Confluence

Motivations

Confluence is an undecidable property.

Decision procedures

- Finite
- Sound
- Terminating

Some techniques to show confluence

- Confluence by strong confluence
- Confluence by equivalence
- Confluence by commutation
- Confluence by interpretation
- Confluence by critical pairs
- Confluence by orthogonality
- Confluence by decreasing diagrams

Confluence by strong confluence

Theorem : If $\mathcal{R}$ is strongly confluent, then $\mathcal{R}$ is confluent.

Example :

$$\mathcal{R} = \begin{cases} f(x,x) & \rightarrow & g(x) \\ f(x,y) & \rightarrow & g(y) \\ g(x) & \rightarrow & f(x,a) \end{cases}$$

We only check diagrams for one-step reduction sequences.

Corollary : If $\mathcal{R}$ has the diamond property, then $\mathcal{R}$ is confluent.

Confluence by equivalence

Theorem : Let $\mathcal{R}$ and $\mathcal{S}$ two rewrite systems such that $\longrightarrow_{\mathcal{R}} \subseteq \longrightarrow_{\mathcal{S}} \subseteq \longrightarrow^*_{\mathcal{R}}$ and $\mathcal{S}$ is strongly confluent. Then $\mathcal{R}$ is confluent.

Proof.

- If $\longrightarrow_{\mathcal{R}} \subseteq \longrightarrow_{\mathcal{S}} \subseteq \longrightarrow^*_{\mathcal{R}}$, then $\longrightarrow^*_{\mathcal{R}} = \longrightarrow^*_{\mathcal{S}}$.
- If $\mathcal{S}$ is strongly confluent, then $\mathcal{S}$ is confluent.
- If $\longrightarrow^*_{\mathcal{R}} = \longrightarrow^*_{\mathcal{S}}$, then $\mathcal{R}$ is confluent iff $\mathcal{S}$ is confluent.

∎

(Famous) Example : confluence of β

Show that $(\lambda x.M)N \rightarrow_{\beta} M\{x/N\}$ is confluent.

Define a relation $\gg$ as follows :

$$\frac{}{x \gg x} \qquad \frac{t \gg t'}{\lambda x.t \gg \lambda x.t'}$$

$$\frac{t \gg t' \text{ and } u \gg u'}{t\ u \gg t'\ u'} \qquad \frac{t \gg t' \text{ and } u \gg u'}{(\lambda x.t)\ u \gg t'\{x/u'\}}$$

Let $\mathcal{R} = \beta$ and let $\mathcal{S} = \gg$. Now,

1. Show that $\beta \subseteq \gg \subseteq \beta^*$.
2. Show that $\gg$ is strongly confluent.
3. Conclude that β is confluent (on all the terms).

Confluence by commutation [Hindley-Rosen]

Two systems $\mathcal{R}$ and $\mathcal{S}$ commute iff

$$\longrightarrow^{*}_{\mathcal{R}}$$

$$\downarrow *_{\mathcal{S}} \qquad\qquad\qquad \downarrow *_{\mathcal{S}}$$

$$\longrightarrow^{*}_{\mathcal{R}}$$

Two systems $\mathcal{R}$ and $\mathcal{S}$ strongly commute iff

$$\longrightarrow_{\mathcal{R}}$$

$$\downarrow_{\mathcal{S}} \qquad\qquad\qquad \downarrow *_{\mathcal{S}}$$

$$\longrightarrow^{=}_{\mathcal{R}}$$

Strong Commutation

Theorem :

– If $\mathcal{R}$ and $\mathcal{S}$ strongly commute , then they commute.

– If $\mathcal{R}$ and $\mathcal{S}$ are confluents and commute, then $\mathcal{R} \cup \mathcal{S}$ is confluent.

(Famous) Example : confluence of $\beta\eta$

Example :

$$\begin{array}{lll} (\lambda x.M)N & \rightarrow_\beta & M\{x/N\} \\ \lambda x.M\ x & \rightarrow_\eta & M \qquad\qquad \text{If } x \notin \texttt{fv}(M) \end{array}$$

Let $\mathcal{R} = \beta$ and $\mathcal{S} = \eta$. Now,

- Show that β is confluent (done).
- Show that η is confluent.
- Show that β and η strongly commute.
- Conclude that $\beta \cup \eta$ is confluent.

Confluence by interpretation

Theorem :

Let $\mathcal{R}$ and $\mathcal{S}$ be two relations s.t. $\mathcal{R}$ is confluent and terminating. If there is a relation $\mathcal{T}$ on the set of $\mathcal{R}$-normal forms s.t.

1. $\longrightarrow^*_{\mathcal{T}} \subseteq \longrightarrow^*_{\mathcal{R}\cup\mathcal{S}}$ and
2. $a \longrightarrow_{\mathcal{S}} b$ implies $\mathcal{R}(a) \longrightarrow^*_{\mathcal{T}} \mathcal{R}(b)$

then if $\mathcal{T}$ is confluent, $\mathcal{R} \cup \mathcal{S}$ is also confluent.

(Famous) Example : confluence of λx

$$
\begin{array}{llll}
(\lambda x.t)\ u & \rightarrow_B & t[x/u] & \\
(t\ u)[x/v] & \rightarrow_{\mathtt{x}} & (t[x/v]\ u[x/v]) & \\
(\lambda y.t)[x/v] & \rightarrow_{\mathtt{x}} & \lambda y.t[x/v] & \text{if } x \neq y \,\&\, y \notin FV(v) \\
y[x/v] & \rightarrow_{\mathtt{x}} & y & \text{if } x \neq y \\
x[x/v] & \rightarrow_{\mathtt{x}} & v &
\end{array}
$$

Let $\mathcal{R} = \mathtt{x}$ and $\mathcal{S} = B$ and $\mathcal{T} = \beta$, Now,

- Show that $\mathtt{x}$ is confluent and terminating.
- Show that $\beta \subseteq (\mathtt{x} \cup B)^*$.
- Show that $a \rightarrow_B b$ implies $\mathtt{x}(a) \rightarrow^*_\beta \mathtt{x}(b)$.
- Since β is confluent $\lambda\mathrm{x} = \mathtt{x} \cup B$ is confluent.

Confluence by critical pairs

Lemm

locall

$$\frac{[\text{for all } y \text{ s.t. } x \to y \text{ we have } y \in SN] \Rightarrow x \in SN}{\forall x.\ x \in SN}$$

Proof. (By Huet) By well-founded induction on $s \in SN$.

$$\begin{array}{ccccc}
s & \to & t' & \to^* & \\
\downarrow & LC & & & \downarrow_* \\
u' & \to & & I.H. & \downarrow_* \\
\downarrow_* & I.H. & \downarrow_* & & \downarrow_* \\
u & \to^* & p & \to^* & p'
\end{array}$$

$t' < s$

$u' < s$

∎

Important remark

The following (infinite) system on natural numbers :

$$\mathcal{R} = \begin{cases} 2.n & \rightarrow & 2.n+1 \\ 2.n & \rightarrow & a \\ 2.m+1 & \rightarrow & 2.m+2 \\ 2.m+1 & \rightarrow & b \end{cases}$$

is locally confluent but not confluent : $a \leftarrow 0 \rightarrow^{*} b$

In fact it is not SN

$$0 \rightarrow\ 1 \rightarrow\ 2 \rightarrow\ 3 \rightarrow\ \ldots$$

Towards local confluence : critical pairs

A critical pair between two variable disjoint rules $l \longrightarrow r$ and $g \longrightarrow d$ of $\mathcal{R}$ (not necessarily distinct rules) is a pair $\langle \sigma(r), \sigma(l)[\sigma(d)]_p \rangle$ s.t.

1. $p \in Pos(l)$ and $l|_p$ is not a variable.
2. σ is a principal unifier of $l|_p$ and g.

Observe that

$$\begin{aligned} \sigma(r) \leftarrow \sigma(l) &= \sigma(l)[\sigma(l)|_p]_p \\ &= \sigma(l)[\sigma(l|_p)]_p \\ &= \sigma(l)[\sigma(g)]_p \longrightarrow \sigma(l)[\sigma(d)]_p \end{aligned}$$

Example :

$$\mathcal{R} = \begin{cases} f(g(x), g(y), a) & \rightarrow & j(x, y, a) \\ g(b) & \rightarrow & b \\ h(b) & \rightarrow & b \end{cases}$$

The critical pairs are :

$$j(b, y, a) \leftarrow f(g(b), g(y), a) \rightarrow \; f(b, g(y), a)$$

$$j(x, b, a) \leftarrow f(g(x), g(b), a) \rightarrow \; f(g(x), b, a)$$

Example :

$$\mathcal{R} = \begin{cases} f(f(x)) & \rightarrow & g(x) \\ f(b) & \rightarrow & b \\ b & \rightarrow & a \end{cases}$$

The critical pairs are :

$$\begin{array}{ccccc} g(b) & \leftarrow & f(f(b)) & \rightarrow & f(b) \\ g(f(x)) & \leftarrow & f(f(f(x))) & \rightarrow & f(g(x)) \\ b & \leftarrow & f(b) & \rightarrow & f(a) \end{array}$$

Local confluence by critical pairs

Theorem : Let $\mathcal{R}$ be a rewrite system. Then $\mathcal{R}$ is locally confluent iff every critical pair of $\mathcal{R}$ is joinable.

Proof.

The *only if* implication is trivial.

For the *if* implication, let us take any case of the form

$$v \leftarrow t \rightarrow \ u$$

Three cases are possible :

– Disjoint reductions :

$$f(b, h(b), a) \quad \leftarrow \quad f(g(b), h(b), a) \quad \rightarrow \quad f(g(b), b, a)$$
$$f(b, h(b), a) \quad \rightarrow \quad f(b, b, a) \quad \leftarrow \quad f(g(b), b, a)$$

– Not disjoint and not critical :

$$j(h(b), h(b), a) \quad \leftarrow \quad f(g(h(b)), g(h(b)), a) \quad \rightarrow \quad f(g(h(b)), g(b), a)$$
$$j(h(b), h(b), a) \qquad\qquad\qquad\qquad\qquad\qquad f(g(h(b)), g(b), a)$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$
$$j(b, h(b), a) \quad \rightarrow \quad j(b, b, a) \quad \leftarrow \quad f(g(b), g(b), a)$$

– Not disjoint and critical : we close the diagram by the hypothesis.

∎

Decidable case of confluence

Theorem : Let $\mathcal{R}$ be a finite and SN rewrite system. Then confluence of $\mathcal{R}$ is decidable.

Proof. The algorithm :

1. Generate all the critical pairs.
2. For each critical pair $\langle u, v \rangle$, compute arbitrary some normal form $\widehat{u}$ of u and some normal form $\widehat{v}$ of v. If $\widehat{u} \neq \widehat{v}$, then fail.
3. Otherwise (no fail for some critical pair), succeed.

∎

Remark that

- If the algorithm fails, then there is a critical pair which is not joinable, so $\mathcal{R}$ is not confluent by the previous theorem.
- If the algorithm succeeds, then every critical pair is joinable, so that $\mathcal{R}$ is locally confluent by the previous theorem. To obtain confluence, apply Newmann's Lemma.

Ortho

no duplication of variables on the left of rules

A system is orthogonal iff it is left linear and has no critical pairs.

Example :

$$\begin{array}{lcl} 0 + y & \rightarrow & y \\ s(x) + y & \rightarrow & s(x + y) \\ 0 * y & \rightarrow & 0 \\ s(x) * y & \rightarrow & (x * y) + y \end{array}$$

Confluence by orthogonality

Theorem : If $\mathcal{R}$ is orthogonal, then it is confluent.

Proof. We define a notion of parallel reduction associated to $\mathcal{R}$ for first-order terms :

$$\frac{}{s \gg s} \quad \text{(reflexivity)} \qquad \frac{l \to r \in \mathcal{R} \text{ and } \sigma \text{ a subst.}}{\sigma(l) \gg \sigma(r)} \quad \text{(head)}$$

$$\frac{s_1 \gg t_1 \ldots\ldots s_n \gg t_n}{f(s_1, \ldots, s_n) \gg f(t_1, \ldots, t_n)} \quad \text{(context)}$$

Example :

For $\mathcal{R} = \left\{ \begin{array}{lcl} f(x,y) & \rightarrow & h(y,y) \\ a & \rightarrow & b \end{array} \right.$ we have
$$\begin{array}{c} f(f(a,c),a,f(a,c)) \\ \gg \\ f(h(a,a),b,f(b,c)) \end{array}$$

Now use the confluence by equivalence technique :

1. Observe that $\rightarrow \ \subseteq \ \gg \ \subseteq \ \rightarrow^*$.
2. Show that $\gg$ has the diamond property.
3. Conclude that $\rightarrow$ is confluent.

■

$\gg$ has the diamond property

Define $\sigma \gg \delta$ iff $dom(\sigma) = dom(\delta)$ and $\sigma x \gg \delta x \ \forall x \in dom(\sigma)$.

Lemma : If $\sigma \gg \delta$, then $\sigma t \gg \delta t$ for every term t.

Proof. By induction on t. ∎

Lemma : Let $\mathcal{R}$ be an orthogonal rewriting system. Let s be a strict subterm of l, where $l \to r \in \mathcal{R}$. If $\sigma s \gg t$, then there is δ s.t. $t = \delta s$ and $\sigma \gg \delta$.

Proof. By induction on s.

- If $s = x$, then define $\delta x = t$.
- If $s = f(s_1, \ldots, s_n)$, we distinguish two cases according to the case applied to obtain $\sigma s \gg t$.
  - If $\sigma s = \tau g$, for some $g \to d \in \mathcal{R}$. Then $\sigma \cup \tau$ unifies s and g

contradicting orthogonality of $\mathcal{R}$.

– If $t = f(t_1, \ldots, t_n)$, where $\sigma_i s_i \gg t_i$, for σ_i equal to σ restricted to the variables of s_i. Then the i.h. gives $t_i = \delta_i s_i$ and $\sigma_i \gg \delta_i$. Since l (and s) are linear, then $\sigma = \bigcup \sigma_i$. We take $\delta = \bigcup \delta_i$. It is easy to check $\sigma \gg \delta$ and $t = \delta s$.

∎

Theorem : Let $\mathcal{R}$ be an orthogonal system and $\gg$ its associated parallel reduction relation. The reduction relation $\gg$ has the diamond property.

Proof. Supose $t \gg t_1$ and $t \gg t_2$. We reason by cases.

– If one of these reductions is by reflexivity, then we trivially close the diagram.

– If both of them use substitution, then $t = \sigma l$, $t_1 = \sigma r$, $t = \delta g$

and $t_2 = \delta d$ for $l \to r$ and $g \to d$ in $\mathcal{R}$. If $l \to r = g \to d$, then $\sigma = \delta$ and we trivially close the diagram. Otherwise we can assume that the rules do not share variables so that $(\delta \cup \sigma)$ gives a unification $(\delta \cup \sigma)l = t = (\sigma \cup \delta)g$ which contradicts orthogonality of $\mathcal{R}$.

– If both of them use context, the property holds by the induction hypothesis.

– If one uses context and the other one uses substitution. We have $t = f(t_1, \ldots, t_n) \gg f(u_1, \ldots, u_n) = t_1$ where $t_i \gg u_i$ and $t = \sigma l \gg \sigma r = t_2$. By the second Lemma there is a substitution δ such that $t_1 = \delta l$ and $\sigma \gg \delta$. By the first Lemma $t_2 = \sigma r \gg \delta r$. We close the diagram with $t_1 = \delta l \gg \delta r$.

▪

Important remark (I)

Left linearity alone is not sufficient for confluence.

Example :

$$
\begin{array}{ccccccccc}
a & \longrightarrow & b & b & \longrightarrow & a & c & \longrightarrow & a \\
a & \longrightarrow & c & b & \longrightarrow & d & c & \longrightarrow & e
\end{array}
$$

Two not joinable terms :

$$e \;{}^{*}\!\longleftarrow a \longrightarrow^{*} \; d$$

Important remark (II)

Absence of critical pairs is not sufficient for confluence.

Example :

$$\begin{array}{lcl} f(x,x) & \rightarrow & a \\ f(x,g(x)) & \rightarrow & b \\ c & \rightarrow & g(c) \end{array}$$

Two not joinable terms :

$$b \ {}^{*}\!\!\leftarrow f(c,c) \rightarrow^{*} a$$

Relaxing orthogonality

$$\begin{array}{rcl} \mathtt{por}(\mathtt{t}, x) & \rightarrow & \mathtt{t} \\ \mathtt{por}(x, \mathtt{t}) & \rightarrow & \mathtt{t} \end{array}$$

This system is not orthogonal but the critical pair is trivial.

A system $\mathcal{R}$ is parallel closed iff for every critical pair $\langle u, v \rangle$ of $\mathcal{R}$ we have $v \gg u$.

Theorem :

If $\mathcal{R}$ is left-linear and parallel closed then it is confluent.