\Sigma_r}_{\Sigma}(R^{\Psi_r}_{\Psi}(R^{\Sigma}_{\Sigma_l}(\mathtt{AR}_{\mathcal{A}}(u'))R^{\Sigma,\Psi}_{\Sigma_r,\Psi_r}(\mathtt{AR}_{\mathcal{A}}(v))))) \\ = & \mathcal{W}_{\Xi}(\mathcal{C}^{\Sigma_l|\Sigma_r}_{\Sigma}(R^{\Sigma}_{\Sigma_l}(\mathtt{AR}_{\mathcal{A}}(u'))R^{\Sigma}_{\Sigma_r}(\mathtt{AR}_{\mathcal{A}}(v)))) \end{array}$$

Then it suffices to notice that $\Xi = \mathtt{fv}(uv) \setminus \mathtt{fv}(u'v)$.

- The case $uv \Rightarrow_\beta uv'$ is similar to the previous one.

For instance, if $t = (\lambda z.y)\ w \to_\beta y = t'$ then $\mathtt{AR}_{\mathcal{A}}(t) = (\lambda z.\mathcal{W}_z(y))\ w \to_\beta \mathcal{W}_w(y) = \mathcal{W}_{\mathtt{fv}(t)\setminus\mathtt{fv}(t')}(\mathtt{AR}_{\mathcal{A}}(t'))$.

Since meta-level substitution can also be simulated by the explicit one by Lemma 9, then we obtain a more general simulation result.

Corollary 12 (Simulation (ii)). *Let $t \in \mathcal{T}_\emptyset$ such that $t \to_\emptyset t'$. Let $\mathcal{B} = \mathcal{A} \cup \{\mathtt{s}\}$, where $\mathcal{A} \subseteq \mathcal{R}$.*

- *If $\mathtt{w} \in \mathcal{A}$, then $\mathtt{AR}_{\mathcal{A}}(t) \to^+_{\mathcal{B}} \mathcal{W}_{\mathtt{fv}(t)\setminus\mathtt{fv}(t')}(\mathtt{AR}_{\mathcal{A}}(t'))$.*
- *If $\mathtt{w} \notin \mathcal{A}$, then $\mathtt{AR}_{\mathcal{A}}(t) \to^+_{\mathcal{B}} \mathtt{AR}_{\mathcal{A}}(t')$.*

For instance, if $t = (\lambda z.y)\ w \to_\beta y = t'$ then $\mathtt{AR}_{\mathtt{W}}(t) = (\lambda z.\mathcal{W}_z(y))\ w \to_{\mathtt{SW}} \mathcal{W}_w(y) = \mathcal{W}_{\mathtt{fv}(t)\setminus\mathtt{fv}(t')}(\mathtt{AR}_{\mathtt{W}}(t'))$.

While Corollary 12 states that adding resources to the $\lambda_\emptyset$-calculus is well behaved, this does not necessarily hold for *any* arbitrary calculus of the prismoid. Thus for example, what happens when the $\lambda_{\mathtt{S}}$-calculus is enriched with resource $\mathtt{w}$? Is it possible to simulate each $\mathtt{s}$-reduction step by a sequence of $\mathtt{sw}$-reduction steps? Unfortunately the answer is no: suppose the function $\mathtt{AR}_{\mathcal{A}}(\_)$ is extended to $\mathtt{s}$-terms in a natural way; then we have $t_1 = (x\ y)[z/v] \to_{\mathtt{S}} x\ y[z/v] = t_2$ but $\mathtt{AR}_{\mathtt{W}}(t_1) = \mathcal{W}_z(x\ y)[z/v] \not\to_{\mathtt{SW}} x\ \mathcal{W}_z(y)[z/v] = \mathtt{AR}_{\mathtt{W}}(t_2)$.

4. Removing Resources

In this section we give a mechanism to remove resources, that is, to change the status of weakening and/or contraction from explicit to implicit. This is dual to the operation adding resources to terms presented in Section 3. Whereas adding is only defined within the implicit base, removing is defined in both bases. As adding, removing is not only done on a static level, but also on a dynamic one. Thus for example, removing translates any $\mathtt{csw}$-reduction sequence into a $\mathcal{B}$-reduction sequence, for any $\mathcal{B} \in \{\mathtt{s}, \mathtt{cs}, \mathtt{sw}\}$.

We first define the **collapsing** function $\mathtt{S}^{\Gamma}_{z}(\_)$ of a well-formed term t without contractions s.t. $z \notin \mathtt{fv}(t)$ as follows:

$$
\begin{array}{lcl}
\mathtt{S}^{\Gamma}_{z}(w) & := & \begin{cases} w & \text{if } w \notin \Gamma \\ z & \text{if } w \in \Gamma \end{cases} \\
\mathtt{S}^{\Gamma}_{z}(uv) & := & \mathtt{S}^{\Gamma}_{z}(u)\mathtt{S}^{\Gamma}_{z}(v) \\
\mathtt{S}^{\Gamma}_{z}(\lambda w.u) & := & \lambda w.\mathtt{S}^{\Gamma}_{z}(u), \text{ if } w \notin \Gamma \\
\mathtt{S}^{\Gamma}_{z}(u[w/v]) & := & \mathtt{S}^{\Gamma}_{z}(u)[w/\mathtt{S}^{\Gamma}_{z}(v)], \text{ if } w \notin \Gamma \\
\mathtt{S}^{\Gamma}_{z}(\mathcal{W}_w(v)) & := & \begin{cases} \mathtt{S}^{\Gamma}_{z}(v) & \mathtt{S}^{\Gamma}_{z}(w) \in \mathtt{fv}(\mathtt{S}^{\Gamma}_{z}(v)) \\ \mathcal{W}_{\mathtt{S}^{\Gamma}_{z}(w)}(\mathtt{S}^{\Gamma}_{z}(v)) & \text{otherwise} \end{cases}
\end{array}
$$

The collapsing function renames the variables of a term by removing also the weakened ones that do not respect well-formedness. Indeed, if $\mathcal{W}_x(u)$ appears in the image term, then $x \notin \mathtt{fv}(u)$. Thus for example $\mathtt{S}^{y,z}_{x}(\mathcal{W}_y(\mathcal{W}_z(x))) = x$.

Lemma 13. *Let* $\mathtt{c} \notin \mathcal{B}$ *and* $t \in \mathcal{T}_{\mathcal{B}}$. *Then,*

1. $\mathtt{S}^{\Gamma}_{z}(t) = t$ *if* $\Gamma \cap \mathtt{fv}(t) = \emptyset$.
2. $\mathtt{S}^{x,y}_{z}(t) = R^{x}_{z}(t)$ *if* $y \notin \mathtt{fv}(t)$.
3. $\mathtt{del}_x(\mathtt{S}^{x_1,x_2}_{x}(t)) = \mathtt{S}^{x_1,x_2}_{x}(\mathtt{del}_{x_1,x_2}(t))$.
4. $\mathtt{S}^{x,x_3}_{z}(\mathtt{S}^{x_1,x_2}_{x}(t)) = \mathtt{S}^{x_1,x_2,x_3}_{z}(t)$.
5. $\mathtt{S}^{x_3,x_4}_{z}(\mathtt{S}^{x_1,x_2}_{x}(t)) = \mathtt{S}^{x_1,x_2}_{x}(\mathtt{S}^{x_3,x_4}_{z}(t))$ *if* $x \neq x_3, x_4$.
6. $\mathtt{S}^{\Gamma}_{z}(t)_{[x:=y]} = \mathtt{S}^{\Gamma}_{z}(t_{[x:=y]})$ *if* $x, y \notin \Gamma, z$.

Proof. All the statements are straightforward by induction on $\mathtt{size}(t)$.

A well-formed term t is said to be **well-signed** iff for every variable $x \in \mathtt{fv}(t)$, $x \in \mathtt{fv}^{+}(t)$ implies $|t|_x = |t|^{+}_{x}$. Thus for example, $\mathcal{W}_x(y)\mathcal{W}_x(z)$ and $x(yx)$ are well-signed while $\mathcal{W}_x(y)x$ does not.

Lemma 14. *Let* $\mathtt{c} \notin \mathcal{B}$. *Suppose* $t, u \in \mathcal{T}_{\mathcal{B}}$ *are well-signed. Then,*

1. $\mathtt{del}_{\Gamma}(t) = t$ *if* $x \in \Gamma$ *implies* $x \in \mathtt{fv}^{+}(t)$.
2. $\mathtt{S}^{\Gamma,y}_{z}(t) = R^{y}_{z}(\mathtt{S}^{\Gamma}_{z}(t))$ *if* $y \in \mathtt{fv}^{+}(t)$.
3. $\mathtt{del}_x(\mathtt{S}^{\Gamma}_{z}(t)) = \mathtt{S}^{\Gamma}_{z}(\mathtt{del}_x(t))$ *with* $x \notin \Gamma$ *and* $x \neq z$.
4. $\mathtt{S}^{y,z}_{x}(t)\{x/u\} = t\{\!\{y/u\}\!\}\{\!\{z/u\}\!\}$ *if* ($|t|^{+}_{y} \geq 1$ *or* $|t|^{+}_{z} \geq 1$) *and* $\mathtt{fv}(t) \cap \mathtt{fv}(u) = \emptyset$.
5. $\mathtt{S}^{x,y}_{z}(t\{w/u\}) = \mathtt{S}^{x,y}_{z}(t)\{w/\mathtt{S}^{x,y}_{z}(u)\}$ *if* $\mathtt{fv}(t) \cap \mathtt{fv}(u) = \emptyset$ *and* x, y *cannot be both in* t *or in* u.

6. *If* $t \to_{\mathcal{B}} t'$, *then* $\mathtt{S}^{\Gamma}_{z}(t) \to_{\mathcal{B}} \mathtt{S}^{\Gamma}_{z}(t')$.

Proof. All the properties can be shown by induction on $\mathtt{size}(t)$, except the last one which can be shown by induction on the reduction relation.

The function $\mathtt{RR}_{\mathcal{A}}(\_) : \mathcal{T}_{\mathcal{B}} \mapsto \mathcal{T}_{\mathcal{B}\setminus\mathcal{A}}$ removes $\mathcal{A} \subseteq \mathcal{R}$ from a $\mathcal{B}$-term.

$$
\begin{array}{lll}
\mathtt{RR}_{\mathcal{A}}(x) & := & x \\
\mathtt{RR}_{\mathcal{A}}(\lambda x.t) & := & \lambda x.\mathtt{RR}_{\mathcal{A}}(t) \\
\mathtt{RR}_{\mathcal{A}}(t\ u) & := & \mathtt{RR}_{\mathcal{A}}(t)\ \mathtt{RR}_{\mathcal{A}}(u) \\
\mathtt{RR}_{\mathcal{A}}(t[x/u]) & := & \mathtt{RR}_{\mathcal{A}}(t)[x/\mathtt{RR}_{\mathcal{A}}(u)] \\
\mathtt{RR}_{\mathcal{A}}(\mathcal{W}_x(t)) & := & \left\{\begin{array}{ll} \mathcal{W}_x(\mathtt{RR}_{\mathcal{A}}(t)) & \text{if } \mathtt{w} \notin \mathcal{A} \\ \mathtt{RR}_{\mathcal{A}}(t) & \text{if } \mathtt{w} \in \mathcal{A} \end{array}\right. \\
\mathtt{RR}_{\mathcal{A}}(\mathcal{C}^{y|z}_x(t)) & := & \left\{\begin{array}{ll} \mathcal{C}^{y|z}_x(\mathtt{RR}_{\mathcal{A}}(t)) & \text{if } \mathtt{c} \notin \mathcal{A} \\ \mathtt{S}^{y,z}_x(\mathtt{del}_{y,z}(\mathtt{RR}_{\mathcal{A}}(t))) & \text{if } \mathtt{c} \in \mathcal{A}\ \&\ x \in \mathtt{fv}^+(\mathcal{C}^{y|z}_x(t)) \\ \mathtt{S}^{y,z}_x(\mathtt{RR}_{\mathcal{A}}(t)) & \text{if } \mathtt{c} \in \mathcal{A}\ \&\ x \notin \mathtt{fv}^+(\mathcal{C}^{y|z}_x(t)) \end{array}\right.
\end{array}
$$

It is worth noticing that $\mathtt{RR}_{\mathcal{A}}(t)$ is always a well-signed term when $\mathtt{c} \in \mathcal{A}$.

For example, $\mathtt{RR}_{\mathtt{C}}(\mathcal{C}^{x_1|x_2}_x(\mathcal{C}^{y_1|y_2}_y(\mathcal{W}_{y_1}(\mathcal{W}_{y_2}(x_1)))[y/x_2])) = \mathcal{W}_y(x)[y/x]$ and $\mathtt{RR}_{\mathtt{W}}(\mathcal{W}_x(z_1)\ \mathcal{W}_y(z_2)) = z_1\ z_2$. More interestingly, $\mathtt{RR}_{\mathtt{C}}(\mathcal{C}^{x_1|x_3}_{y_2}(\mathcal{W}_{x_1}(y_1)x_3))$ is $y_1 y_2$ and not $\mathcal{W}_{y_2}(y_1)y_2$. This is because when projecting contractions, we do not want to leave negative variables whose positive occurrences come from the image of the projection. This is particularly useful when projecting a SCa-reduction step. Indeed, let us suppose

$$
\underbrace{\mathcal{C}^{y_1|y_2}_x(\mathcal{C}^{x_1|x_3}_{y_2}(\mathcal{W}_{x_1}(y_1)x_3))[x/z]}_{t_0} \to_{\mathtt{SCa}} \underbrace{\mathcal{C}^{z_1|z_2}_z(\mathcal{C}^{x_1|x_3}_{y_2}(\mathcal{W}_{x_1}(y_1)x_3)[y_1/z_1][y_2/z_2])}_{t_1}
$$

Then, projecting contractions gives

$$
\mathtt{RR}_{\mathtt{C}}(t_0) = (xx)[x/z] \to_{\mathtt{SDup}} (y_1 y_2)[y_1/z][y_2/z] = \mathtt{RR}_{\mathtt{C}}(t_1)
$$

Remark that the removing function $\mathtt{RR}_{\mathcal{A}}(\_)$ is the identity if the resources $\mathcal{A}$ to be removed are not in the term, i.e. $\mathtt{RR}_{\mathcal{A}}(t) = t$ if $t \in \mathcal{T}_{\mathcal{B}\setminus\mathcal{A}}$.

The operation $\mathtt{RR}_{\mathcal{A}}(\_)$ enjoys the following properties:

Lemma 15. *Let* $t \in \mathcal{T}_{\mathcal{B}}$. *Then, for all* $\mathcal{A} \subseteq \mathcal{R}$

1. $R^{\Gamma}_{\Delta}(\mathtt{RR}_{\mathcal{A}}(t)) = \mathtt{RR}_{\mathcal{A}}(R^{\Gamma}_{\Delta}(t))$.
2. $\mathtt{fv}^+(\mathtt{RR}_{\mathcal{A}}(t)) = \mathtt{fv}^+(t)$.
3. $\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t)) = \mathtt{fv}(t)$ *if* $\mathtt{w} \in \mathcal{B} \setminus \mathcal{A}$, $\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t)) \subseteq \mathtt{fv}(t)$ *otherwise.*
4. $\mathtt{RR}_{\mathcal{A}}(t)_{[x:=y_1\ldots y_n]} = \mathtt{RR}_{\mathcal{A}}(t_{[x:=y_1\ldots y_n]})$ *if* $\mathtt{c} \notin \mathcal{B}$.
5. $\mathtt{del}_{\Gamma}(\mathtt{RR}_{\mathcal{A}}(t)) = \mathtt{RR}_{\mathcal{A}}(\mathtt{del}_{\Gamma}(t))$.

PROOF. By induction on $\mathtt{size}(t)$.

Lemma 16. *Let $t, u \in \mathcal{T}_\mathcal{B}$ and $\mathcal{A} \subseteq \mathcal{R}$. If $t\{x/u\} \in \mathcal{T}_\mathcal{B}$, then $\mathtt{RR}_\mathcal{A}(t\{x/u\}) = \mathtt{RR}_\mathcal{A}(t)\{x/\mathtt{RR}_\mathcal{A}(u)\}$.*

PROOF. If $x \notin \mathtt{fv}(t)$ then the property is straightforward so that suppose $x \in \mathtt{fv}(t)$. We first prove $\mathtt{RR}_\mathcal{A}(t\{x/u\}) = \mathtt{RR}_\mathcal{A}(t)\{x/\mathtt{RR}_\mathcal{A}(u)\}$ when $|t|_x^+ \leq 1$. Now, to prove in the general case that $\mathtt{RR}_\mathcal{A}(t\{x/u\}) = \mathtt{RR}_\mathcal{A}(t)\{x/\mathtt{RR}_\mathcal{A}(u)\}$ we proceed by induction on $|t|_x^+$.

- If $|t|_x^+ = n+1 \geq 2$, then $\mathtt{c} \notin \mathcal{B}$. We have

  $$\begin{array}{ll} \mathtt{RR}_\mathcal{A}(t\{x/u\}) & = \\ \mathtt{RR}_\mathcal{A}(t_{[x:=x_1\ldots x_n]}\{x_1/u\}\ldots\{x_n/u\}\{x/u\}) & =_{i.h.} \\ \mathtt{RR}_\mathcal{A}(t_{[x:=x_1\ldots x_n]})\{x_1/\mathtt{RR}_\mathcal{A}(u)\}\ldots\{x_n/\mathtt{RR}_\mathcal{A}(u)\}\{x/\mathtt{RR}_\mathcal{A}(u)\} & =_{L.\ 15:4} \\ \mathtt{RR}_\mathcal{A}(t)_{[x:=x_1\ldots x_n]}\{x_1/\mathtt{RR}_\mathcal{A}(u)\}\ldots\{x_n/\mathtt{RR}_\mathcal{A}(u)\}\{x/\mathtt{RR}_\mathcal{A}(u)\} & = \\ \mathtt{RR}_\mathcal{A}(t)\{x/\mathtt{RR}_\mathcal{A}(u)\} & \end{array}$$

  We now show $\mathtt{RR}_\mathcal{A}(t\{x/u\}) = \mathtt{RR}_\mathcal{A}(t)\{x/\mathtt{RR}_\mathcal{A}(u)\}$ when $|t|_x^+ \leq 1$. We proceed by induction on $\langle \mathtt{o}_x(t), \mathtt{size}(t)\rangle$.

- If $|t|_x^+ = 0$ we have three cases.
  - If $|\mathtt{fv}(t)|_x = 0$ or $\mathtt{w} \notin \mathcal{B}$ then : $\mathtt{RR}_\mathcal{A}(t\{x/u\}) = \mathtt{RR}_\mathcal{A}(\mathtt{del}_x(t)) =_{L.\ 15:5} \mathtt{del}_x(\mathtt{RR}_\mathcal{A}(t)) = \mathtt{RR}_\mathcal{A}(t)\{x/\mathtt{RR}_\mathcal{A}(u)\}$.
  - If $|\mathtt{fv}(t)|_x > 0$ and $\mathtt{w} \in \mathcal{B}$ and $\mathtt{w} \notin \mathcal{A}$ then :

    $$\begin{array}{ll} \mathtt{RR}_\mathcal{A}(t\{x/u\}) & = \\ \mathtt{RR}_\mathcal{A}(\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{del}_x(t))) & = \\ \mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{RR}_\mathcal{A}(\mathtt{del}_x(t))) & =_{L.\ 15:5} \\ \mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{del}_x(\mathtt{RR}_\mathcal{A}(t))) & =_{L.\ 15:3} \\ \mathcal{W}_{\mathtt{fv}(\mathtt{RR}_\mathcal{A}(u))\setminus\mathtt{fv}(\mathtt{RR}_\mathcal{A}(t))}(\mathtt{del}_x(\mathtt{RR}_\mathcal{A}(t))) & = \mathtt{RR}_\mathcal{A}(t)\{x/\mathtt{RR}_\mathcal{A}(u)\} \end{array}$$

  - If $|\mathtt{fv}(t)|_x > 0$ and $\mathtt{w} \in \mathcal{B}$ and $\mathtt{w} \in \mathcal{A}$ then :

    $$\begin{array}{ll} \mathtt{RR}_\mathcal{A}(t\{x/u\}) & = \\ \mathtt{RR}_\mathcal{A}(\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(t)}(\mathtt{del}_x(t))) & = \\ \mathtt{RR}_\mathcal{A}(\mathtt{del}_x(t)) & =_{L.\ 15:5} \\ \mathtt{del}_x(\mathtt{RR}_\mathcal{A}(t)) & = \mathtt{RR}_\mathcal{A}(t)\{x/\mathtt{RR}_\mathcal{A}(u)\} \end{array}$$

- We now consider the case where $|t|_x^+ = 1$
  - If $t = x$ then $\mathtt{RR}_\mathcal{A}(x)\{x/\mathtt{RR}_\mathcal{A}(u)\} = \mathtt{RR}_\mathcal{A}(u) = \mathtt{RR}_\mathcal{A}(x\{x/u\})$.
  - The case $t = \lambda y.v$ is straightforward by induction.
  - Cases $t = v\ w$, $t = v[y/w]$, $t = \mathcal{W}_y(v)$ are easily done by the i.h. and Lemma 15.

– $t = \mathcal{C}_y^{y_1|y_2}(v)$. Most of the cases are done using the i.h. and Lemma 15 except the one where $y = x$ & $\mathtt{c} \in \mathcal{A}$. We use the following notations: $\Gamma = \mathtt{fv}(u)$, Δ, Π are sets of fresh variables, $\Gamma_1 = \{x \in \Gamma || \Gamma|_x^+ \geq 1\}$, $\Gamma_0 = \Gamma \setminus \Gamma_1$, $\Delta_1, \Pi_1, \Delta_0, \Pi_0$ are similarly defined.

$$
\begin{array}{ll}
\mathtt{RR}_{\mathtt{C}}(\mathcal{C}_x^{y_1|y_2}(v))\{x/\mathtt{RR}_{\mathtt{C}}(u)\} & = \\
\mathtt{S}_x^{y_1,y_2}(\mathtt{del}_{y_1,y_2}(\mathtt{RR}_{\mathtt{C}}(v)))\{x/\mathtt{RR}_{\mathtt{C}}(u)\} & =_{L.\ 14:4} \\
\mathtt{del}_{y_1,y_2}(\mathtt{RR}_{\mathtt{C}}(v))\{\!\{y_1/\mathtt{RR}_{\mathtt{C}}(u)\}\!\}\{\!\{y_2/\mathtt{RR}_{\mathtt{C}}(u)\}\!\} & =_{L.\ 4:7} \\
\mathtt{RR}_{\mathtt{C}}(v)\{y_1/\mathtt{RR}_{\mathtt{C}}(u)\}\{y_2/\mathtt{RR}_{\mathtt{C}}(u)\} & = \\
\mathtt{RR}_{\mathtt{C}}(v)\{y_1/R_\Gamma^\Delta(R_\Delta^\Gamma(\mathtt{RR}_{\mathtt{C}}(u)))\}\{y_2/R_\Gamma^\Pi(R_\Pi^\Gamma(\mathtt{RR}_{\mathtt{C}}(u)))\} & =_{L.\ 15:1} \\
\mathtt{RR}_{\mathtt{C}}(v)\{y_1/R_\Gamma^\Delta(\mathtt{RR}_{\mathtt{C}}(R_\Delta^\Gamma(u)))\}\{y_2/R_\Gamma^\Pi(\mathtt{RR}_{\mathtt{C}}(R_\Pi^\Gamma(u)))\} & =_{L.\ 13:2} \\
\mathtt{RR}_{\mathtt{C}}(v)\{y_1/\mathtt{S}_\Gamma^{\Delta,\Pi}(\mathtt{RR}_{\mathtt{C}}(R_\Delta^\Gamma(u)))\}\{y_2/\mathtt{S}_\Gamma^{\Delta,\Pi}(\mathtt{RR}_{\mathtt{C}}(R_\Pi^\Gamma(u)))\} & =_{L.\ 14:5\ \&\ L.\ 13:1} \\
\mathtt{S}_\Gamma^{\Delta,\Pi}(\mathtt{RR}_{\mathtt{C}}(v)\{y_1/\mathtt{RR}_{\mathtt{C}}(R_\Delta^\Gamma(u))\}\{y_2/\mathtt{RR}_{\mathtt{C}}(R_\Pi^\Gamma(u))\}) & =_{i.h.} \\
\mathtt{S}_{\Gamma_0}^{\Delta_0,\Pi_0}(\mathtt{S}_{\Gamma_1}^{\Delta_1,\Pi_1}(\mathtt{RR}_{\mathtt{C}}(v\{y_1/R_\Delta^\Gamma(u)\}\{y_2/R_\Pi^\Gamma(u)\}))) & =_{L.\ 14:1} \\
\mathtt{S}_{\Gamma_0}^{\Delta_0,\Pi_0}(\mathtt{S}_{\Gamma_1}^{\Delta_1,\Pi_1}(\mathtt{del}_{\Delta_1,\Pi_1}(\mathtt{RR}_{\mathtt{C}}(v\{y_1/R_\Delta^\Gamma(u)\}\{y_2/R_\Pi^\Gamma(u)\})))) & = \\
\mathtt{RR}_{\mathtt{C}}(\mathcal{C}_\Gamma^{\Delta|\Pi}(v\{y_1/R_\Delta^\Gamma(u)\}\{y_2/R_\Pi^\Gamma(u)\})) & = \\
\mathtt{RR}_{\mathtt{C}}(\mathcal{C}_x^{y_1|y_2}(v)\{x/u\}) &
\end{array}
$$

To illustrate Lemma 16, let us consider the terms $t = \mathcal{C}_x^{y|z}(\mathcal{W}_y(z))$ and $u = \mathcal{W}_a(\lambda w.w)$. Then $t\{x/u\} = \mathcal{C}_a^{a_1|a_2}(\mathcal{W}_{a_1}(\mathcal{W}_{a_2}(\lambda w.w)))$. We thus have:

$$\mathtt{RR}_{\mathtt{C}}(t\{x/u\}) = \mathtt{S}_a^{a_1,a_2}(\mathcal{W}_{a_1}(\mathcal{W}_{a_2}(\lambda w.w))) = \mathcal{W}_a(\mathtt{S}_a^{a_1,a_2}(\lambda w.w)) = \mathcal{W}_a(\lambda w.w)$$

and

$$\mathtt{RR}_{\mathtt{C}}(t)\{x/\mathtt{RR}_{\mathtt{C}}(u)\} = x\{x/\mathcal{W}_a(\lambda w.w)\} = \mathcal{W}_a(\lambda w.w)$$

Calculi of the prismoid include rules/equations to handle substitution but also other rules/equations to handle resources $\{\mathtt{c}, \mathtt{w}\}$. Moreover, implicit (resp. explicit) substitution is managed by the β-rule (resp. the whole system $\mathtt{s}$). We can then split the reduction relation $\rightarrow_{\mathcal{B}}$ in two different parts: one for (implicit or explicit) substitution, which can be *s*trictly projected into itself, and another one for weakening and contraction, which can be projected into a more subtle way given by the following statement.

Theorem 2 (Projection). *Let $\mathcal{A} \subseteq \mathcal{R}$ such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{S}$ and let $t \in \mathcal{T}_{\mathcal{B}}$. If $t \equiv_{\mathcal{B}} u$, then $\mathtt{RR}_{\mathcal{A}}(t) \equiv_{\mathcal{B}\setminus\mathcal{A}} \mathtt{RR}_{\mathcal{A}}(u)$. Otherwise:*

- *If $\mathtt{s} \notin \mathcal{B}$:*
  - *If $t \Rightarrow_\beta u$, then $\mathtt{RR}_{\mathcal{A}}(t) \rightarrow_\beta^+ \mathtt{RR}_{\mathcal{A}}(u)$.*
  - *If $t \Rightarrow_{\mathcal{B}\setminus\beta} u$, then $\mathtt{RR}_{\mathcal{A}}(t) \rightarrow^*_{\mathcal{B}\setminus\beta\setminus\mathcal{A}} \mathtt{RR}_{\mathcal{A}}(u)$ and $\mathtt{RR}_{\mathcal{B}}(t) = \mathtt{RR}_{\mathcal{B}}(u)$.*
- *Otherwise,*
  - *If $t \Rightarrow_{\mathtt{s}} u$, then $\mathtt{RR}_{\mathcal{A}}(t) \rightarrow_{\mathtt{s}}^+ \mathtt{RR}_{\mathcal{A}}(u)$.*
  - *If $t \Rightarrow_{\mathcal{B}\setminus\mathtt{s}} u$, then $\mathtt{RR}_{\mathcal{A}}(t) \rightarrow^*_{\mathcal{B}\setminus\mathtt{s}\setminus\mathcal{A}} \mathtt{RR}_{\mathcal{A}}(u)$.*

PROOF. By induction on the reduction relation. For the points involving $\mathtt{RR}_{\mathcal{A}}(\_)$, one can first consider the case where $\mathcal{A}$ is a singleton. Then the general result follows from two successive applications of the simpler property.

We only show here the following interesting case where $\mathtt{c} \in \mathcal{A}$.

Let $t = \mathcal{C}_x^{y|z}(t_1)[x/u] \rightarrow_{\mathtt{SCa}} \mathcal{C}_\Gamma^{\Delta|\Pi}(t_1[y/R_\Delta^\Gamma(u)][z/R_\Pi^\Gamma(u)]) = t'$, with $y, z \in \mathtt{fv}^+(t_1)$, $\Gamma = \mathtt{fv}(u)$ and Π, Δ fresh. Then,

$$
\begin{array}{ll}
\mathtt{RR}_{\mathcal{A}}(t) & = \\
\mathtt{S}_x^{y,z}(\mathtt{del}_{y,z}(\mathtt{RR}_{\mathcal{A}}(t_1)))[x/\mathtt{RR}_{\mathcal{A}}(u)] & =_{L.\ 14:1} \\
\mathtt{S}_x^{y,z}(\mathtt{RR}_{\mathcal{A}}(t_1))[x/\mathtt{RR}_{\mathcal{A}}(u)] & =_{L.\ 14:2} \\
R_x^z(R_x^y(\mathtt{RR}_{\mathcal{A}}(t_1)))[x/\mathtt{RR}_{\mathcal{A}}(u)] & \rightarrow_{\mathtt{SDup}} \\
\mathtt{RR}_{\mathcal{A}}(t_1)[y/\mathtt{RR}_{\mathcal{A}}(u)][z/\mathtt{RR}_{\mathcal{A}}(u)] & = \\
\mathtt{RR}_{\mathcal{A}}(t_1)[y/R_\Gamma^\Delta(R_\Delta^\Gamma(\mathtt{RR}_{\mathcal{A}}(u)))][z/R_\Gamma^\Pi(R_\Pi^\Gamma(\mathtt{RR}_{\mathcal{A}}(u)))] & =_{L.\ 13:2} \\
\mathtt{RR}_{\mathcal{A}}(t_1)[y/\mathtt{S}_{\Gamma_0}^{\Delta_0}(\mathtt{S}_{\Gamma_1}^{\Delta_1}(\mathtt{RR}_{\mathcal{A}}(R_\Delta^\Gamma(u))))][z/\mathtt{S}_{\Gamma_0}^{\Pi_0}(\mathtt{S}_{\Gamma_1}^{\Pi_1}(\mathtt{RR}_{\mathcal{A}}(R_\Pi^\Gamma(u))))] & =_{L.\ 13:1} \\
\mathtt{S}_{\Gamma_0}^{\Delta_0,\Pi_0}(\mathtt{S}_{\Gamma_1}^{\Delta_1,\Pi_1}(\mathtt{RR}_{\mathcal{A}}(t_1)[y/\mathtt{RR}_{\mathcal{A}}(R_\Delta^\Gamma(u))][z/\mathtt{RR}_{\mathcal{A}}(R_\Pi^\Gamma(u))])) & =_{L.\ 14:1} \\
\mathtt{S}_{\Gamma_0}^{\Delta_0,\Pi_0}(\mathtt{S}_{\Gamma_1}^{\Delta_1,\Pi_1}(\mathtt{del}_{\Delta_1,\Pi_1}(\mathtt{RR}_{\mathcal{A}}(t_1)[y/\mathtt{RR}_{\mathcal{A}}(R_\Delta^\Gamma(u))][z/\mathtt{RR}_{\mathcal{A}}(R_\Pi^\Gamma(u))]))) & = \mathtt{RR}_{\mathcal{A}}(t')
\end{array}
$$

The other cases use Lemmas 13, 14, 15, and 16.

For instance, the reduction $t = \mathcal{C}_x^{y|z}(y\ z)[x/a] \rightarrow_{\mathtt{SCa}} \mathcal{C}_a^{a_1|a_2}((y\ z)[y/a_1][z/a_2]) = t'$ is projected into $\mathtt{RR}_{\mathtt{C}}(t) = (x\ x)[x/a] \rightarrow_{\mathtt{SDup}} (x\ y)[x/a][y/a] =_\alpha (y\ z)[y/a][z/a] = \mathtt{RR}_{\mathtt{C}}(t')$.

It is now time to discuss the need of positive conditions (conditions involving positive free variables) in the specification of the reduction rules of the prismoid. For that, let us consider a relaxed form of the $\mathtt{SS}_1$-rule: $t[x/u][y/v] \rightarrow t[x/u[y/v]]$ if $y \in \mathtt{fv}(u) \setminus \mathtt{fv}(t)$ (instead of $y \in \mathtt{fv}^+(u) \setminus \mathtt{fv}(t)$).

The need for the condition $y \in \mathtt{fv}(u)$ is well-known [Blo97], otherwise PSN does not hold. The need for the condition $y \notin \mathtt{fv}(t)$ is also natural if one wants to preserve well-formed terms. Now, the reduction step $t_1 = x[x/\mathcal{W}_y(z)][y/y'] \rightarrow_{\mathtt{SS}_1} x[x/\mathcal{W}_y(z)[y/y']] = t_2$ in the calculus with sorts $\{\mathtt{s}, \mathtt{w}\}$ cannot be projected into $\mathtt{RR}_{\mathtt{W}}(t_1) = x[x/z][y/y'] \rightarrow_{\mathtt{SS}_1} x[x/z[y/y']] = \mathtt{RR}_{\mathtt{W}}(t_2)$ since $y \notin \mathtt{fv}(z)$. Similar examples can be given to justify positive conditions in rules $\mathtt{SDup}, \mathtt{SCa}$ and $\mathtt{CS}$.

Lemma 17. *Let $t \in \mathcal{T}_\emptyset$ and let $\mathcal{A} \subseteq \mathcal{R}$. Then $\mathtt{RR}_{\mathcal{A}}(\mathtt{AR}_{\mathcal{A}}(t)) = t$.*

PROOF. By induction on $\mathtt{size}(t)$.

The following property states that administration of weakening and/or contraction is terminating in any calculus.

Lemma 18. *If $\mathtt{s} \notin \mathcal{B}$, then the reduction relation $\rightarrow_{\mathcal{B}\setminus\beta}$ is terminating. If $\mathtt{s} \in \mathcal{B}$, then the reduction relation $\rightarrow_{\mathcal{B}\setminus\mathtt{S}}$ is terminating.*

PROOF. The reduction relation $\rightarrow_{\mathcal{B}\setminus\beta}$ is contained in $\rightarrow_{\mathcal{B}\setminus\mathtt{S}}$ so it is sufficient to show termination of the biggest relation. We show that $w \rightarrow_{\mathcal{B}\setminus\mathtt{S}} w'$ implies $\langle \mathtt{S}(w'), \mathtt{I}(w'), \mathtt{L}(w') \rangle <_{\mathtt{lex}} \langle \mathtt{S}(w), \mathtt{I}(w), \mathtt{L}(w) \rangle$ where $\mathtt{S}(t)$, $\mathtt{I}(t)$ and $\mathtt{L}(t)$ are defined by induction as follows :

$$\begin{array}{lcl}
\mathtt{S}(x) & := & 1\\
\mathtt{S}(\lambda x.t) & := & \mathtt{S}(t)\\
\mathtt{S}(v\ w) & := & \mathtt{S}(v)+\mathtt{S}(w)\\
\mathtt{S}(\mathcal{W}_x(t)) & := & \mathtt{S}(t)\\
\mathtt{S}(\mathcal{C}^{y|z}_x(t)) & := & \mathtt{S}(t)\\
\mathtt{S}(t[x/u]) & := & \mathtt{S}(t)+\mathtt{M}_x(t).\mathtt{S}(u)
\end{array}
\qquad
\begin{array}{lcl}
\mathtt{L}(x) & := & 1\\
\mathtt{L}(\lambda x.t) & := & \mathtt{L}(t)\\
\mathtt{L}(t\ u) & := & \mathtt{L}(t)+\mathtt{L}(u)\\
\mathtt{L}(\mathcal{W}_x(t)) & := & \mathtt{L}(t)\\
\mathtt{L}(\mathcal{C}^{y|z}_x(t)) & := & \mathtt{L}(t)+1\\
\mathtt{L}(t[x/u]) & := & \mathtt{L}(t).(\mathtt{L}(u)+1)
\end{array}$$

$$\begin{array}{lcl}
\mathtt{I}(x) & := & 2\\
\mathtt{I}(\lambda x.t) & := & 2.\mathtt{I}(t)+2\\
\mathtt{I}(t\ u) & := & 2.(\mathtt{I}(t)+\mathtt{I}(u))+2\\
\mathtt{I}(\mathcal{W}_x(t)) & := & \mathtt{I}(t)+1\\
\mathtt{I}(\mathcal{C}^{y|z}_x(t)) & := & 2.\mathtt{I}(t)\\
\mathtt{I}(t[x/u]) & := & \mathtt{I}(t).(\mathtt{I}(u)+1)
\end{array}$$

with $\mathtt{M}_x(t)$ defined as follows :
If $x \notin \mathtt{fv}(t)$ then $\mathtt{M}_x(t) := 1$, otherwise :

$$\begin{array}{lcl}
\mathtt{M}_x(x) & := & 1\\
\mathtt{M}_x(\lambda y.t) & := & \mathtt{M}_x(t)\\
\mathtt{M}_x(t\ u) & := & \begin{cases} \mathtt{M}_x(t) & \text{if } x \in \mathtt{fv}(t)\setminus\mathtt{fv}(u)\\ \mathtt{M}_x(u) & \text{if } x \in \mathtt{fv}(u)\setminus\mathtt{fv}(t)\\ \mathtt{M}_x(t)+\mathtt{M}_x(u) & \text{if } x \in \mathtt{fv}(t)\cap\mathtt{fv}(u)\end{cases}\\
\mathtt{M}_x(\mathcal{W}_y(t)) & := & \begin{cases} 1 & \text{if } x = y\\ \mathtt{M}_x(t) & \text{if } x \neq y\end{cases}\\
\mathtt{M}_x(\mathcal{C}^{y_1|y_2}_y(t)) & := & \begin{cases} 1+\mathtt{M}_{y_1}(t)+\mathtt{M}_{y_2}(t) & \text{if } x = y\\ \mathtt{M}_x(t) & \text{if } x \neq y\end{cases}\\
\mathtt{M}_x(t[y/u]) & := & \begin{cases} \mathtt{M}_x(t)+\mathtt{M}_y(t).(\mathtt{M}_x(u)+1) & \text{if } x \in \mathtt{fv}(u)\cap\mathtt{fv}(t)\\ \mathtt{M}_y(t).(\mathtt{M}_x(u)+1) & \text{if } x \in \mathtt{fv}(u)\setminus\mathtt{fv}(t)\\ \mathtt{M}_x(t) & \text{otherwise}\end{cases}
\end{array}$$

We conclude this section by relating adding and removing resources :

Lemma 19. *Let $\emptyset \neq \mathcal{A} \subseteq \mathcal{R}$. If $t \in \mathcal{T}_{\mathcal{A}}$ is in $\mathcal{A}$-normal form then $\mathtt{w} \in \mathcal{A}$ implies $t \equiv_{\mathcal{A}} \mathcal{W}_{\mathtt{fv}(t)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t))}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(t)))$ and $\mathtt{w} \notin \mathcal{A}$ implies $t \equiv_{\mathcal{A}} \mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(t))$.*

Proof. By induction on $\mathtt{size}(t)$.

- If $t = x$, then $x = \mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(x))$ and $\mathtt{fv}(t)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t)) = \emptyset$
- If $t = \lambda x.u$, then we reason by cases.
  - $\mathtt{w} \in \mathcal{A}$. We know $u \equiv_{\mathcal{A}} \mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(u))}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(u)))$ by the i.h. But t is in $\mathcal{A}$-normal form, so $\mathtt{fv}(u)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(u)) \subseteq \{x\}$, otherwise it can be reduced by LW. Now, if $\mathtt{fv}(u)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(u)) = \emptyset$, then also $\mathtt{fv}(t)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t)) = \emptyset$ and the claim $t \equiv_{\mathcal{A}} \mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(\lambda x.u))$ immediately holds. Otherwise, $\mathtt{fv}(u)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(u)) = \{x\}$ and $t \equiv_{\mathcal{A}} \lambda x.\mathcal{W}_x(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(u))) = \mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(t))$.

– $\mathtt{w} \notin \mathcal{A}$. Then $\lambda x.u \equiv_{\mathcal{A}}$ (*i.h.*) $\lambda x.\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(u)) = \mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(\lambda x.u))$.

- If $t = u\ v$, then we reason by cases.
  - $\mathtt{w} \in \mathcal{A}$. Then,

    $$t \equiv_{\mathcal{A}} \mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(u))}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(u)))\ \mathcal{W}_{\mathtt{fv}(v)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(v))}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(v)))$$

    by the i.h. But t is an $\mathcal{A}$-normal form, thus $\mathtt{fv}(u) \setminus \mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(u)) = \mathtt{fv}(v)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(v)) = \emptyset$, (otherwise it could be reduced by $\mathtt{AW_l}$ or $\mathtt{AW_r}$). Hence, $\mathtt{fv}(t) = \mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t))$ and $t \equiv_{\mathcal{A}} \mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(u))\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(v))$. If $\mathtt{c} \in \mathcal{A}$ then $t \equiv_{\mathcal{A}} \mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(t))$ since $\mathtt{RR}_{\mathcal{A}}(u)$ and $\mathtt{RR}_{\mathcal{A}}(v)$ have no variable in common. If $\mathtt{c} \notin \mathcal{A}$ then $t \equiv_{\mathcal{A}} \mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(t))$ by definition of the function $\mathtt{AR}_{\mathcal{A}}(\_)$.
  - $\mathtt{w} \notin \mathcal{A}$. Then, $t \equiv_{\mathcal{A}} \mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(u))\ \mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(v))$ by i.h. We have $t \equiv_{\mathcal{A}} \mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(t))$ since $\mathtt{RR}_{\mathcal{A}}(u)$ and $\mathtt{RR}_{\mathcal{A}}(v)$ have no variable in common.

- If $t = \mathcal{W}_x(u)$, then $t \equiv_{\mathcal{A}} \mathcal{W}_x(\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(u))}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(u))))$ by the i.h. This last term is equal to $\mathcal{W}_{\mathtt{fv}(t)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t))}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(t)))$ since $x \in \mathtt{fv}(t)$ but $x \notin \mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t))$.

- If $t = \mathcal{C}_x^{y|z}(u)$, then $t \equiv_{\mathcal{A}} \mathcal{C}_x^{y|z}(\mathcal{W}_{\mathtt{fv}(u)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(u))}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(u))))$ by the i.h. We know also that $y, z \in \mathtt{fv}^+(u)$ since otherwise t could be reduced by $\mathtt{CW_2}$ or $\mathtt{CW_1}$. We now reason by cases.
  - $\mathtt{w} \in \mathcal{A}$. Since t is in $\mathcal{A}$-normal form, we have $\mathtt{fv}(u) \setminus \mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(u)) = \emptyset$, otherwise t could be reduced by $\mathtt{CW_2}$ or $\mathtt{CW_1}$. Thus we get $t \equiv_{\mathcal{A}} \mathcal{C}_x^{y|z}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(u)))$. But t is well-formed, so that $y, z \in \mathtt{fv}(u)$ and $x \notin \mathtt{fv}(u)$. Since $y, z \in \mathtt{fv}^+(u)$, then $y, z \in \mathtt{fv}^+(\mathtt{RR}_{\mathcal{A}}(u)) \subseteq \mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(u))$ and also $x \notin \mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(u))$.

    Since $\mathtt{c} \in \mathcal{A}$, then by definition $\mathtt{RR}_{\mathcal{A}}(t) = \mathtt{S}_x^{y,z}(\mathtt{del}_{y,z}(\mathtt{RR}_{\mathcal{A}}(u)))$, so that $x \in \mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t))$ and we get $\mathtt{fv}(t) = \mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t))$.

    Notice that $\mathtt{RR}_{\mathcal{A}}(u)$ can be neither a variable (otherwise t would not be well-formed) nor an abstraction (otherwise t could be reduced by $\mathtt{CL}$), so that $\mathtt{RR}_{\mathcal{A}}(u) = w\ v$, and thus $\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(u)) = \mathcal{C}_{\Phi}^{\Upsilon|\Psi}(R_{\Upsilon}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(w))\ R_{\Psi}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(v)))$ for $\Phi = \mathtt{fv}(w) \cap \mathtt{fv}(v)$ and Υ and Ψ fresh sets of variables.

    Hence, $t \equiv_{\mathcal{A}} \mathcal{C}_x^{y|z}(\mathcal{C}_{\Phi}^{\Upsilon|\Psi}(R_{\Upsilon}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(w))\ R_{\Psi}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(v))))$.

    Now it would suffice that $y \in \mathtt{fv}(w) \setminus \mathtt{fv}(v)$ and $z \in \mathtt{fv}(v) \setminus \mathtt{fv}(w)$ (the symmetric case is similar) to prove that this term is in fact:

    $$\begin{array}{ll}
\mathcal{C}_x^{y|z}(\mathcal{C}_{\Phi}^{\Upsilon|\Psi}(R_{\Upsilon}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(w))\ R_{\Psi}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(v)))) & = \\
\mathcal{C}_x^{y|z}(\mathcal{C}_{\Phi}^{\Upsilon|\Psi}(\mathtt{AR}_{\mathcal{A}}(R_{\Upsilon,y}^{\Phi,x}(R_x^y(w)))\ \mathtt{AR}_{\mathcal{A}}(R_{\Psi,z}^{\Phi,x}(R_x^z(v))))) & = \\
\mathcal{C}_x^{y|z}(\mathcal{C}_{\Phi}^{\Upsilon|\Psi}(R_{\Upsilon,y}^{\Phi,x}(\mathtt{AR}_{\mathcal{A}}(R_x^y(w)))\ R_{\Psi,z}^{\Phi,x}(\mathtt{AR}_{\mathcal{A}}(R_x^z(v))))) & = \\
\mathtt{AR}_{\mathcal{A}}(R_x^y(w)\ R_x^z(v)) & =_{L.\ 13:2} \\
\mathtt{AR}_{\mathcal{A}}(\mathtt{S}_x^{y,z}(\mathtt{RR}_{\mathcal{A}}(u))) & =_{L.\ 14:1} \\
\mathtt{AR}_{\mathcal{A}}(\mathtt{S}_x^{y,z}(\mathtt{del}_{y,z}(\mathtt{RR}_{\mathcal{A}}(u)))) & = \\
\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(t)) &
\end{array}$$

By well-formedness we know that $y, z \in \mathtt{fv}(w\ v)$.
Suppose that one of them, say y, is both in w and in v. Then $y \in \Phi$, so that

$$t \equiv_{\mathcal{A}} \mathcal{C}_x^{y|z}(\mathcal{C}_{\Phi',y}^{(\Upsilon',y')|(\Psi',y'')}(R_{\Upsilon}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(w)))\ R_{\Psi}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(v)))$$

which we can rearrange using $\equiv_{\mathtt{CC}_{\mathcal{A}}}$ into

$$t \equiv_{\mathcal{A}} \mathcal{C}_x^{y|y''}(\mathcal{C}_{\Phi',y}^{(\Upsilon',y')|(\Psi',z)}(R_{\Upsilon}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(w))\ R_{\Psi}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(v))))$$

if $z \in \mathtt{fv}(w) \setminus \mathtt{fv}(v)$, or into

$$t \equiv_{\mathcal{A}} \mathcal{C}_x^{y|y'}(\mathcal{C}_{\Phi',y}^{(\Upsilon',z)|(\Psi',y'')}(R_{\Upsilon}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(w))\ R_{\Psi}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(v))))$$

if $z \in \mathtt{fv}(v) \setminus \mathtt{fv}(w)$, or into

$$t \equiv_{\mathcal{A}} \mathcal{C}_x^{y|z}(\mathcal{C}_{\Phi'',y,z}^{(\Upsilon'',y',y'')|(\Psi'',z',z'')}(R_{\Upsilon}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(w))\ R_{\Psi}^{\Phi}(\mathtt{AR}_{\mathcal{A}}(v))))$$

if $z \in \mathtt{fv}(v) \cap \mathtt{fv}(w)$.
In the first (resp. second and third) case, t can be $\mathtt{CA_L}$ (resp. $\mathtt{CA_R}$ and ($\mathtt{CA_L}$ or $\mathtt{CA_R}$))-reduced on $\mathcal{C}_y^{y'|z}()$ (resp. $\mathcal{C}_y^{z|y''}()$ and ($\mathcal{C}_y^{y'|z'}()$ or $\mathcal{C}_z^{y''|z''}()$)). In both cases, it contradicts the fact that t is in $\mathcal{A}$-normal form. Hence, $y \notin \Phi$ (and similarly $z \notin \Phi$).
Now suppose that both y and z are on the same side, say in w. Then t can be $\mathtt{CA_L}$-reduced on $\mathcal{C}_x^{y|z}()$. Similarly, they cannot be both in v. Hence one of them is only in w, and the other is only in v, as required.

– $\mathtt{w} \notin \mathcal{A}$. Then, we have $y, z \in \mathtt{fv}(u)$, otherwise t could be reduced by $\mathtt{CGc}$. The reasoning is then similar to the previous case except that here $\mathtt{RR}_{\mathcal{A}}(u)$ cannot be a variable otherwise it would be $\mathtt{CGc}$-reducible; and $y, z \in \mathtt{RR}_{\mathcal{A}}(u)$ by the i.h. and the fact that $\mathtt{AR}_{\mathcal{A}}()$ preserves free variables.

To illustrate Lemma 19 let us consider the term $t = \mathcal{W}_w(\lambda x.\mathcal{C}_x^{y|z}(y\ z))$. Then, $\mathtt{RR}_{\{\mathtt{C},\mathtt{W}\}}(t) = \lambda x.x\ x$, $\mathtt{AR}_{\{\mathtt{C},\mathtt{W}\}}(\mathtt{RR}_{\{\mathtt{C},\mathtt{W}\}}(t)) = \lambda x.\mathcal{C}_x^{y|z}(y\ z)$. We can conclude since $\mathtt{fv}(t) \setminus \mathtt{fv}(\mathtt{RR}_{\{\mathtt{C},\mathtt{W}\}}(t)) = w$.

Corollary 20. *Let $\emptyset \neq \mathcal{A} \subseteq \mathcal{R}$. Then, the unique $\mathcal{A}$-normal form of $t \in \mathcal{T}_{\mathcal{A}}$ is $\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(t))$ if $\mathtt{w} \notin \mathcal{A}$, and $\mathcal{W}_{\mathtt{fv}(t)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t))}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(t)))$ if $\mathtt{w} \in \mathcal{A}$.*

Proof. Suppose $\mathtt{w} \in \mathcal{A}$. Termination of $\rightarrow_{\mathcal{A}}$ (Lemma 18) implies that there is t' in $\mathcal{A}$-normal form such that $t \rightarrow_{\mathcal{A}}^{*} t'$. By Lemma 7, $\mathtt{fv}(t) = \mathtt{fv}(t')$ and by Theorem 2, $\mathtt{RR}_{\mathcal{A}}(t) = \mathtt{RR}_{\mathcal{A}}(t')$. Since t' is in $\mathcal{A}$-normal form, then $t' \equiv_{\mathcal{A}} \mathcal{W}_{\mathtt{fv}(t')\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t'))}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(t')))$ by Lemma 19 and thus we have that $t' \equiv_{\mathcal{A}} \mathcal{W}_{\mathtt{fv}(t)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t))}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(t)))$. To show uniqueness, let us consider two $\mathcal{A}$-normal forms t'_1 and t'_2 of t. By the previous remark, both t'_1 and t'_2 are congruent to the term $\mathcal{W}_{\mathtt{fv}(t)\setminus\mathtt{fv}(\mathtt{RR}_{\mathcal{A}}(t))}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(t)))$ which concludes the case. The case $\mathtt{w} \notin \mathcal{A}$ is similar.

5. Untyped Properties

We first show PSN for all the calculi of the prismoid. The proof will be split in two different subcases, one for each base. This dissociation comes from the fact that redexes are erased by β-reduction in base $\mathfrak{B}_I$ while they are erased by $\mathtt{SGc}$ and/or $\mathtt{SW_1}$-reduction in base $\mathfrak{B}_E$.

Theorem 3 (PSN). *Let $\mathcal{B} \subseteq \mathcal{S}$ and $\mathcal{A} = \mathcal{B} \setminus \{\mathtt{s}\}$. If $t \in \mathcal{T}_\emptyset$ & $t \in \mathcal{SN}_\emptyset$, then $\mathtt{AR}_\mathcal{A}(t) \in \mathcal{SN}_\mathcal{B}$.*

Proof. There are three cases, one for $\mathfrak{B}_I$ and two subcases for $\mathfrak{B}_E$.

- Suppose $\mathtt{s} \notin \mathcal{B}$. We first show that $u \in \mathcal{T}_\mathcal{B}$ & $\mathtt{RR}_\mathcal{B}(u) \in \mathcal{SN}_\emptyset$ imply $u \in \mathcal{SN}_\mathcal{B}$. For that we apply Theorem 6 in the appendix with $\mathtt{A}_1 = \to_\beta$, $\mathtt{A}_2 = \to_{\mathcal{B}\setminus\beta}$, $\mathtt{A} = \to_\beta$ and $\mathcal{R} = \mathtt{RR}_\mathcal{B}(\_)$, using Theorem 2 and Lemma 18. Take $u = \mathtt{AR}_\mathcal{B}(t)$. Then $\mathtt{RR}_\mathcal{B}(\mathtt{AR}_\mathcal{B}(t)) =_{L.17} t \in \mathcal{SN}_\emptyset$ by hypothesis. Thus, $\mathtt{AR}_\mathcal{B}(t) \in \mathcal{SN}_\mathcal{B}$.

- Suppose $\mathcal{B} = \{\mathtt{s}\}$. The proof of $\mathtt{AR}_\mathtt{S}(t) = t \in \mathcal{SN}_\mathtt{S}$ follows a modular proof technique to show PSN of calculi with full composition which is completely developed in [Kes08]. Details concerning the $\mathtt{s}$-calculus can be found in [Ren08].

- Suppose $\mathtt{s} \in \mathcal{B}$. Then $\mathcal{B} = \{\mathtt{s}\} \cup \mathcal{A}$. We show that $u \in \mathcal{T}_\mathcal{B}$ & $\mathtt{RR}_\mathcal{A}(u) \in \mathcal{SN}_\mathtt{S}$ imply $u \in \mathcal{SN}_\mathcal{B}$. For that we apply Theorem 6 in the appendix with $\mathtt{A}_1 = \to_\mathtt{S}$, $\mathtt{A}_2 = \to_{\mathcal{B}\setminus\mathtt{S}}$, $\mathtt{A} = \to_\mathtt{S}$ and $\mathcal{R} = \mathtt{RR}_\mathcal{A}(\_)$, using Theorem 2 and Lemma 18.

  Now, take $u = \mathtt{AR}_\mathcal{A}(t)$. We have $\mathtt{RR}_\mathcal{A}(\mathtt{AR}_\mathcal{A}(t)) =_{L.17} t \in \mathcal{SN}_\emptyset$ by hypothesis and $t \in \mathcal{SN}_\mathtt{S}$ by the previous point. Thus, $\mathtt{AR}_\mathcal{A}(t) \in \mathcal{SN}_\mathcal{B}$.

Confluence of each calculus of the prismoid is based on that of the $\lambda_\emptyset$-calculus [Bar84]. For any $\mathcal{A} \subseteq \mathcal{R}$, consider $\mathtt{xc} : \mathcal{T}_{\{\mathtt{S}\}\cup\mathcal{A}} \mapsto \mathcal{T}_\mathcal{A}$ which replaces explicit by implicit substitution.

$$
\begin{array}{lll@{\qquad}lll}
\mathtt{xc}(y) & := & y & \mathtt{xc}(\mathcal{W}_y(t)) & := & \mathcal{W}_y(\mathtt{xc}(t)) \\
\mathtt{xc}(t\ u) & := & \mathtt{xc}(t)\ \mathtt{xc}(u) & \mathtt{xc}(\mathcal{C}_y^{y_1|y_2}(t)) & := & \mathcal{C}_y^{y_1|y_2}(\mathtt{xc}(t)) \\
\mathtt{xc}(\lambda y.t) & := & \lambda y.\mathtt{xc}(t) & \mathtt{xc}(t[y/u]) & := & \mathtt{xc}(t)\{y/\mathtt{xc}(u)\}
\end{array}
$$

Lemma 21. *Let $t \in \mathcal{T}_\mathcal{B}$. Then $t \to^*_\mathcal{B} \mathtt{xc}(t)$.*

Proof. By induction on $\mathtt{size}(t)$ using Lemma 9.

Lemma 22. *Let $t \in \mathcal{T}_\mathcal{B}$. Then $\mathtt{RR}_{\mathcal{B}\setminus\mathtt{S}}(\mathtt{xc}(t)) = \mathtt{xc}(\mathtt{RR}_{\mathcal{B}\setminus\mathtt{S}}(t))$.*

Proof. By induction on $\mathtt{size}(t)$ using Lemma 16.

Lemma 23. *Let $t \in \mathcal{T}_\mathtt{S}$. If $t \to_\mathtt{S} u$, then $\mathtt{xc}(t) \to^*_\beta \mathtt{xc}(u)$.*

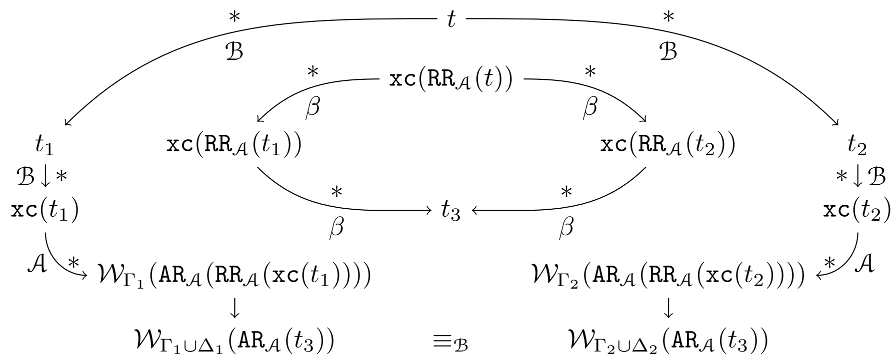


Figure 3: Confluence diagram

Proof. By induction on $t \rightarrow_{\mathtt{s}} u$ using the simplified (but equivalent) notion of substitution on $\mathtt{s}$-terms given in Section 2.

Theorem 4 (Confluence). *Every calculus $\lambda_{\mathcal{B}}$ of the prismoid is confluent modulo $\equiv_{\mathcal{B}}$.*

Proof. The proof is diagrammatically described in Figure 3.

Let $t \rightarrow^*_{\mathcal{B}} t_1$ and $t \rightarrow^*_{\mathcal{B}} t_2$. We remark that $\mathcal{B} = \mathcal{A}$ or $\mathcal{B} = \{\mathtt{s}\} \cup \mathcal{A}$, with $\mathcal{A} \subseteq \mathcal{R}$. We have $\mathtt{RR}_{\mathcal{A}}(t) \rightarrow^*_{\mathcal{B}\setminus\mathcal{A}} \mathtt{RR}_{\mathcal{A}}(t_i)$ (i=1,2) by Theorem 2. Furthermore $\mathtt{xc}(\mathtt{RR}_{\mathcal{A}}(t)) \rightarrow^*_{\beta} \mathtt{xc}(\mathtt{RR}_{\mathcal{A}}(t_i))$ (i=1,2) by Lemma 23 and $\mathtt{xc}(\mathtt{RR}_{\mathcal{A}}(t_i)) \rightarrow^*_{\beta} t_3$ (i=1,2) for some $t_3 \in \mathcal{T}_{\emptyset}$ by confluence of the λ-calculus [Bar84]. We also have $\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(\mathtt{xc}(t_i))) =_{L.\ 22} \mathtt{AR}_{\mathcal{A}}(\mathtt{xc}(\mathtt{RR}_{\mathcal{A}}(t_i))) \rightarrow^*_{\mathcal{A}} \mathcal{W}_{\Delta_i}(\mathtt{AR}_{\mathcal{A}}(t_3))$ for some Δ_i (i=1,2) by Theorem 1.

Lemmas 21 and Corollary 20 give $t_i \rightarrow^*_{\mathcal{B}} \mathtt{xc}(t_i) \rightarrow^*_{\mathcal{A}} \mathcal{W}_{\Gamma_i}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(\mathtt{xc}(t_i))))$ for some Γ_i (i=1,2). Then we get $\mathcal{W}_{\Gamma_i}(\mathtt{AR}_{\mathcal{A}}(\mathtt{RR}_{\mathcal{A}}(\mathtt{xc}(t_i)))) \rightarrow^*_{\mathcal{A}} \mathcal{W}_{\Gamma_i \cup \Delta_i}(\mathtt{AR}_{\mathcal{A}}(t_3))$ (i=1,2). Now, $\rightarrow^*_{\mathcal{A}} \subseteq \rightarrow^*_{\mathcal{B}}$ so in order to close the diagram we reason as follows.

If $\mathtt{w} \notin \mathcal{B}$, then $\Gamma_1 \cup \Delta_1 = \Gamma_2 \cup \Delta_2 = \emptyset$ and we are done. If $\mathtt{w} \in \mathcal{B}$, then $\rightarrow_{\mathcal{B}}$ preserves free variables by Lemma 7 so that $\mathtt{fv}(t) = \mathtt{fv}(t_i) = \mathtt{fv}(\mathcal{W}_{\Gamma_i \cup \Delta_i}(\mathtt{AR}_{\mathcal{A}}(t_3)))$ (i=1,2) which gives $\Gamma_1 \cup \Delta_1 = \Gamma_2 \cup \Delta_2$.

6. Typing

We now introduce **simply typed terms** for all the calculi of the prismoid, and show that they all enjoy strong normalisation. **Types** are built over a countable set of atomic symbols and the type constructor $\rightarrow$.

An **environment** is a finite set of pairs of the form $x : T$. If $\Gamma = \{x_1 : T_1, ..., x_n : T_n\}$ is an environment then the domain of Γ is $\mathtt{dom}(\Gamma) = \{x_1, ..., x_n\}$. The **renaming of an environment** is the renaming of its domain. Thus for example $R^{x,y}_{x',y'}(x : A, y : B) = x' : A, y' : B$. Two environments Γ and Δ are said to be **compatible** if $x : T \in \Gamma$ and $x : U \in \Delta$ imply $T = U$. Two environments Γ and Δ are said to be **disjoint** if there is no common variable

$$\frac{}{x : T \vdash_{\mathcal{B}} x : T}$$

$$\frac{\Gamma \vdash_{\mathcal{B}} t : U}{\Gamma \setminus\!\!\setminus_{\mathcal{B}} x : T \vdash_{\mathcal{B}} \lambda x.t : T \to U} \qquad \frac{\Gamma \vdash_{\mathcal{B}} u : U \quad \Delta \vdash_{\mathcal{B}} t : T}{\Gamma \uplus_{\mathcal{B}} (\Delta \setminus\!\!\setminus_{\mathcal{B}} x : U) \vdash_{\mathcal{B}} t[x/u] : T} (\mathtt{s} \in \mathcal{B})$$

$$\frac{\Gamma \vdash_{\mathcal{B}} t : T \to U \quad \Delta \vdash_{\mathcal{B}} u : T}{\Gamma \uplus_{\mathcal{B}} \Delta \vdash_{\mathcal{B}} tu : U} \qquad \frac{\Gamma \vdash_{\mathcal{B}} t : T}{\Gamma; x : U \vdash_{\mathcal{B}} \mathcal{W}_x(t) : T} (\mathtt{w} \in \mathcal{B})$$

$$\frac{\Gamma \vdash_{\mathcal{B}} t : T}{x : U; (\Gamma \setminus\!\!\setminus_{\mathcal{B}} \{y : U, z : U\}) \vdash_{\mathcal{B}} \mathcal{C}_x^{y|z}(t) : T} (\mathtt{c} \in \mathcal{B})$$

Figure 4: Typing rules

in their environments. **Compatible union** (**resp. disjoint union**) is defined to be the union of compatible (resp. disjoint) environments.

Typing judgements have the form $\Gamma \vdash t : T$ for t a term, T a type and Γ an environment. **Typing rules** described in Figure 6 extend the inductive rules for well-formed terms (Section 2) with type annotations. Thus, typed terms are necessarily well-formed and each set of sorts $\mathcal{B}$ has its own set of typing rules.

A term $t \in \mathcal{T}_{\mathcal{B}}$ **has type** T (written $t \in \mathcal{T}_{\mathcal{B}}^T$) iff there is Γ s.t. $\Gamma \vdash_{\mathcal{B}} t : T$. A term $t \in \mathcal{T}_{\mathcal{B}}$ is said to be **well-typed** iff there is a type T s.t. $t \in \mathcal{T}_{\mathcal{B}}^T$.

Lemma 24. *If $\Gamma \vdash_{\mathcal{B}} t : T$, then*

1. $\mathtt{fv}(t) = \mathtt{dom}(\Gamma)$,
2. $\Lambda; R_{\mathcal{S}}^{\mathtt{dom}(\Pi)}(\Pi) \vdash_{\mathcal{B}} R_{\mathcal{S}}^{\mathtt{dom}(\Pi)}(t) : T$, *where* $\Gamma = \Lambda; \Pi$ *and* $\mathcal{S}$ *is a fresh set of variables.*
3. $\mathtt{RR}_{\mathcal{A}}(t) \in \mathcal{T}_{\mathcal{B}\setminus\mathcal{A}}^T$, *for every* $\mathcal{A} \subseteq \mathcal{R}$.

Proof. By induction on $\Gamma \vdash_{\mathcal{B}} t : T$.

Theorem 5 (Subject Reduction). *If $t \in \mathcal{T}_{\mathcal{B}}^T$ & $t \to_{\mathcal{B}} u$, then $u \in \mathcal{T}_{\mathcal{B}}^T$.*

Proof. By induction on the reduction relation using Lemma 24. The proof is very similar to that of Lemma 7.

We consider the case where $\mathcal{C}_x^{y|z}(s)[x/v] \to_{\mathtt{SCa}} \mathcal{C}_{\Gamma}^{\Delta|\Pi}(s[y/R_{\Delta}^{\Gamma}(v)][z/R_{\Pi}^{\Gamma}(v)])$, with $\Gamma = \mathtt{fv}(u)$ & Δ, Π fresh. Since $\mathtt{c} \in \mathcal{B}$ we know that $\uplus_{\mathcal{B}}$ is disjoint union so that the type derivation of t looks like:

$$\frac{\Gamma \vdash v : C \qquad \dfrac{\Lambda \vdash s : T}{x : C; \Lambda \setminus\!\!\setminus_{\mathcal{B}} \{y : C, z : C\} \vdash \mathcal{C}_x^{y|z}(s) : T}}{\Gamma; (\Lambda \setminus\!\!\setminus_{\mathcal{B}} \{y : C, z : C\}) \vdash \mathcal{C}_x^{y|z}(s)[x/v] : T}$$

We then construct the following type derivation:

$$
\dfrac{\dfrac{\Gamma \vdash v : C}{\Pi \vdash R^{\Gamma}_{\Pi}(v) : C} \qquad \dfrac{\dfrac{\Gamma \vdash v : C}{\Delta \vdash R^{\Gamma}_{\Delta}(v) : C} \qquad \Lambda \vdash s : T}{\Delta; (\Lambda \mathbin{\backslash\!\!\backslash_{\mathcal{B}}} y : C) \vdash s[y/R^{\Gamma}_{\Delta}(v)] : T}}{\dfrac{\Pi; \Delta; ((\Lambda \mathbin{\backslash\!\!\backslash_{\mathcal{B}}} y : C) \mathbin{\backslash\!\!\backslash_{\mathcal{B}}} z : C) \vdash s[y/R^{\Gamma}_{\Delta}(v)][z/R^{\Gamma}_{\Pi}(v)] : T}{\overline{\Gamma; (\Lambda \mathbin{\backslash\!\!\backslash_{\mathcal{B}}} y : C \mathbin{\backslash\!\!\backslash_{\mathcal{B}}} z : C) \vdash \mathcal{C}^{\Delta|\Pi}_{\Gamma}(s[y/R^{\Gamma}_{\Delta}(v)][z/R^{\Gamma}_{\Pi}(v)]) : T}}}
$$

We conclude since $\Lambda \mathbin{\backslash\!\!\backslash_{\mathcal{B}}} \{y : C, z : C\} = \Lambda \mathbin{\backslash\!\!\backslash_{\mathcal{B}}} y : C \mathbin{\backslash\!\!\backslash_{\mathcal{B}}} z : C$.

Corollary 25 (Strong Normalisation). *Let* $t \in \mathcal{T}^{T}_{\mathcal{B}}$*, then* $t \in \mathcal{SN}_{\mathcal{B}}$.

Proof. Let $\mathcal{A} \subseteq \mathcal{R}$ so that $\mathcal{B} = \mathcal{A}$ or $\mathcal{B} = \mathcal{A} \cup \{\mathtt{s}\}$. It is well-known that (simply) typed $\lambda_{\emptyset}$-calculus is strongly normalising (see for example [Bar84]). It is also straightforward to show that PSN for the $\lambda_{\mathtt{S}}$-calculus implies strong normalisation for well-typed $\mathtt{s}$-terms (see for example [Kes07]). By Theorem 2 any infinite $\mathcal{B}$-reduction sequence starting at t can be projected into an infinite $(\mathcal{B} \setminus \mathcal{A})$-reduction sequence starting at $\mathtt{RR}_{\mathcal{A}}(t)$. By Lemma 24 $\mathtt{RR}_{\mathcal{A}}(t)$ is a well-typed $(\mathcal{B} \setminus \mathcal{A})$-term, that is, a well-typed term in $\lambda_{\emptyset}$ or $\lambda_{\mathtt{S}}$. This leads to a contradiction.

7. Conclusion and Future Work

The prismoid of resources is an homogeneous framework to define λ-calculi being able to control weakening, contraction and linear substitution. The formalism is based on MELL Proof-Nets so that the computational behaviour of substitution is not only based on the propagation of substitution through terms but also on the decreasingness of the multiplicity of variables that are affected by substitutions. All calculi of the prismoid enjoy sanity properties such as simulation of β-reduction, confluence, preservation of β-strong normalisation and strong normalisation for typed terms.

The technology used in the prismoid could also be applied to implement higher-order rewriting systems. Indeed, it seems possible to extend these ideas to different frameworks such as CRSs [Klo80], ERSs [Kha90] or HRSs [Nip91].

Another open problem concerns meta-confluence, that is, confluence for terms with meta-variables. This could be useful in the framework of Proof Assistants.

Finally, a more technical question is related to the operational semantics of the calculi of the prismoid. It seems possible to extend the ideas in [AG09] to our framework in order to identify those reduction rules of the prismoid that could be transformed into equations. Equivalence classes will be bigger, but reduction rules will coincide exactly with those of the graphical formalism in [AG09].

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A. Appendix

Theorem 6. *Let* $\mathtt{A}_1$ *and* $\mathtt{A}_2$ *(resp.* $\mathcal{E}$*) be two reduction (resp. equivalence) relations on* $\mathtt{s}$*. Let* $\mathtt{A}$ *be a reduction relation on* $\mathtt{S}$ *and let consider a relation* $\mathcal{R} \subseteq \mathtt{s} \times \mathtt{S}$*. Suppose that for all* u, v, U

(P0) $u \; \mathcal{R} \; U$ & $u \; \mathcal{E} \; v$ *imply* $\exists V$ *s.t.* $v \; R \; V$ & $U \; = V$.

(P1) $u \; \mathcal{R} \; U$ & $u \; \mathtt{A}_1 \; v$ *imply* $\exists V$ *s.t.* $v \; \mathcal{R} \; V$ & $U \; \mathtt{A}^* \; V$.

(P2) $u \; \mathcal{R} \; U$ & $u \; \mathtt{A}_2 \; v$ *imply* $\exists V$ *s.t.* $v \; \mathcal{R} \; V$ & $U \; \mathtt{A}^+ \; V$.

(P3) *The relation* $\mathtt{A}_1$ *modulo* $\mathcal{E}$ *is well-founded.*

Then, $t \; \mathcal{R} \; T$ & $T \in \mathcal{SN}_{\mathtt{A}}$ *imply* $t \in \mathcal{SN}_{(\mathtt{A}_1 \cup \mathtt{A}_2)/\mathcal{E}}$.

Proof. A proof by contradiction can be easily done as follows. Suppose $t \notin \mathcal{SN}_{(\mathtt{A}_1 \cup \mathtt{A}_2)/\mathcal{E}}$. Then, there is an infinite $(\mathtt{A}_1 \cup \mathtt{A}_2)/\mathcal{E}$ -reduction sequence starting at t, and since $\mathtt{A}_1/\mathcal{E}$ is a well-founded relation by P3, this reduction sequence has necessarily the form

$$t(\mathtt{A}_1/\mathcal{E})^* t_1 (\mathtt{A}_2/\mathcal{E})^+ t_2 (\mathtt{A}_1/\mathcal{E})^* t_3 (\mathtt{A}_2/\mathcal{E})^+ \ldots \infty$$

and can be projected by P0, P1 and P2 into an infinite $\mathtt{A}$-reduction sequence as follows:

$$\begin{array}{llllllll} t_1 & (\mathtt{A}_1/\mathcal{E})^* & t_2 & (\mathtt{A}_2/\mathcal{E})^+ & t_3 & (\mathtt{A}_1/\mathcal{E})^* & \ldots & \infty \\ T_1 & \mathtt{A}^* & T_2 & \mathtt{A}^+ & T_3 & \mathtt{A}^* & \ldots & \infty \end{array}$$

We thus get a contradiction with the fact the $T \in \mathcal{SN}_{\mathtt{A}}$.

B. The $\boldsymbol{\lambda_\emptyset}$-calculus

Rules :

$$(\beta) \quad (\lambda x.t) \; u \quad \rightarrow \quad t\{x/u\}$$

C. The $\boldsymbol{\lambda}_{\mathtt{C}}$-calculus

Equations :

$(\mathtt{CC}_{\mathcal{A}})$	$\mathcal{C}_w^{x\|z}(\mathcal{C}_x^{y\|p}(t))$	$\equiv$	$\mathcal{C}_w^{x\|y}(\mathcal{C}_x^{z\|p}(t))$	
$(\mathtt{C}_{\mathcal{C}})$	$\mathcal{C}_x^{y\|z}(t)$	$\equiv$	$\mathcal{C}_x^{z\|y}(t)$	
$(\mathtt{CC}_{\mathcal{C}})$	$\mathcal{C}_{x'}^{y'\|z'}(\mathcal{C}_x^{y\|z}(t))$	$\equiv$	$\mathcal{C}_x^{y\|z}(\mathcal{C}_{x'}^{y'\|z'}(t))$	$x \neq y', z'$ & $x' \neq y, z$

Rules :

(β)	$(\lambda x.t)\ u$	$\rightarrow$	$t\{x/u\}$	
$(\mathtt{CL})$	$\mathcal{C}_w^{y\|z}(\lambda x.t)$	$\rightarrow$	$\lambda x.\mathcal{C}_w^{y\|z}(t)$	
$(\mathtt{CA}_{\mathtt{L}})$	$\mathcal{C}_w^{y\|z}(t\ u)$	$\rightarrow$	$\mathcal{C}_w^{y\|z}(t)\ u$	$y, z \notin fv(u)$
$(\mathtt{CA}_{\mathtt{R}})$	$\mathcal{C}_w^{y\|z}(t\ u)$	$\rightarrow$	$t\ \mathcal{C}_w^{y\|z}(u)$	$y, z \notin fv(t)$
$(\mathtt{CGc})$	$\mathcal{C}_w^{y\|z}(t)$	$\rightarrow$	$R_w^z(t)$	$y \notin fv(t)$

D. The $\boldsymbol{\lambda}_{\mathtt{S}}$-calculus

Equations :

$(\mathtt{SS}_{\mathcal{C}})$	$t[x/u][y/v]$	$\equiv$	$t[y/v][x/u]$	$y \notin fv(u)$ & $x \notin fv(v)$

Rules :

$(\mathtt{B})$	$(\lambda x.t)\ u$	$\rightarrow$	$t[x/u]$	
$(\mathtt{V})$	$x[x/u]$	$\rightarrow$	u	
$(\mathtt{SGc})$	$t[x/u]$	$\rightarrow$	t	$x \notin fv(t)$
$(\mathtt{SDup})$	$t[x/u]$	$\rightarrow$	$t_{[y]_x}[x/u][y/u]$	$\|t\|_x > 1$ & y fresh
$(\mathtt{SL})$	$(\lambda y.t)[x/u]$	$\rightarrow$	$\lambda y.t[x/u]$	
$(\mathtt{SA}_{\mathtt{L}})$	$(t\ v)[x/u]$	$\rightarrow$	$t[x/u]\ v$	$x \notin fv(v)$
$(\mathtt{SA}_{\mathtt{R}})$	$(t\ v)[x/u]$	$\rightarrow$	$t\ v[x/u]$	$x \notin fv(t)$
$(\mathtt{SS})$	$t[y/v][x/u]$	$\rightarrow$	$t[y/v[x/u]]$	$x \notin fv(t)$ & $x \in fv(v)$

E. The $\boldsymbol{\lambda}_{\mathtt{W}}$-calculus

Equations :

$(\mathtt{WW}_{\mathcal{C}})$	$\mathcal{W}_x(\mathcal{W}_y(t))$	$\equiv$	$\mathcal{W}_y(\mathcal{W}_x(t))$

Rules :

(β)	$(\lambda x.t)\ u$	$\rightarrow$	$t\{x/u\}$	
$(\mathtt{LW})$	$\lambda x.\mathcal{W}_y(t)$	$\rightarrow$	$\mathcal{W}_y(\lambda x.t)$	$x \neq y$
$(\mathtt{AW_l})$	$\mathcal{W}_y(u)v$	$\rightarrow$	$\mathcal{W}_{y \setminus \mathtt{fv}(v)}(uv)$	
$(\mathtt{AW_r})$	$u\mathcal{W}_y(v)$	$\rightarrow$	$\mathcal{W}_{y \setminus \mathtt{fv}(u)}(uv)$	

F. The $\lambda_{\mathtt{CS}}$-calculus

Equations :

$(\mathtt{CC}_{\mathcal{A}})$	$\mathcal{C}^{x\mid z}_{w}(\mathcal{C}^{y\mid p}_{x}(t))$	$\equiv$	$\mathcal{C}^{x\mid y}_{w}(\mathcal{C}^{z\mid p}_{x}(t))$	
$(\mathtt{C}_{\mathcal{C}})$	$\mathcal{C}^{y\mid z}_{x}(t)$	$\equiv$	$\mathcal{C}^{z\mid y}_{x}(t)$	
$(\mathtt{CC}_{\mathcal{C}})$	$\mathcal{C}^{y'\mid z'}_{x'}(\mathcal{C}^{y\mid z}_{x}(t))$	$\equiv$	$\mathcal{C}^{y\mid z}_{x}(\mathcal{C}^{y'\mid z'}_{x'}(t))$	$x \neq y', z'$ & $x' \neq y, z$
$(\mathtt{SS}_{\mathcal{C}})$	$t[x/u][y/v]$	$\equiv$	$t[y/v][x/u]$	$y \notin \mathtt{fv}(u)$ & $x \notin \mathtt{fv}(v)$

Rules :

$(\mathtt{B})$	$(\lambda x.t)\ u$	$\rightarrow$	$t[x/u]$	
$(\mathtt{CL})$	$\mathcal{C}^{y\mid z}_{w}(\lambda x.t)$	$\rightarrow$	$\lambda x.\mathcal{C}^{y\mid z}_{w}(t)$	
$(\mathtt{CA}_{\mathtt{L}})$	$\mathcal{C}^{y\mid z}_{w}(tu)$	$\rightarrow$	$\mathcal{C}^{y\mid z}_{w}(t)u$	$y, z \notin \mathtt{fv}(u)$
$(\mathtt{CA}_{\mathtt{R}})$	$\mathcal{C}^{y\mid z}_{w}(tu)$	$\rightarrow$	$t\mathcal{C}^{y\mid z}_{w}(u)$	$y, z \notin \mathtt{fv}(t)$
$(\mathtt{CGc})$	$\mathcal{C}^{y\mid z}_{w}(t)$	$\rightarrow$	$R^{z}_{w}(t)$	$y \notin \mathtt{fv}(t)$
$(\mathtt{V})$	$x[x/u]$	$\rightarrow$	u	
$(\mathtt{SGc})$	$t[x/u]$	$\rightarrow$	t	$x \notin \mathtt{fv}(t)$
$(\mathtt{SL})$	$(\lambda y.t)[x/u]$	$\rightarrow$	$\lambda y.t[x/u]$	
$(\mathtt{SA}_{\mathtt{L}})$	$(tv)[x/u]$	$\rightarrow$	$t[x/u]v$	$x \notin \mathtt{fv}(v)$
$(\mathtt{SA}_{\mathtt{R}})$	$(tv)[x/u]$	$\rightarrow$	$tv[x/u]$	$x \notin \mathtt{fv}(t)$
$(\mathtt{SS})$	$t[x/u][y/v]$	$\rightarrow$	$t[x/u[y/v]]$	$y \notin \mathtt{fv}(t)$ & $y \in \mathtt{fv}^{+}(u)$
$(\mathtt{SCa})$	$\mathcal{C}^{y\mid z}_{x}(t)[x/u]$	$\rightarrow$	$\mathcal{C}^{\Delta\mid\Pi}_{\Gamma}(t[y/R^{\Gamma}_{\Delta}(u)][z/R^{\Gamma}_{\Pi}(u)])$	$\begin{cases} y, z \in \mathtt{fv}^{+}(t) \\ \Gamma = \mathtt{fv}(u) \\ \Delta, \Pi \text{ fresh} \end{cases}$
$(\mathtt{CS})$	$\mathcal{C}^{y\mid z}_{w}(t[x/u])$	$\rightarrow$	$t[x/\mathcal{C}^{y\mid z}_{w}(u)]$	$y, z \in \mathtt{fv}^{+}(u)$
$(\mathtt{SCb})$	$\mathcal{C}^{y\mid z}_{w}(t)[x/u]$	$\rightarrow$	$\mathcal{C}^{y\mid z}_{w}(t[x/u])$	$x \neq w$ & $y, z \notin \mathtt{fv}(u)$

G. The $\boldsymbol{\lambda}_{\texttt{CW}}$-calculus

Equations :

$(\texttt{CC}_{\mathcal{A}})$	$\mathcal{C}_w^{x\|z}(\mathcal{C}_x^{y\|p}(t))$	$\equiv$	$\mathcal{C}_w^{x\|y}(\mathcal{C}_x^{z\|p}(t))$	
$(\texttt{C}_{\mathcal{C}})$	$\mathcal{C}_x^{y\|z}(t)$	$\equiv$	$\mathcal{C}_x^{z\|y}(t)$	
$(\texttt{CC}_{\mathcal{C}})$	$\mathcal{C}_{x'}^{y'\|z'}(\mathcal{C}_x^{y\|z}(t))$	$\equiv$	$\mathcal{C}_x^{y\|z}(\mathcal{C}_{x'}^{y'\|z'}(t))$	$x \neq y', z'$ & $x' \neq y, z$
$(\texttt{WW}_{\mathcal{C}})$	$\mathcal{W}_x(\mathcal{W}_y(t))$	$\equiv$	$\mathcal{W}_y(\mathcal{W}_x(t))$	

Rules :

(β)	$(\lambda x.t)\ u$	$\rightarrow$	$t\{x/u\}$	
$(\texttt{LW})$	$\lambda x.\mathcal{W}_y(t)$	$\rightarrow$	$\mathcal{W}_y(\lambda x.t)$	$x \neq y$
$(\texttt{AW}_\texttt{l})$	$\mathcal{W}_y(u)v$	$\rightarrow$	$\mathcal{W}_{y\setminus\texttt{fv}(v)}(uv)$	
$(\texttt{AW}_\texttt{r})$	$u\mathcal{W}_y(v)$	$\rightarrow$	$\mathcal{W}_{y\setminus\texttt{fv}(u)}(uv)$	
$(\texttt{CL})$	$\mathcal{C}_w^{y\|z}(\lambda x.t)$	$\rightarrow$	$\lambda x.\mathcal{C}_w^{y\|z}(t)$	
$(\texttt{CA}_\texttt{L})$	$\mathcal{C}_w^{y\|z}(tu)$	$\rightarrow$	$\mathcal{C}_w^{y\|z}(t)u$	$y, z \notin \texttt{fv}(u)$
$(\texttt{CA}_\texttt{R})$	$\mathcal{C}_w^{y\|z}(tu)$	$\rightarrow$	$t\mathcal{C}_w^{y\|z}(u)$	$y, z \notin \texttt{fv}(t)$
$(\texttt{CW}_1)$	$\mathcal{C}_w^{y\|z}(\mathcal{W}_y(t))$	$\rightarrow$	$R_w^z(t)$	
$(\texttt{CW}_2)$	$\mathcal{C}_w^{y\|z}(\mathcal{W}_x(t))$	$\rightarrow$	$\mathcal{W}_x(\mathcal{C}_w^{y\|z}(t))$	$x \neq y, z$
$(\texttt{CGc})$	$\mathcal{C}_w^{y\|z}(t)$	$\rightarrow$	$R_w^z(t)$	$y \notin \texttt{fv}(t)$

H. The $\boldsymbol{\lambda}_{\texttt{SW}}$-calculus

Equations :

$(\texttt{WW}_{\mathcal{C}})$	$\mathcal{W}_x(\mathcal{W}_y(t))$	$\equiv$	$\mathcal{W}_y(\mathcal{W}_x(t))$	
$(\texttt{SS}_{\mathcal{C}})$	$t[x/u][y/v]$	$\equiv$	$t[y/v][x/u]$	$y \notin \texttt{fv}(u)$ & $x \notin \texttt{fv}(v)$

Rules :

$(\texttt{B})$	$(\lambda x.t)\ u$	$\rightarrow$	$t[x/u]$	
$(\texttt{LW})$	$\lambda x.\mathcal{W}_y(t)$	$\rightarrow$	$\mathcal{W}_y(\lambda x.t)$	$x \neq y$
$(\texttt{AW}_\texttt{l})$	$\mathcal{W}_y(u)v$	$\rightarrow$	$\mathcal{W}_{y\setminus\texttt{fv}(v)}(uv)$	
$(\texttt{AW}_\texttt{r})$	$u\mathcal{W}_y(v)$	$\rightarrow$	$\mathcal{W}_{y\setminus\texttt{fv}(u)}(uv)$	
$(\texttt{V})$	$x[x/u]$	$\rightarrow$	u	
$(\texttt{SGc})$	$t[x/u]$	$\rightarrow$	t	$x \notin fv(t)$
$(\texttt{SDup})$	$t[x/u]$	$\rightarrow$	$t_{[y]_x}[x/u][y/u]$	$\|t\|_x^+ > 1$ & y fresh
$(\texttt{SL})$	$(\lambda y.t)[x/u]$	$\rightarrow$	$\lambda y.t[x/u]$	
$(\texttt{SA}_\texttt{L})$	$(t\ v)[x/u]$	$\rightarrow$	$t[x/u]\ v$	$x \notin fv(v)$
$(\texttt{SA}_\texttt{R})$	$(t\ v)[x/u]$	$\rightarrow$	$t\ v[x/u]$	$x \notin fv(t)$
$(\texttt{SS})$	$t[y/v][x/u]$	$\rightarrow$	$t[y/v[x/u]]$	$x \notin fv(t)$ & $x \in fv(v)$
$(\texttt{SW}_1)$	$\mathcal{W}_x(t)[x/u]$	$\rightarrow$	$\mathcal{W}_{\texttt{fv}(u)\setminus\texttt{fv}(t)}(t)$	
$(\texttt{SW}_2)$	$\mathcal{W}_y(t)[x/u]$	$\rightarrow$	$\mathcal{W}_{y\setminus\texttt{fv}(u)}(t[x/u])$	$x \neq y$
$(\texttt{SW})$	$t[x/\mathcal{W}_y(u)]$	$\rightarrow$	$\mathcal{W}_{y\setminus\texttt{fv}(t)}(t[x/u])$	

I. The $\lambda_{\texttt{CSW}}$-calculus

Equations :

$(\texttt{CC}_{\mathcal{A}})$	$\mathcal{C}_w^{x\vert z}(\mathcal{C}_x^{y\vert p}(t))$	$\equiv$	$\mathcal{C}_w^{x\vert y}(\mathcal{C}_x^{z\vert p}(t))$	
$(\texttt{C}_{\mathcal{C}})$	$\mathcal{C}_x^{y\vert z}(t)$	$\equiv$	$\mathcal{C}_x^{z\vert y}(t)$	
$(\texttt{CC}_{\mathcal{C}})$	$\mathcal{C}_{x'}^{y'\vert z'}(\mathcal{C}_x^{y\vert z}(t))$	$\equiv$	$\mathcal{C}_x^{y\vert z}(\mathcal{C}_{x'}^{y'\vert z'}(t))$	$x \neq y', z'$ & $x' \neq y, z$
$(\texttt{WW}_{\mathcal{C}})$	$\mathcal{W}_x(\mathcal{W}_y(t))$	$\equiv$	$\mathcal{W}_y(\mathcal{W}_x(t))$	
$(\texttt{SS}_{\mathcal{C}})$	$t[x/u][y/v]$	$\equiv$	$t[y/v][x/u]$	$y \notin \texttt{fv}(u)$ & $x \notin \texttt{fv}(v)$

Rules :

$(\texttt{B})$	$(\lambda x.t)\ u$	$\rightarrow$	$t[x/u]$	
$(\texttt{V})$	$x[x/u]$	$\rightarrow$	u	
$(\texttt{SDup})$	$t[x/u]$	$\rightarrow$	$t_{[y]_x}[x/u][y/u]$	$\vert t\vert_x^+ > 1$ & y fresh
$(\texttt{SL})$	$(\lambda y.t)[x/u]$	$\rightarrow$	$\lambda y.t[x/u]$	
$(\texttt{SA}_{\texttt{L}})$	$(tv)[x/u]$	$\rightarrow$	$t[x/u]v$	$x \notin \texttt{fv}(v)$
$(\texttt{SA}_{\texttt{R}})$	$(tv)[x/u]$	$\rightarrow$	$tv[x/u]$	$x \notin \texttt{fv}(t)$
$(\texttt{SS})$	$t[x/u][y/v]$	$\rightarrow$	$t[x/u[y/v]]$	$y \notin \texttt{fv}(t)$ & $y \in \texttt{fv}^+(u)$
$(\texttt{SW}_1)$	$\mathcal{W}_x(t)[x/u]$	$\rightarrow$	$\mathcal{W}_{\texttt{fv}(u)\setminus\texttt{fv}(t)}(t)$	
$(\texttt{SW}_2)$	$\mathcal{W}_y(t)[x/u]$	$\rightarrow$	$\mathcal{W}_{y\setminus\texttt{fv}(u)}(t[x/u])$	$x \neq y$
$(\texttt{LW})$	$\lambda x.\mathcal{W}_y(t)$	$\rightarrow$	$\mathcal{W}_y(\lambda x.t)$	$x \neq y$
$(\texttt{AW}_1)$	$\mathcal{W}_y(u)v$	$\rightarrow$	$\mathcal{W}_{y\setminus\texttt{fv}(v)}(uv)$	
$(\texttt{AW}_{\texttt{r}})$	$u\mathcal{W}_y(v)$	$\rightarrow$	$\mathcal{W}_{y\setminus\texttt{fv}(u)}(uv)$	
$(\texttt{SW})$	$t[x/\mathcal{W}_y(u)]$	$\rightarrow$	$\mathcal{W}_{y\setminus\texttt{fv}(t)}(t[x/u])$	
$(\texttt{SCa})$	$\mathcal{C}_x^{y\vert z}(t)[x/u]$	$\rightarrow$	$\mathcal{C}_{\Gamma}^{\Delta\vert\Pi}(t[y/R_{\Delta}^{\Gamma}(u)][z/R_{\Pi}^{\Gamma}(u)])$	$\begin{cases} y, z \in \texttt{fv}^+(t) \\ \Gamma = \texttt{fv}(u) \\ \Delta, \Pi \text{ fresh} \end{cases}$
$(\texttt{CL})$	$\mathcal{C}_w^{y\vert z}(\lambda x.t)$	$\rightarrow$	$\lambda x.\mathcal{C}_w^{y\vert z}(t)$	
$(\texttt{CA}_{\texttt{L}})$	$\mathcal{C}_w^{y\vert z}(tu)$	$\rightarrow$	$\mathcal{C}_w^{y\vert z}(t)u$	$y, z \notin \texttt{fv}(u)$
$(\texttt{CA}_{\texttt{R}})$	$\mathcal{C}_w^{y\vert z}(tu)$	$\rightarrow$	$t\mathcal{C}_w^{y\vert z}(u)$	$y, z \notin \texttt{fv}(t)$
$(\texttt{CS})$	$\mathcal{C}_w^{y\vert z}(t[x/u])$	$\rightarrow$	$t[x/\mathcal{C}_w^{y\vert z}(u)]$	$y, z \in \texttt{fv}^+(u)$
$(\texttt{SCb})$	$\mathcal{C}_w^{y\vert z}(t)[x/u]$	$\rightarrow$	$\mathcal{C}_w^{y\vert z}(t[x/u])$	$x \neq w$ & $y, z \notin \texttt{fv}(u)$
$(\texttt{CW}_1)$	$\mathcal{C}_w^{y\vert z}(\mathcal{W}_y(t))$	$\rightarrow$	$R_w^z(t)$	
$(\texttt{CW}_2)$	$\mathcal{C}_w^{y\vert z}(\mathcal{W}_x(t))$	$\rightarrow$	$\mathcal{W}_x(\mathcal{C}_w^{y\vert z}(t))$	$x \neq y, z$