
From Explicit Substitution Calculi to MELL Proof-Nets

Agenda for Today

- 1 Typing Systems for Explicit Substitutions
- 2 Strong Normalisation for Typed Terms with ES
- 3 From Typed Terms to MELL - Static Translation
- 4 From Typed Terms to MELL - Dynamic Translation

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- Capital greek letters $\Gamma, \Delta, \Pi, \dots$ are used to denote **finite contexts**, defined as partial functions from variables to types.
- Alternative notation for contexts: $x_1 : A_1, \dots, x_n : A_n$.
- **Domain of contexts:** $\text{dom}(\Gamma) = \{x \mid \Gamma(x) \text{ defined}\}$.
- **Union of contexts:**

$$(\Gamma \cup \Delta)(x) := \begin{cases} \Gamma(x) & \text{if } x \in \text{dom}(\Gamma) \ \& \ x \notin \text{dom}(\Delta) \\ \Delta(x) & \text{if } x \notin \text{dom}(\Gamma) \ \& \ x \in \text{dom}(\Delta) \\ \Delta(x) & \text{if } x \in \text{dom}(\Gamma) \ \& \ x \in \text{dom}(\Delta) \ \& \ \text{and } \Gamma(x) = \Delta(x) \\ \text{undefined} & \text{otherwise} \end{cases}$$

- **Union of disjoint contexts:** $\Gamma, \Delta := \Gamma \cup \Delta$ when $\text{dom}(\Gamma) \cap \text{dom}(\Delta) = \emptyset$.

Simple Types for Explicit Substitutions - Additive System

$$\frac{}{x_1 : A_1, \dots, x_n : A_n \vdash x_i : A_i} \text{ (axiom)}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x. t : A \rightarrow B} (\rightarrow \text{ intro}) \quad \frac{\Gamma \vdash t : A \rightarrow B \quad \Gamma \vdash u : A}{\Gamma \vdash tu : B} (\rightarrow \text{ elim})$$

$$\frac{\Gamma \vdash u : B \quad \Gamma, x : B \vdash t : A}{\Gamma \vdash t[x \setminus u] : A} \text{ (cut)}$$

Notation We write $\Gamma \vdash_a t : A$ if $\Gamma \vdash t : A$ is derivable in this system.

Remark If $\Gamma \vdash_a t : A$ is derivable, then $\text{fv}(t) \subseteq \text{dom}(\Gamma)$.

Simple Types for Explicit Substitutions - Multiplicative System

$$\frac{}{x : A \vdash x : A} \text{ (ax)} \quad \frac{\Gamma \vdash t : A \rightarrow B \quad \Delta \vdash u : A}{\Gamma \cup \Delta \vdash tu : B} (\rightarrow \text{ e})$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \rightarrow B} (\rightarrow \text{ i1}) \quad \frac{\Gamma \vdash t : B \quad x \notin \mathbf{fv}(t)}{\Gamma \vdash \lambda x.t : A \rightarrow B} (\rightarrow \text{ i2})$$

$$\frac{\Gamma \vdash u : B \quad \Delta, x : B \vdash t : A}{\Gamma \cup \Delta \vdash t[x \setminus u] : A} \text{ (cut1)}$$

$$\frac{\Gamma \vdash u : B \quad \Delta \vdash t : A \quad x \notin \mathbf{fv}(t)}{\Gamma \cup \Delta \vdash t[x \setminus u] : A} \text{ (cut2)}$$

Notation We write $\Gamma \vdash_m t : A$ if $\Gamma \vdash t : A$ is derivable in this system.

Remark If $\Gamma \vdash_m t : A$, then $\mathbf{fv}(t) = \mathbf{dom}(\Gamma)$.

Relating the Additive and the Multiplicative System

Lemma

If $\Gamma \vdash_a t : A$, then $\Gamma|_{\text{fv}(t)} \vdash_m t : A$.

Lemma

If $\Gamma \vdash_m t : A$, then $\Gamma \vdash_a t : A$.

Exercise:

Give a typing derivation for the term $\lambda w.(xy)[x \setminus a][z \setminus b]$ in the additive as well as in the multiplicative system.

Theorem (Subject Reduction)

- If $\Gamma \vdash_a t : A$ and $t \rightarrow_{\lambda_{\text{pn}}} t'$, then $\Gamma \vdash_a t' : A$.
- If $\Gamma \vdash_m t : A$ and $t \rightarrow_{\lambda_{\text{pn}}} t'$, then $\Gamma' \vdash_m t' : A$ for some $\Gamma' \subseteq \Gamma$.

Proof.

The proof of the first point is by induction on $t \rightarrow_{\lambda_{\text{pn}}} t'$. The proof of the second point uses the following property:

Lemma

Let $\Gamma \vdash_m t : A$.

- If $t \equiv u$, then $\Gamma \vdash_m u : A$.
- If $t \rightarrow_{\text{dB,var,arg,dup}} u$, then $\Gamma \vdash_m u : A$.
- If $t \rightarrow_{\text{gc}} u$, then $\Gamma' \vdash_m u : A$ for some $\Gamma' \subseteq \Gamma$.

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Lemma

- If $\Gamma, x : B \vdash_a t : A$, $\Gamma \vdash_a u : B$, and $t, u \in SN(\lambda pn)$, then $t\{x \setminus u\} \in SN(\lambda pn)$.
- If $\Gamma, x : B \vdash_m t : A$, $\Delta \vdash_m u : B$, and $t, u \in SN(\lambda pn)$, then $t\{x \setminus u\} \in SN(\lambda pn)$.

Proof.

Induction on the lexicographic 3-tuple $\langle B, \eta_{\lambda pn}(t), t \rangle$, where $\eta_{\lambda pn}(t)$ denotes the maximal length of an λpn -reduction sequence starting at t . The proof also uses subject reduction. □

Theorem (SN for λ_{pn})

- If $\Gamma \vdash_a t : A$, then $t \in SN(\lambda_{pn})$.
- If $\Gamma \vdash_m t : A$, then $t \in SN(\lambda_{pn})$.

Proof 1: Induction on typing derivability and previous lemma using ISN.

Proof 2: Using Intersection Types.

Proof 3:

- Define a translation T from λ_{pn} to λ by unfolding substitutions.
- Show that t typable implies $T(t)$ typable in λ -calculus.
- Apply SN Theorem of λ -calculus to get $T(t) \in SN(\beta)$.
- Conclude $T(t) \in SN(\lambda_{pn})$ by PSN.
- Since $T(t) \rightarrow_{\lambda_{pn}}^* t$, then $t \in SN(\lambda_{pn})$.

Proof 4: In a few more slides

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(Call-by-Name) Translation of Formulae

$$\begin{aligned} \iota^+ &:= \iota \\ (A \rightarrow B)^+ &:= ?(A^-) \wp B^+ \\ A^- &:= (A^+)^{\perp} \end{aligned}$$

Observation:

- $(?(A^-) \wp B^+)^{\perp} = !A^+ \otimes B^-$.
- $(?A^-)^{\perp} = !A^+$.

Translation of Derivations

Let $\Gamma = x_1 : B_1, \dots, x_n : B_n$. Then $\Gamma \vdash_m t : A$ translates to a MELL Proof-Net written $(\Gamma \vdash t : A)^{\circ}$ with interface $?\Gamma^-, A^+$, where $?\Gamma^-$ means $?B_1^-, \dots, ?B_n^-$

Alternative Multiplicative Presentation of Simple Types for Explicit Substitutions

$$\frac{}{x : A \vdash x : A} \text{ (ax)} \quad \frac{\Gamma_t, \Gamma_{tu} \vdash t : A \rightarrow B \quad \Gamma_u, \Gamma_{tu} \vdash u : A \quad \Gamma_t \# \Gamma_u}{\Gamma_t, \Gamma_u, \Gamma_{tu} \vdash tu : B} (\rightarrow \text{ e})$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x. t : A \rightarrow B} (\rightarrow \text{ i1}) \quad \frac{\Gamma \vdash t : B \quad x \notin \mathbf{fv}(t)}{\Gamma \vdash \lambda x. t : A \rightarrow B} (\rightarrow \text{ i2})$$

$$\frac{\Gamma_u, \Gamma_{tu} \vdash u : B \quad \Gamma_t, \Gamma_{tu}, x : B \vdash t : A \quad \Gamma_t \# \Gamma_u}{\Gamma_t, \Gamma_u, \Gamma_{tu} \vdash t[x \setminus u] : A} (\text{cut1})$$

$$\frac{\Gamma_u, \Gamma_{tu} \vdash u : B \quad \Gamma_t, \Gamma_{tu} \vdash t : A \quad x \notin \mathbf{fv}(t) \quad \Gamma_t \# \Gamma_u}{\Gamma_t, \Gamma_u, \Gamma_{tu} \vdash t[x \setminus u] : A} (\text{cut2})$$

Translating (ax)

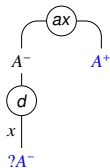
Original derivation:

$$\frac{}{x : A \vdash x : A} \text{ (ax)}$$

Sequent Translation:

$$\frac{}{\vdash A^-, A^+} \\ \frac{}{\vdash ?A^-, A^+}$$

Proof-Net Translation:



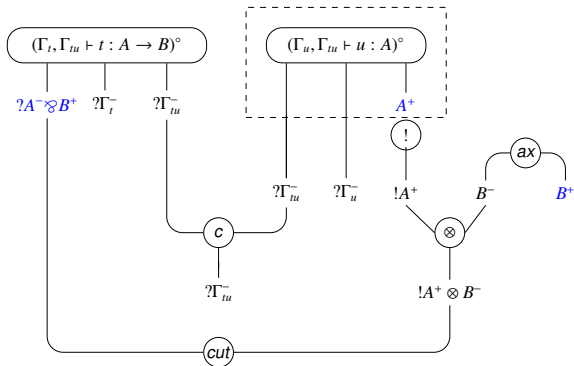
Original derivation:

$$\frac{\Gamma_t, \Gamma_{tu} \vdash t : A \rightarrow B \quad \Gamma_u, \Gamma_{tu} \vdash u : A \quad \Gamma_t \# \Gamma_u}{\Gamma_t, \Gamma_u, \Gamma_{tu} \vdash tu : B} (\rightarrow e)$$

Sequent Translation:

$$\frac{\frac{\frac{i.h.}{\vdash ?\Gamma_t^-, ?\Gamma_{tu}^-, ?A^- \wp B^+} \quad \frac{\frac{i.h.}{\vdash ?\Gamma_u^-, ?\Gamma_{tu}^-, !A^+} \quad \frac{\vdash B^-, B^+}{\vdash ?\Gamma_u^-, ?\Gamma_{tu}^-, !A^+ \otimes B^-, B^+}}{\vdash ?\Gamma_t^-, ?\Gamma_u^-, ?\Gamma_{tu}^-, ?\Gamma_{tu}^-, B^+}}{\vdash ?\Gamma_t^-, ?\Gamma_u^-, ?\Gamma_{tu}^-, B^+}}$$

Proof-Net Translation:



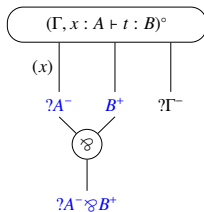
Original derivation:

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x. t : A \rightarrow B} (\rightarrow \text{ i1})$$

Sequent Translation:

$$\frac{\frac{i.h.}{\vdash ?\Gamma^-, ?A^-, B^+}}{\vdash ?\Gamma^-, ?A^- \wp B^+}$$

Proof-Net Translation:



Translating (\rightarrow i2)

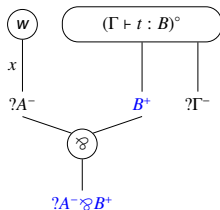
Original derivation:

$$\frac{\Gamma \vdash t : B \quad x \notin \mathbf{fv}(t)}{\Gamma \vdash \lambda x.t : A \rightarrow B} (\rightarrow \text{ i2})$$

Sequent Translation:

$$\frac{\frac{\frac{i.h.}{\vdash ?\Gamma^-, B^+}}{\vdash ?\Gamma^-, ?A^-, B^+}}{\vdash ?\Gamma^-, ?A^- \wp B^+}$$

Proof-Net Translation:



Translating (cut1)

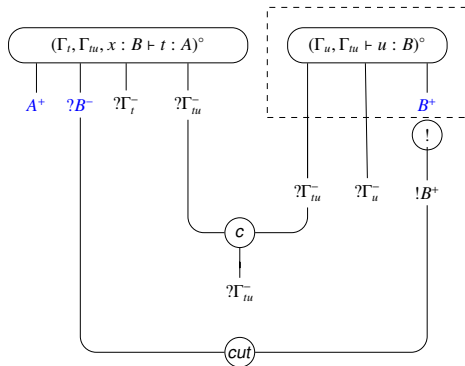
Original derivation:

$$\frac{\Gamma_u, \Gamma_{tu} \vdash u : B \quad \Gamma_t, \Gamma_{tu}, x : B \vdash t : A \quad \Gamma_t \# \Gamma_u}{\Gamma_u, \Gamma_t, \Gamma_{tu} \vdash t[x \setminus u] : A} \text{ (cut1)}$$

Sequent Translation:

$$\frac{\frac{\frac{i.h.}{\vdash ?\Gamma_u^-, ?\Gamma_{tu}^-, B^+}}{\vdash ?\Gamma_u^-, ?\Gamma_{tu}^-, !B^+} \quad \frac{i.h.}{\vdash ?\Gamma_t^-, ?\Gamma_{tu}^-, ?B^-, A^+}}{\vdash ?\Gamma_u^-, ?\Gamma_t^-, ?\Gamma_{tu}^-, A^+}}{\vdash ?\Gamma_u^-, ?\Gamma_t^-, ?\Gamma_{tu}^-, A^+}$$

Proof-Net Translation:



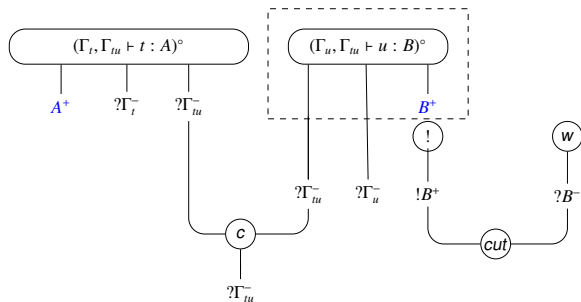
Original derivation:

$$\frac{\Gamma_u, \Gamma_{tu} \vdash u : B \quad \Gamma_t, \Gamma_{tu} \vdash t : A \quad \Gamma_t \# \Gamma_u \quad x \notin \mathbf{fv}(t)}{\Gamma_u, \Gamma_t, \Gamma_{tu} \vdash t[x \setminus u] : A} \text{ (cut2)}$$

Sequent Translation:

$$\frac{\frac{\frac{i.h.}{\vdash ?\Gamma_u^-, ?\Gamma_{tu}^-, B^+}}{\vdash ?\Gamma_u^-, ?\Gamma_{tu}^-, !B^+} \quad (!) \quad \frac{\frac{i.h.}{\vdash ?\Gamma_t^-, ?\Gamma_{tu}^-, A^+}}{\vdash ?\Gamma_t^-, ?\Gamma_{tu}^-, ?B^-, A^+}}{\vdash ?\Gamma_u^-, ?\Gamma_t^-, ?\Gamma_{tu}^-, ?\Gamma_{tu}^-, A^+}}{\vdash ?\Gamma_u^-, ?\Gamma_t^-, ?\Gamma_{tu}^-, A^+}$$

Proof-Net Translation:



- Arguments of applications are translated to boxes.
- Arguments of substitutions are translated to boxes.

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Theorem

Let $\Gamma \vdash_{\text{m}} t : A$ and $t \rightarrow_{\lambda_{\text{PN}}} u$. Consider the derivation $\Gamma' \vdash_{\text{m}} u : A$ (for some $\Gamma' \subseteq \Gamma$) given by the subject reduction property. Then $(\Gamma \vdash t : A)^{\circ} \rightarrow_{\mathcal{R}/\mathcal{E}}^{*} W[(\Gamma' \vdash_{\text{m}} u : A)^{\circ}]$ for some MELL-context W made only of weakenings.

The proof uses the following property:

Lemma

Let $\Gamma \vdash_{\text{m}} t : A$ and $t \rightarrow u$ and $\Gamma' \vdash_{\text{m}} u : A$ as in the statement of the theorem.

- If $t \equiv u$, then $(\Gamma \vdash t : A)^{\circ} \simeq_{\mathcal{E}} (\Gamma \vdash u : A)^{\circ}$.
- If $t \rightarrow_{\text{dB}} u$, then $(\Gamma \vdash t : A)^{\circ} \rightarrow_{\mathcal{C}(\mathcal{I}, \otimes, \mathcal{C}(a))}^{+} (\Gamma \vdash u : A)^{\circ}$.
- If $t \rightarrow_{\text{var}} u$, then $(\Gamma \vdash t : A)^{\circ} \rightarrow_{\mathcal{C}(d,b), \mathcal{C}(a)}^{+} (\Gamma \vdash u : A)^{\circ}$.
- If $t \rightarrow_{\text{arg}} u$, then $(\Gamma \vdash t : A)^{\circ} \rightarrow_{\mathcal{C}(b,b)}^{+} (\Gamma \vdash u : A)^{\circ}$.
- If $t \rightarrow_{\text{dup}} u$, then $(\Gamma \vdash t : A)^{\circ} \rightarrow_{\mathcal{C}(c,b)}^{+} (\Gamma \vdash u : A)^{\circ}$.
- If $t \rightarrow_{\text{gc}} u$, then $(\Gamma \vdash t : A)^{\circ} \rightarrow_{\mathcal{C}(w,b), w^{-b}, w^{-c}}^{+} W[(\Gamma' \vdash u : A)^{\circ}]$.

Theorem (SN for λpn)

If $\Gamma \vdash_m t : A$, then $t \in SN(\lambda\text{pn})$.

Proof 4: using the previous theorem and SN of $\rightarrow_{\mathcal{R}/\mathcal{E}}$.

The notion of σ -equivalence in λ -calculus

$$\begin{array}{llll} (\lambda x. \lambda y. u)v & \equiv_{\sigma_1} & \lambda y. (\lambda x. u)v & \text{if } y \notin \mathbf{fv}(v) \\ (\lambda x. uv)t & \equiv_{\sigma_2} & ((\lambda x. u)t)v & \text{if } x \notin \mathbf{fv}(v) \end{array}$$

Lemma

If $t \equiv_{\sigma} t'$, then $t \equiv_{\beta} t'$.

Theorem (Regnier'90)

If $t \equiv_{\sigma} t'$, then $\eta_{\beta}(t) = \eta_{\beta}(t')$.

Theorem

If $t \equiv_{\sigma} t'$, then t and t' translate to the same MELL Proof-Net (modulo multiplicative cuts).

The notion of σ -equivalence for calculi with ES

Recall the relation \equiv on λpn -terms:

$$\begin{aligned}(\lambda y.u)[x \setminus v] &\equiv \lambda y.u[x \setminus v] && \text{if } y \notin \mathbf{fv}(v) \\(tu)[x \setminus v] &\equiv t[x \setminus v]u && \text{if } x \notin \mathbf{fv}(u) \\t[y \setminus u][x \setminus v] &\equiv t[x \setminus v][y \setminus u] && \text{if } x \notin \mathbf{fv}(u), y \notin \mathbf{fv}(v)\end{aligned}$$

Theorem

If $t \equiv t'$, then $\eta_{\lambda\text{pn}}(t) = \eta_{\lambda\text{pn}}(t')$.

Theorem

If $t \equiv t'$, then t and t' translate to the same MELL Proof-Net (modulo multiplicative cuts).

The λ_{PN} -calculus is a computational interpretation of Natural Deduction plus cut enjoying the following properties:

- Full composition.
- Confluence on terms.
- Confluence on open terms.
- Simulation of one-step β -reduction.
- Preservation of β -strong normalization.
- Strong normalization of well-typed terms.
- Admits a (nice) translation into MELL Proof-Nets.