Term Algebras

Signatures and finite terms

Σ : Set of function symbols having an arity $n \in \mathbb{N}$.

$\mathcal{X}$: Set of variables.

$\mathcal{T}(\mathcal{X}, \Sigma)$: Set of terms over $\mathcal{X}$ and Σ :

$$\frac{x \in \mathcal{X}}{x \in \mathcal{T}(\mathcal{X}, \Sigma)} \qquad \frac{t_1, \ldots, t_n \in \mathcal{T}(\mathcal{X}, \Sigma) \quad f/n \in \Sigma}{f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{X}, \Sigma)}$$

We note $Var(t)$ the set of variables of the term t. A term t is closed if $Var(t) = \emptyset$.

Example :

Signature :

z :	arity 0	s :	arity 1
$+$:	arity 2	eq :	arity 2
$true$:	arity 0	$false$:	arity 0

Terms :

$$eq(s(x+z),z) \quad eq(true,false) \quad eq(s(x+z),false)$$

closed

Positions of terms

$\mathbb{N}^*$: positions over $\mathbb{N}$

$$\frac{}{\Lambda \in \mathbb{N}^*} \qquad \frac{i \in \mathbb{N} \text{ and } p \in \mathbb{N}^*}{ip \in \mathbb{N}^*}$$

$Pos(t)$: positions of a term t

$$\frac{}{\Lambda \in Pos(t)} \qquad \frac{p \in Pos(t_i) \text{ and } 1 \leq i \leq n}{ip \in Pos(f(t_1, \ldots, t_n))}$$

Example :

$$Pos(f(g(a, h(b)), x, c)) = \{\Lambda, 1, 2, 3, 11, 12, 121\}$$

The relation $\leq_{pref}$ over positions

Concatenation of positions : $\begin{cases} \Lambda.q & = & q \\ (ip).q & = & i(p.q) \end{cases}$

Comparing positions : $p \leq_{pref} q$ iff $\exists r \in \mathbb{N}^* \; p.r = q$

Example :

1	$\leq_{pref}$	1211	"1 is [illegible]"
231	$\geq_{pref}$	23	"231 is [illegible]"
12	$\bowtie$	2	"12 is parallel to 2"

$12 \not\leq_{pref} 2$ & $2 \not\leq_{pref} 12$

Sub-terms

$v \trianglelefteq t$: v is a subterm/subtree of t :

$$\frac{}{t \trianglelefteq t} \qquad \frac{v \trianglelefteq t_i}{v \trianglelefteq f(t_1, \ldots, t_n)}$$

$v \triangleleft t$: v is a strict subterm of t.

$ST(t)$: All the subterms of t.

Example : $g(x, y) \triangleleft f(g(x, y), a)$ and $a \triangleleft f(g(x, y), a)$ but $f(x, a) \not\triangleleft f(g(x, y), a)$.

Sub-term X at position Y

$t|_p$: subterm of t at position p

$$\frac{}{t|_\Lambda = t} \qquad \frac{t_i|_q = v}{f(t_1, \ldots, t_n)|_{iq} = v}$$

Example : $f(g(a, h(b)), x, c)|_{11} = a$ but $f(g(a, h(b)), x, c)|_{21}$ is not defined.

Exercise : Show that $p.q \in Pos(t)$ implies $t|_{p.q} = (t|_p)_q$.

Replacement

$t[p/\!/v]$: replacement of the subterm $t|_p$ by the term v

- $t[\Lambda/\!/v] = v$
- $f(t_1, \ldots, t_n)[ip/\!/v] = f(t_1, \ldots, t_i[p/\!/v], \ldots, t_n)$

Other notations :$t[v]_p$ or $t[v]$ if p is clear from the context.

Example : $f(h(x, y), a)[12/\!/b] = f(h(x, b), a)$ and $f(h(x, y), a)[1/\!/b] = f(b, a)$.

Exercise : Show the following properties :

- If $p \in Pos(s)$ and $q \in Pos(t)$, then $(s[t]_p)|_{p.q} = t|_q$ and $(s[t]_p)[r]_{p.q} = s[t[r]_q]_p$.
- If $p.q \in Pos(s)$, then $(s[t]_{p.q})|_p = (s|_p)[t]_q$ and $(s[t]_{p.q})[r]_p = s[r]_p$.
- If $p, q \in Pos(s)$ and $p \bowtie q$, then $(s[t]_p)|_q = s|_q$ and $(s[t]_p)[r]_q = (s[r]_q)[t]_p$.

Σ-algebras

A Σ-algebra $\mathcal{A}$ is defined by :

– A non-empty domain $\mathbf{A}$.

– An interpretation function $f^{\mathcal{A}} : \mathbf{A^n} \mapsto \mathbf{A}$, for each f/n.

Example : Let $\Sigma = \{z, s, p\}$.

1. $\mathbf{A}$ is the set $\mathbb{N}$, $z^{\mathcal{A}} = 0$, $s^{\mathcal{A}}(n) = n + 1$ and $p^{\mathcal{A}}(n, m) = n + m$.
2. $\mathbf{A}$ is the set $\mathbb{Z}$, $z^{\mathcal{A}} = -5$, $s^{\mathcal{A}}(n) = n * 13$ and $p^{\mathcal{A}}(n, m) = n * m$.
3. Syntactic Algebra : $\mathbf{A}$ is the set of all the terms over $\mathcal{X}$ and $\Sigma = \{z, s, p\}$ such that $z^{\mathcal{A}} = z$, $s^{\mathcal{A}}(t) = s(t)$ and $p^{\mathcal{A}}(t, u) = p(t, u)$.

Congruence

The symbol $f \in \Sigma$ is monotonic w.r.t the relation R iff

$a_i \; R \; b_i$ implies $f^{\mathcal{A}}($ $a_n)$.

reflexive, symmetric, transitive

A congruence $\sim$ is an equivalence relation, s.t. every symbol $f \in \Sigma$ is monotonic w.r.t. $\sim$.

Example : $\sim = \{(x, y) \mid 4 \text{ divides } x - y\}$ is a congruence.

Notation : $\mathcal{A}/_{\sim}$ is the set of equivalence classes of a $\mathcal{A}$ modulo the congruence $\sim$.

The quotient algebra

Quotient algebra $\mathcal{A}^{\sim}$ over $\mathcal{A}$:

- Domain $\mathcal{A}/_{\sim}$
- Interpretations $f^{\mathcal{A}^{\sim}}([a_1], \ldots, [a_n]) = [f^{\mathcal{A}}(a_1, \ldots, a_n)]$.

Example : Let $\Sigma = \{z, suc, pred, add\}$ and let $\mathcal{A}$ be the Σ-algebra given by $\mathbf{A} = \mathrm{Z}$, $suc^{\mathcal{A}}(n) = n + 1$, $pred^{\mathcal{A}}(n) = n - 1$ and $add^{\mathcal{A}}(n, m) = n + m$.

Let $\sim = \{(x, y) \mid 4 \text{ divides } x - y\}$.

We have $\mathcal{A}/_{\sim} = \{[0], [1], [2], [3]\}$, where

$$\begin{aligned} [0] &= \{\ldots, -4, 0, 4, 8, \ldots\} \\ [1] &= \{\ldots, -3, 1, 5, 9, \ldots\} \\ [2] &= \{\ldots, -6, -2, 2, 6, 10, \ldots\} \\ [3] &= \{\ldots, -5, -1, 3, 7, \ldots\} \end{aligned}$$

We have $z^{\mathcal{A}^{\sim}} = [0]$, $suc^{\mathcal{A}^{\sim}}([n]) = [n+1]$, $pred^{\mathcal{A}^{\sim}}([n]) = [n-1]$ and $add^{\mathcal{A}^{\sim}}([n], [m]) = [n+m]$.

Homomorphism, endomorphism, isomorphism

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-algebras. A morphism is a function $\Phi : \mathcal{A} \to \mathcal{B}$ s.t. for all $n \geq 0$, for all $f/n \in \Sigma$ and for all $a_1, \ldots, a_n \in \mathbf{A}$ we have

$$\Phi(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(\Phi(a_1), \ldots, \Phi(a_n))$$

An endomorphism over a Σ-algebra $\mathcal{A}$ is a morphism from $\mathcal{A}$ to itself. An isomorphism is a bijective morphism.

Exercise : Let $\Sigma = \{a, s, g, h\}$. Define a Σ-algebra $\mathcal{B}$ and a morphism Φ between the syntactic Σ-algebra and $\mathcal{B}$.

Let $\mathcal{B}$ be the Σ-algebra defined by :

- The domain $\mathbf{B} = \{n \in \mathbb{N} \mid n \geq 2\}$
- The interpretations :

$$\begin{array}{llll} a^{\mathcal{B}} & = & 2 & \qquad g^{\mathcal{B}}(t) = t+1 \\ s^{\mathcal{B}}(t) & = & t & \qquad h^{\mathcal{B}}(u,t) = u+t \end{array}$$

Let Φ be the following function :

$$\begin{array}{llll} \Phi(a) & = & 2 & \qquad \Phi(g(t)) = \Phi(t)+1 \\ \Phi(s(t)) & = & \Phi(t) & \qquad \Phi(h(u,t)) = \Phi(u)+\Phi(t) \end{array}$$

One verifies $\Phi(f^{\mathcal{A}}(t_1, \ldots, t_n)) = f^{\mathcal{B}}(\Phi(t_1), \ldots, \Phi(t_n))$ for every f/n in Σ.

Valuations

Let $\mathcal{A}$ be a Σ-algebra and let $\mathcal{X}$ be a set of variables.

A $\mathcal{A}$-valuation is an application $\sigma : \mathcal{X} \to \mathcal{A}$.

Theorem : For every $\mathcal{A}$-valuation $\sigma : \mathcal{X} \to \mathcal{A}$, there is a unique morphism $\widehat{\sigma} : \mathcal{T}(\mathcal{X}, \Sigma) \to \mathcal{A}$, s.t.

$$\begin{aligned} \widehat{\sigma}(x) &= \sigma(x) \\ \widehat{\sigma}(f(t_1, \ldots, t_n)) &= f^{\mathcal{A}}(\widehat{\sigma}(t_1), \ldots, \widehat{\sigma}(t_n)) \end{aligned}$$

Remarque : We usually do not distinguish σ and $\widehat{\sigma}$

$\mathcal{A}$-valuations vs $\mathcal{A}^{\sim}$-valuations

Given a $\mathcal{A}$-valuation σ, we can always construct the $\mathcal{A}^{\sim}$-valuation $\sigma^{\sim}(x) = [\sigma(x)]_{\sim}$.

Given a $\mathcal{A}^{\sim}$-valuation $\sigma^{\sim}$, we can always construct a $\mathcal{A}$-valuation σ s.t. $\sigma^{\sim}(x) = [\sigma(x)]_{\sim}$.

Substitutions as valuations

having a finite domain

the set of all the terms

A [illegible] from $\mathcal{X}$ to the syntactic algebra.

Finite substitutions are denoted $\theta = \{x_1/t_1, \ldots, x_n/t_n\}$.

A Renaming is an isomorphic substitution.

Example :

- $\theta_1 = \{x/y, y/x\}$ and $\theta_2 = \{x/y, y/z, z/w\}$ are renamings.
- Given $t = f(x, g(y))$ and $\theta = \{x/g(a), y/f(x, x)\}$ we have $\theta(t) = f(g(a), g(f(x, x)))$.

Semantical equational reasoning

A Σ-equation is a pair of terms denoted $s \doteq t$.

A Σ-algebra $\mathcal{A}$ is a model of a set of Σ-equations $\mathcal{E}$, noté $\mathcal{A} \models \mathcal{E}$, iff $\mathcal{A}$ is a model of every equation of $\mathcal{E}$ (e.g. $\sigma(s) = \sigma(t)$ holds for every $\mathcal{A}$-valuation σ).

Example : The first Σ-algebra on slide (Σ-algebras) is a model of the equation $p(x, y) \doteq p(y, x)$.

Properties of semantical equational reasoning

- $\mathcal{A} \models s \doteq s$.
- If $\mathcal{A} \models s \doteq t$, then $\mathcal{A} \models t \doteq s$.
- If $\mathcal{A} \models s \doteq t$ and $\mathcal{A} \models t \doteq u$, then $\mathcal{A} \models s \doteq u$.
- If $\mathcal{A} \models s \doteq t$, then $\forall u\ \forall p \in Pos(u)$, $\mathcal{A} \models u[s]_p \doteq u[t]_p$.

  Proof. By induction on p. ∎
- If $\mathcal{A} \models s \doteq t$, then $\mathcal{A} \models \theta(s) \doteq \theta(t)$, for every substitution θ.

  Proof. We want to prove $\sigma(\theta(s)) = \sigma(\theta(t))$ for every $\mathcal{A}$-valuation σ so that let us take an arbitrary $\mathcal{A}$-valuation σ. Let τ_σ be the $\mathcal{A}$-valuation given by $\tau_\sigma(x) = \sigma(\theta(x))$. Show by induction on u that $\tau_\sigma(u) = \sigma(\theta(u))$. Now, $\mathcal{A} \models s \doteq t$ implies $\tau(s) = \tau(t)$ for every $\mathcal{A}$-valuation τ, so that in particular for τ_σ. Thus, $\tau_\sigma(s) = \sigma(\theta(s)) = \sigma(\theta(t)) = \tau_\sigma(t)$. ∎

Semantic equational consequence

The equation $s \doteq t$ is a semantic consequence of a set of equations $\mathcal{E}$, written $\mathcal{E} \models s \doteq t$, iff every model of $\mathcal{E}$ is also a model of $s \doteq t$ iff $\mathcal{A} \models \mathcal{E}$ implies $\mathcal{A} \models s \doteq t$.

Syntactic rules for equational reasoning

$$\frac{s \doteq t \in \mathcal{E}}{s \doteq t} \quad \text{(axiom)} \qquad \frac{}{s \doteq s} \quad \text{(reflexivity)}$$

$$\frac{s \doteq t}{t \doteq s} \quad \text{(symmetry)} \qquad \frac{s \doteq t \quad t \doteq u}{s \doteq u} \quad \text{(transitivity)}$$

$$\frac{s \doteq t}{\sigma(s) \doteq \sigma(t)} \quad \text{(substitution)} \qquad \frac{s \doteq t}{u[s]_p \doteq u[t]_p} \quad \text{(context)}$$

Derivation of $s \doteq t$ from the set $\mathcal{E}$ is denoted by $\mathcal{E} \vdash s \doteq t$.

Example

Let $\mathcal{E} = \{0 + x \doteq x, s(y) + x \doteq s(y + x)\}$.

We derive $s(0) + 3 \doteq 4$ from $\mathcal{E}$ as follows :

$$\dfrac{\dfrac{s(y) + x \doteq s(y + x)}{s(0) + 3 \doteq s(0 + 3)}\,(subst) \qquad \dfrac{\dfrac{0 + x \doteq x}{0 + 3 \doteq 3}\,(subst)}{s(0 + 3) \doteq s(3)}\,(ctxt)}{s(0) + 3 \doteq s(3)}\,(tran)$$

The relation $\leftrightarrow_{\mathcal{E}}^{*}$

$$\frac{s \doteq t \in \mathcal{E}}{\sigma(s) \leftrightarrow_{\mathcal{E}} \sigma(t)} \quad (\forall \text{ subst. } \theta) \qquad \frac{s \leftrightarrow_{\mathcal{E}} t}{u[s]_p \leftrightarrow_{\mathcal{E}} u[t]_p} \quad (\forall u \forall p)$$

$\leftrightarrow_{\mathcal{E}}^{*}$ is the reflexive-transitive closure of $\leftrightarrow_{\mathcal{E}}$.

Exercise :

1. $\leftrightarrow_{\mathcal{E}}^{*}$ is stable by substitution.
2. $\leftrightarrow_{\mathcal{E}}^{*}$ is a congruence over $\mathcal{T}(\mathcal{X}, \Sigma)$.

Properties of $\leftrightarrow_{\mathcal{E}}^{*}$

Exercise : $\mathcal{E} \vdash s \doteq t$ if and only if $s \leftrightarrow_{\mathcal{E}}^{*} t$.

Substitution Lemma

Let $\sigma^{\sim}$ be a $\mathcal{A}^{\sim}$-valuation and let θ be a $\mathcal{A}$-valuation s.t. $\sigma^{\sim} : x \mapsto [\theta(x)]_{\sim}$. Then $\sigma^{\sim}(t) = [\theta(t)]_{\sim}$ for every term t.

Proof. By induction on t. ∎

A particular case of this Lemma is given when θ is a substitution and σ a $\mathcal{T}(\mathcal{X}, \Sigma)^{\leftrightarrow^*_{\varepsilon}}$-valuation.

An even more particular case is when θ is the empty substitution so that $t\ \sigma^{\sim}(t) = [t]$.

$\mathcal{T}(\mathcal{X},\Sigma)^{\leftrightarrow^*_\mathcal{E}}$ is a model of $\mathcal{E}$

Take any $s \doteq t \in \mathcal{E}$. Take an arbitrary $\mathcal{T}(\mathcal{X},\Sigma)^{\leftrightarrow^*_\mathcal{E}}$-valuation $\sigma^{\leftrightarrow^*_\mathcal{E}}$.
We've remarked that there exists a $\mathcal{T}(\mathcal{X},\Sigma)$-valuation θ s.t.
$\sigma^{\leftrightarrow^*_\mathcal{E}}(x) = [\theta(x)]$.

Now, $s \doteq t \in \mathcal{E}$ implies
$\mathcal{E} \vdash s \doteq t$ iff $s \leftrightarrow^*_\mathcal{E} t$ implies
$\theta(s) \leftrightarrow^*_\mathcal{E} \theta(t)$ iff $[\theta(s)] = [\theta(t)]$ iff (Subst. Lemma)
$\sigma^{\leftrightarrow^*_\mathcal{E}}(s) = \sigma^{\leftrightarrow^*_\mathcal{E}}(t)$.

Thus, any $\mathcal{T}(\mathcal{X},\Sigma)^{\leftrightarrow^*_\mathcal{E}}$-valuation satifies the equations of $\mathcal{E}$ so that $\mathcal{T}(\mathcal{X},\Sigma)^{\leftrightarrow^*_\mathcal{E}}$ is a model of $\mathcal{E}$.

Birkhoff's Theorem (1933)

Let $\mathcal{E}$ be a set of Σ-equations.

(Soundness) If $\mathcal{E} \vdash s \doteq t$, then $\mathcal{E} \models s \doteq t$.

Proof. By induction on the derivation $\mathcal{E} \vdash s \doteq t$ using the properties of semantical equational reasoning. ∎

(Completeness) If $\mathcal{E} \models s \doteq t$, then $\mathcal{E} \vdash s \doteq t$.

Proof. If $\mathcal{E} \models s \doteq t$, then every model of $\mathcal{E}$ is a a model of $s \doteq t$. Since $\mathcal{T}(\mathcal{X}, \Sigma)^{\leftrightarrow^*_{\mathcal{E}}}$ is a model of $\mathcal{E}$, then $\mathcal{T}(\mathcal{X}, \Sigma)^{\leftrightarrow^*_{\mathcal{E}}} \models s \doteq t$. This means that $\sigma^{\leftrightarrow^*_{\mathcal{E}}}(s) = \sigma^{\leftrightarrow^*_{\mathcal{E}}}(t)$ for every

$\mathcal{T}(\mathcal{X}, \Sigma)^{\leftrightarrow^*_\mathcal{E}}$-valuation $\sigma^{\leftrightarrow^*_\mathcal{E}}$, so in particular for $\tau : x \mapsto [x]_{\leftrightarrow^*_\mathcal{E}}$.
By the Substitution Lemma $\tau(s) = [s]$ and $\tau(t) = [t]$, thus
$[s] = [t]$ which means $s \leftrightarrow^*_\mathcal{E} t$. We conclude $\mathcal{E} \vdash s \doteq t$.

∎

Unification

Unification

Two terms s and t are unifiable iff there exists a substitution (called unifier) s.t. $\theta(s) = \theta(t)$.

Example :

- $f(x, g(x, a))$ and $f(f(a), y)$ are unifiable with
  $\theta = \{x/f(a), y/g(f(a), a)\}$

$$\begin{aligned} f(x, g(x, a)) &\doteq f(f(a), y) \\ f(f(a), g(f(a), a)) &= f(f(a), g(f(a), a)) \end{aligned}$$

- $f(x, g(x, a))$ and $f(f(a), f(b, a))$ are not unifiable.

Principal substitutions

Let θ and τ be two substitutions and $\mathcal{S}$ be a set of substitutions.

- The composition of θ and τ is $(\theta \circ \tau)(x) = \widehat{\theta}(\tau(x))$ for every variable $x \in \mathcal{X}$.
- θ is an instance of τ (or τ is more general than θ) iff there exists a substitution ρ s.t. for every variable $x \in \mathcal{X}$, $(\rho \circ \tau)(x) = \theta(x)$.
- $\tau \in \mathcal{S}$ is principal (or most general) iff every substitution $\theta \in \mathcal{S}$ is an instance of τ.

Example : Let $\sigma_1 = \{y/b, z/h(c)\}$ and $\sigma_2 = \{x/f(y), y/z\}$.

$\sigma_1 \circ \sigma_2 = \{x/f(b), y/h(c), z/h(c)\}$.

σ_2 is more general than $\sigma_1 \circ \sigma_2$.

Principal unifier

Theorem : Let $\mathcal{S}$ be a non-empty set of unifiers of s and t. Then, there exists a principal unifier $\theta \in \mathcal{S}$ s.t. for every $\tau \in \mathcal{S}$, θ is more general than τ. Moreover, this principal unifier is unique modulo renaming.

Idempotent unifiers

A substitution θ is idempotent iff $\theta \circ \theta = \theta$.

Example : $\{y/b, z/h(c)\}$ is idempotent.

$\{x/f(y), y/z\}$ is not idempotent.

Theorem : If s and t are unifiable, then there exists a principal unifier of s and t which is idempotent.

How we can construct this unifier ?

Equational systems

An (Σ)equational system is a set of (Σ)equations of the form $s \doteq t$.

An equational system E is unifiable iff there exists a unifier (called solution) for all the equations of E.

Finite equational systems are denoted $\{s_1 \doteq t_1, \ldots, s_n \doteq t_n\}$.

Solved forms

The equational system $E = \{s_1 \doteq t_1, \ldots, s_n \doteq t_n\}$ is in solved form iff

- All the s_i are distinct variables.
- No s_i appears in t_j.

Example : $E_0 = \{x \doteq y, z \doteq f(a)\}$ is in solved form but
$E_1 = \{x \doteq y, x \doteq f(a)\}$,
$E_2 = \{x \doteq y, y \doteq f(a)\}$,
$E_3 = \{x \doteq z, y \doteq f(y)\}$ do not.

Notation : For the solved system $E = \{\alpha_1 \doteq t_1, \ldots, \alpha_n \doteq t_n\}$ we note $\vec{E}$ the substitution $\{\alpha_1/t_1, \ldots, \alpha_n/t_n\}$.

The transformation rules

$$\frac{E \cup \{s \doteq s\}}{E} \quad \text{(erase)} \qquad \frac{E \cup \{t \doteq \alpha\} \quad t \notin \mathcal{X}}{E \cup \{\alpha \doteq t\}} \quad \text{(orient)}$$

$$\frac{E \cup \{f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n)\}}{E \cup \{s_1 \doteq t_1, \ldots, s_n \doteq t_n\}} \quad \text{(decompose)}$$

$$\frac{E \cup \{\alpha \doteq s\} \quad \alpha \in Var(E) \quad \alpha \notin Var(s)}{E\{\alpha/s\} \cup \{\alpha \doteq s\}} \quad \text{(substitute)}$$

The unification algorithm

1. Take an equational system E
2. Compute a new system P by applying the transformation rules as far as possible.
3. If the system P is in solved form
   - then send the answer $\vec{P}$
   - else fail

Example : Unification of the system

$\{p(a, x, f(g(y))) \doteq p(z, f(z), f(u))\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p(a, x, f(g(y))) \doteq p(z, f(z), f(u))}{a \doteq z, x \doteq f(z), f(g(y)) \doteq f(u)}\, d}{z \doteq a, x \doteq f(z), f(g(y)) \doteq f(u)}\, o}{z \doteq a, x \doteq f(a), \ldots}\, s}{\ldots}\, d}{z \doteq a, x \doteq f(a), u \doteq g(y)}\, o$$

solved form

yields the (idempotent) substitution $\{z/a, x/f(a), u/g(y)\}$.

Soundness and completeness of the algorithm

Theorem : The algorithm terminates.

Theorem : **(Soundness)** If the algorithm finds a substitution $\vec{S}$ for the problem P, then P is unifiable and $\vec{S}$ is a m.g.u. of P.

That is,

If P is not unifiable, then the algorithm fails.

Theorem : **(Completeness)** If the system P is unifiable, then the algorithm computes the m.g.u. of P.

That is,

If the algorithm fails, then the system P is not unifiable.