
Abstract Rewriting

Formal definition of rewriting

Given a set of **objects** A , an (**abstract**) rewriting system is a **relation** $\mathcal{R} \subseteq A \times A$.

Example :

$A =$ the set of finite sequences over $\{\circ, \bullet, \color{red}\bullet\}$.

$$\text{Rewriting system } \mathcal{R} = \left\{ \begin{array}{l} \circ \bullet \rightarrow_1 \bullet \circ \\ \color{red}\bullet \bullet \rightarrow_2 \bullet \color{red}\bullet \\ \color{red}\bullet \circ \rightarrow_3 \circ \color{red}\bullet \end{array} \right.$$

Closure notions

- An \mathcal{R} -rewrite sequence has the form

$$s = s_0 \rightarrow_{\mathcal{R}} s_1 \rightarrow_{\mathcal{R}} \dots \rightarrow_{\mathcal{R}} s_n = t \text{ (for } n \geq 0 \text{)}.$$

- $s \rightarrow_{\mathcal{R}}^+ t$ is the transitive closure of $\rightarrow_{\mathcal{R}}$.
- $\rightarrow_{\mathcal{R}}^{\overline{}}$ is the reflexive closure.
- $\rightarrow_{\mathcal{R}}^*$ is the reflexive transitive closure of $\rightarrow_{\mathcal{R}}$.

- $\leftrightarrow_{\mathcal{R}}$ is the symmetrique closure.
- $\leftrightarrow_{\mathcal{R}}^*$ is the reflexive, symetrique and transitive closure.

Example of \mathcal{R} -rewrite sequence

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More basic vocabulary

- A term t is **\mathcal{R} -reducible** iff there exists s s.t. $t \rightarrow_{\mathcal{R}} s$.
- A term t is **in \mathcal{R} -normal form** iff it is no \mathcal{R} -reducible.
- A term s is **a \mathcal{R} -normal form** of t iff $t \rightarrow_{\mathcal{R}}^* s$ and s is in \mathcal{R} -normal form.

Different meaning for equivalent terms

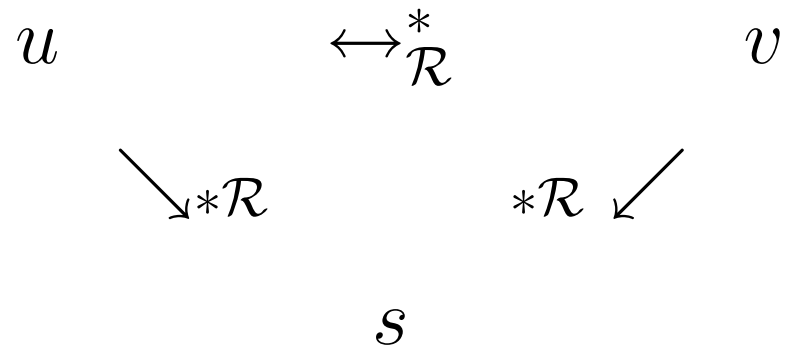
Given again

$$\mathcal{R} = \begin{cases} f(x, x) & \rightarrow c \\ a & \rightarrow b \\ f(x, b) & \rightarrow d \end{cases}$$

we can compute from the same term $f(a, a)$ two different results c and d .

Same meaning for equivalent terms

\mathcal{R} is Church-Rosser iff



Confluence diagrams

A diagram like :

$$\begin{array}{ccccc} t & \mathcal{R}_1 & & & u \\ & & & & \\ \mathcal{R}_2 & & & & \mathcal{R}_3 \\ & & & & \\ v & \mathcal{R}_4 & & & s \end{array}$$

reads :

for all t, u, v such that $t \mathcal{R}_1 u$ and $t \mathcal{R}_2 v$,
there exist s such that $u \mathcal{R}_3 s$ and $v \mathcal{R}_4 s$.

Confluence notions

– \mathcal{R} is **confluent** iff

$$\begin{array}{ccc} t & \xrightarrow{*}_{\mathcal{R}} & u \\ \downarrow *_{\mathcal{R}} & & \downarrow *_{\mathcal{R}} \\ v & \xrightarrow{*}_{\mathcal{R}} & s \end{array}$$

– \mathcal{R} is **locally confluent** iff

$$\begin{array}{ccc} t & \xrightarrow{\mathcal{R}} & u \\ \downarrow \mathcal{R} & & \downarrow *_{\mathcal{R}} \\ v & \xrightarrow{*}_{\mathcal{R}} & s \end{array}$$

– \mathcal{R} is **strongly confluent** iff

$$\begin{array}{ccc} t & \rightarrow_{\mathcal{R}} & u \\ \downarrow_{\mathcal{R}} & & \downarrow_{*\mathcal{R}} \\ v & \rightarrow_{\overline{\mathcal{R}}} & s \end{array}$$

– \mathcal{R} has the **diamond property** iff

$$\begin{array}{ccc} t & \rightarrow_{\mathcal{R}} & u \\ \downarrow_{\mathcal{R}} & & \downarrow_{\mathcal{R}} \\ v & \rightarrow_{\mathcal{R}} & s \end{array}$$

This is a particular case of strongly confluence.

Equivalent notions

Theorem : \mathcal{R} is Church-Rosser iff \mathcal{R} is confluent.

Not equivalent notions

The following system [Curry] :

$$\mathcal{R} = \left\{ \begin{array}{l} c \rightarrow a \\ c \rightarrow d \\ d \rightarrow c \\ d \rightarrow b \end{array} \right.$$

is **locally confluent** but not **confluent** :

$$a \leftarrow c \rightarrow^* b$$

Termination notions

- The **system \mathcal{R} is weakly normalising (WN)** iff every element has at least one \mathcal{R} -normal form.
- The **system \mathcal{R} terminates** or is **strongly normalising (SN)** or **noetherien** or **well-founded (WF)** iff every \mathcal{R} -reduction sequence starting at s is finite. We note $s \in SN_{\mathcal{S}}$.

Weak vs strong normalisation

$$\mathcal{R} = \begin{cases} f(a) & \rightarrow c \\ f(x) & \rightarrow f(a) \end{cases}$$

The system is weakly normalising but not strongly normalising :

$$f(b) \rightarrow f(a) \rightarrow c$$

$$f(b) \rightarrow f(a) \rightarrow f(a) \dots$$

Convergent Systems

Définition : The system \mathcal{R} is **convergent** iff it is confluent and strongly normalising.

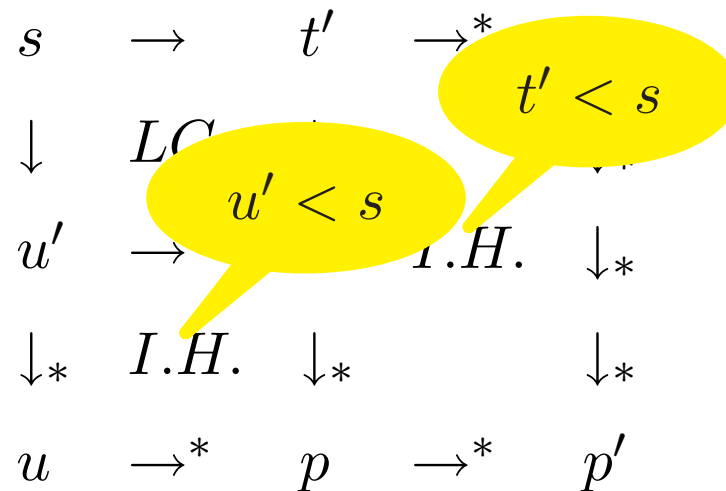
Remarque :

- If \mathcal{R} is confluent, then every element has **at most** a normal form.
- If \mathcal{R} is convergent, then every element has **one and only one** normal form. In this case, we use the *functional* notation $\mathcal{R}(t)$ to denote the \mathcal{R} -normal form of t .

Confluence from local confluence

Lemme : (Newmann) Let \mathcal{R} be a **SN** system. Then \mathcal{R} is locally confluent iff \mathcal{R} is confluent.

Proof. (By Huet) By well-founded induction on $s \in SN$.



Important remark

The following (infinite) system on natural numbers :

$$\mathcal{R} = \left\{ \begin{array}{l} 2.n \quad \rightarrow \quad 2.n + 1 \\ 2.n \quad \rightarrow \quad a \\ 2.m + 1 \quad \rightarrow \quad 2.m + 2 \\ 2.m + 1 \quad \rightarrow \quad b \end{array} \right.$$

is **locally confluent** but not **confluent** : $a \leftarrow 0 \rightarrow^* b$

In fact it is not SN

$$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \dots$$